Capacity of Large Amplify and Forward Relay Networks

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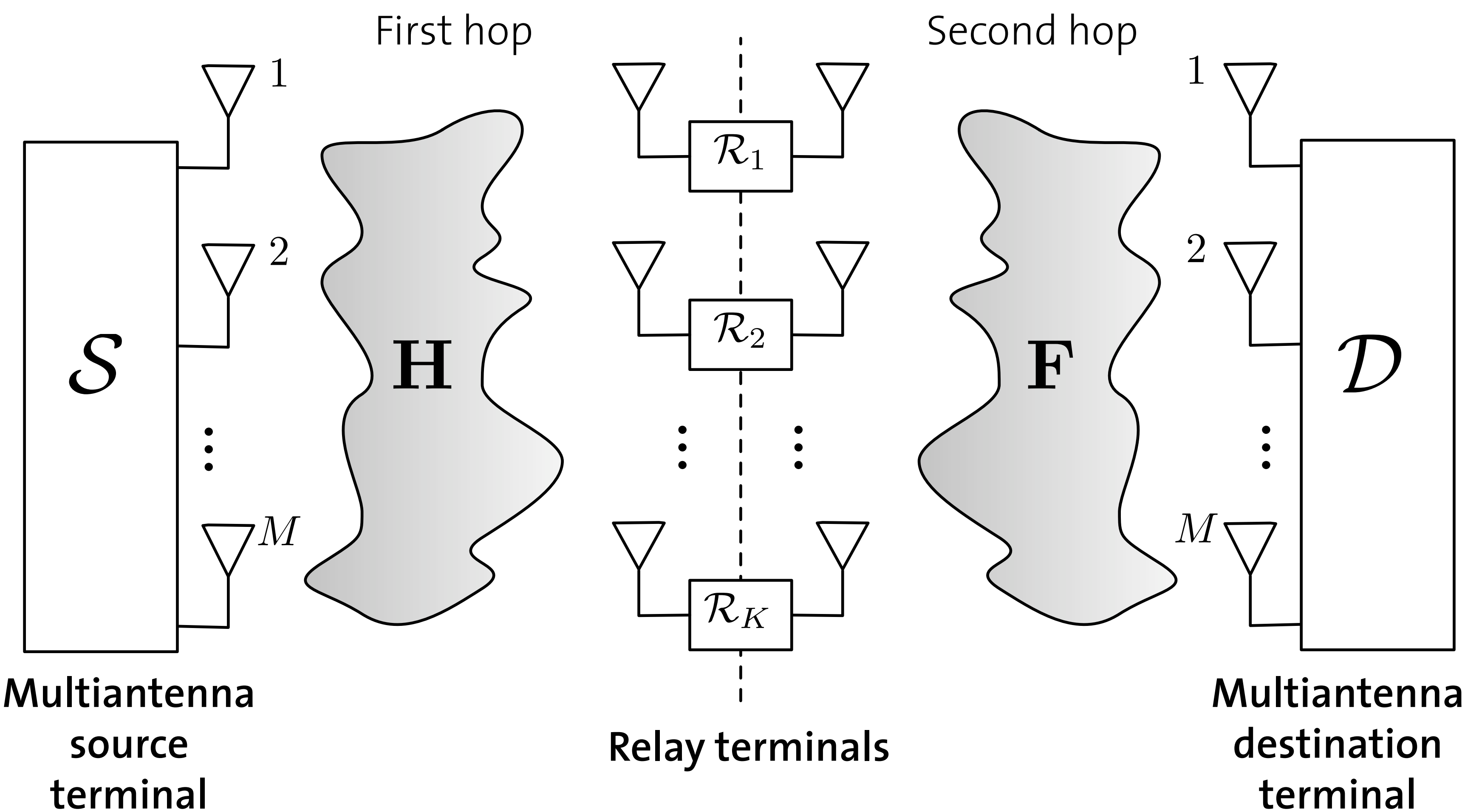
Amplify and Forward (AF) Relay Network



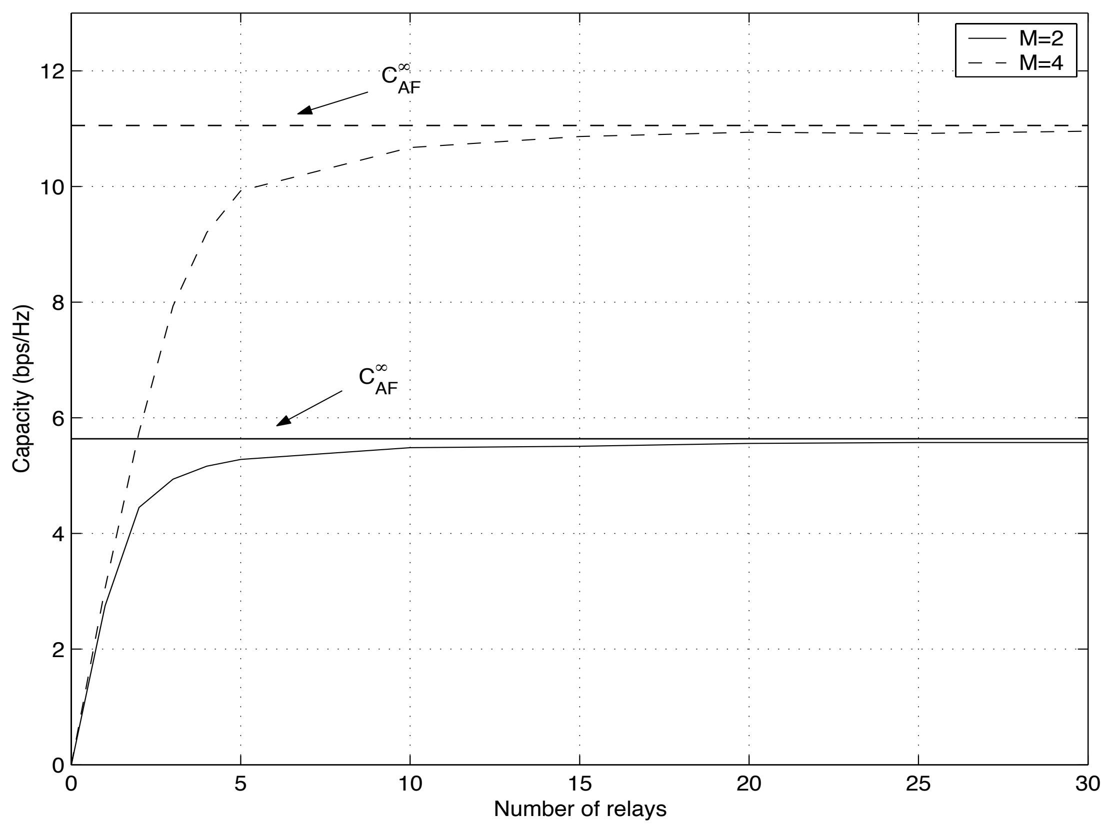
Large K Capacity of AF Relay Network for Finite M

- **Total power constraint** across relays
- **Receiver knows composite MIMO channel**
- For M fixed, in the limit $K \to \infty$, **AF relay network approaches point-to-point MIMO channel** with capacity [HB et. al., 2004]

$$C_{AF}^{\infty} = \frac{1}{2}\mathbb{E}_{\mathbf{H}}\left[\log\det\left(\mathbf{I} + \mathrm{SNR}\,\mathbf{H}_w\mathbf{H}_w^H\right)\right] = \frac{M}{2}\log(\mathsf{SNR}) + O(1)$$

- **Relays** can help to **restore the rank of poor-scattering channels (active (but dumb) scatterers)**

Convergence of Capacity



Capacity vs. number of relays for the AF relay network

Generalization to $M \to \infty$

Assumptions

- Overall I-O relation: $\mathbf{y} = d\mathbf{F}\mathbf{H}\mathbf{s} + d\mathbf{F}\mathbf{n}_r + \mathbf{n}_d$

- Fixed receive SNR at each relay and at each destination node

- $\mathbf{H} \in \mathbb{C}^{K \times M}, \mathbf{F} \in \mathbb{C}^{M \times K}$
  - $\mathbf{H}$ … i.i.d. entries with mean 0 and variance $1/M$
  - $\mathbf{F}$ … i.i.d. entries with mean 0 and variance $1/K$

- $\mathbf{n}_r \in \mathbb{C}^{K \times 1}, \mathbf{n}_d \in \mathbb{C}^{M \times 1}$
  - $\mathbf{n}_r$ … i.i.d. $\mathcal{CN}(0, \sigma_n^2)$ noise at relays
  - $\mathbf{n}_d$ … i.i.d. $\mathcal{CN}(0, \sigma_n^2)$ noise at destination terminal

- Gaussian codebook, receiver knows $\mathbf{F}\mathbf{H}$ and $\mathbf{F}$

Capacity

- **Capacity** of the **effective MIMO channel** is given by

$$C = \frac{1}{2}\mathbb{E}\left[\log\det\left(\mathbf{I} + \frac{d^2}{\sigma_n^2}\mathbf{H}^H\mathbf{F}^H\left(\mathbf{I} + d^2\mathbf{F}\mathbf{F}^H\right)^{-1}\mathbf{F}\mathbf{H}\right)\right]$$
$$= \frac{1}{2}\mathbb{E}\left[\sum_{i=1}^{K}\log\left(1 + \frac{1}{\sigma_n^2}\lambda_i\right)\right]$$

with

$$\lambda_i = \lambda_i(\mathbf{H}\mathbf{H}^H\mathbf{T}) \qquad \text{and} \qquad \mathbf{T} = \mathbf{F}^H\left(\frac{1}{d^2}\mathbf{I} + \mathbf{F}\mathbf{F}^H\right)^{-1}\mathbf{F}$$

- Need to study large M, K-behavior of $\lambda_i(\mathbf{H}\mathbf{H}^H\mathbf{T})$

Brief Review of Large Random Matrix Theory

- For an $M \times M$ random Hermitian matrix $\mathbf{X}$ define the *empirical eigenvalue distribution function (EEDF)* of $\mathbf{X}$ as

$$F_{\mathbf{X}}^{M}(x) = \frac{1}{M}\sum_{i=1}^{M} 1\left\{\lambda_i(\mathbf{X}) \leq x\right\}$$

- From **Large Random Matrix Theory [Wigner, Silverstein, Bai, ...]:**

  Under certain assumptions on $\mathbf{X}$, when $M \to \infty$, $F_{\mathbf{X}}^{M}(x)$ converges almost surely to a deterministic limit, i.e.,

$$F_{\mathbf{X}}^{M}(x) \xrightarrow{\text{a.s.}} F_{\mathbf{X}}(x)$$

  where $F_{\mathbf{X}}(x)$ is the *asymptotic EEDF*

Proof Program (for simplicity of exposition $K = M$)

Goal: Prove convergence of $F^{M}_{\mathbf{HH}^H\mathbf{T}}(x)$ and compute the corresponding asymptotic PDF $f_{\mathbf{HH}^H\mathbf{T}}(x)$

1. [Theorem (Silverstein, 1995)]: If $F^{M}_{\mathbf{T}}(x) \xrightarrow{\text{a.s.}} F_{\mathbf{T}}(x)$, then $F^{M}_{\mathbf{HH}^H\mathbf{T}}(x) \xrightarrow{\text{a.s.}} F_{\mathbf{HH}^H\mathbf{T}}(x)$ with the Stieltjes transform $m_{\mathbf{HH}^H\mathbf{T}}(z)$ given by the unique solution of a fixed-point equation (depends on $F_{\mathbf{T}}(x)$)

2. **Solve the fixed-point equation** and find $m_{\mathbf{HH}^H\mathbf{T}}(z)$

3. **Use the Stieltjes inversion formula** to compute $f_{\mathbf{HH}^H\mathbf{T}}(x)$

4. **Asymptotic per antenna capacity** given by

$$\frac{C}{M} = \frac{1}{2}\int_0^\infty \log\left(1 + \frac{1}{\sigma_n^2}x\right) f_{\mathbf{HH}^H\mathbf{T}}(x)\, dx$$

Computing $f_{\mathbf{T}}(x)$

- Singular value decomposition $\mathbf{F} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H$
- $\mathbf{T}$ can be written as

$$\mathbf{T} = \mathbf{F}^H \left(\frac{1}{d^2}\mathbf{I} + \mathbf{F}\mathbf{F}^H\right)^{-1} \mathbf{F} = \mathbf{V}\,\mathrm{diag}\left\{\frac{\lambda_i}{1/d^2 + \lambda_i}\right\}_{i=1}^{M} \mathbf{V}^H$$

- Marčenko ($\stackrel{\text{a.s.}}{\to}$ Marchenko)-Pastur law [Marčenko and Pastur, 1967] gives asymptotic PDF of eigenvalues of

$$\mathbf{F}^H\mathbf{F} = \mathbf{V}\,\mathrm{diag}\{\lambda_i\}_{i=1}^{M}\,\mathbf{V}^H$$

- $\mathbf{F}^H\mathbf{F}$ and $\mathbf{T}$ are related through a bijection $\Rightarrow$

$$f_{\mathbf{T}}(x) = \frac{1}{d^2(1-x)^2}\, f_{\mathbf{F}^H\mathbf{F}}\left(\frac{x}{d^2(1-x)}\right)$$

Computing $f_{\mathbf{T}}(x)$ (Cont'd)

Lemma 1. [Marčenko-Pastur] *If the matrix* $\mathbf{F} \in \mathbb{C}^{M,M}$ *has i.i.d. entries with mean* 0 *and variance* $1/M$*, then* $F^M_{\mathbf{F}^H\mathbf{F}}(x)$ *converges a.s., as* $M \to \infty$*, to a non-random* $F_{\mathbf{F}^H\mathbf{F}}(x)$ *with corresponding PDF*

$$f_{\mathbf{F}^H\mathbf{F}}(x) = \begin{cases} \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}, & 0 \le x \le 4 \\ 0, & \textit{otherwise} \end{cases}$$

Lemma 2. *Under the same conditions* $F_{\mathbf{T}}(x)$ *converges a.s., as* $M \to \infty$*, to a non-random* $F_{\mathbf{T}}(x)$ *with corresponding PDF*

$$f_{\mathbf{T}}(x) = \begin{cases} \frac{1}{2\pi d^2}\frac{1}{(1-x)^2}\sqrt{\frac{4d^2-(4d^2+1)x}{x}}, & 0 \le x \le 4d^2/(1+4d^2) \\ 0, & \textit{otherwise} \end{cases}$$

Brief Review of Stieltjes Transform

Let $F(x)$ be a distribution function

- **Stieltjes transform:**

$$m_F(z) := \int \frac{f(x)}{x-z}\,dx, \quad z \in \mathbb{C}^+ := \{z \in \mathbb{C} : \Im z > 0\}$$

- **Inversion formula:**

$$f(x) = \frac{1}{\pi} \lim_{y \to 0^+} \Im\left[m_F(x+iy)\right]$$

[Silverstein, 1995]

Assume that

- $\mathbf{H} \in \mathbb{C}^{M \times M}$ has i.i.d. elements with mean 0 and variance $1/M$
- $\mathbf{T} \in \mathbb{C}^{M \times M}$ is a random Hermitian nonnegative definite matrix, with $F_{\mathbf{T}}^{M}(x) \xrightarrow{\text{a.s.}} F_{\mathbf{T}}(x)$ on $[0, \infty)$ as $M \to \infty$
- $\mathbf{H}$ and $\mathbf{T}$ are independent

Then, $F_{\mathbf{H}\mathbf{H}^H\mathbf{T}}^{M}(x) \xrightarrow{\text{a.s.}} F_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(x)$, as $M \to \infty$, with Stieltjes transform satisfying

$$m_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(z) = -\int_{-\infty}^{\infty} \frac{f_{\mathbf{T}}(x)dx}{z\left(x\, m_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(z) + 1\right)}, \quad z \in \mathbb{C}^+$$

The solution of this equation is unique in $\mathbb{C}^+$

Putting the Pieces Together

- Putting the pieces together, we get

$$m_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(z) = -\frac{1}{2\pi d^2}\int_0^{\frac{4d^2}{(4d^2+1)}} \frac{\sqrt{4d^2-(4d^2+1)x}}{(1-x)^2\sqrt{x}} \frac{dx}{z\left(x\, m_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(z)+1\right)}$$

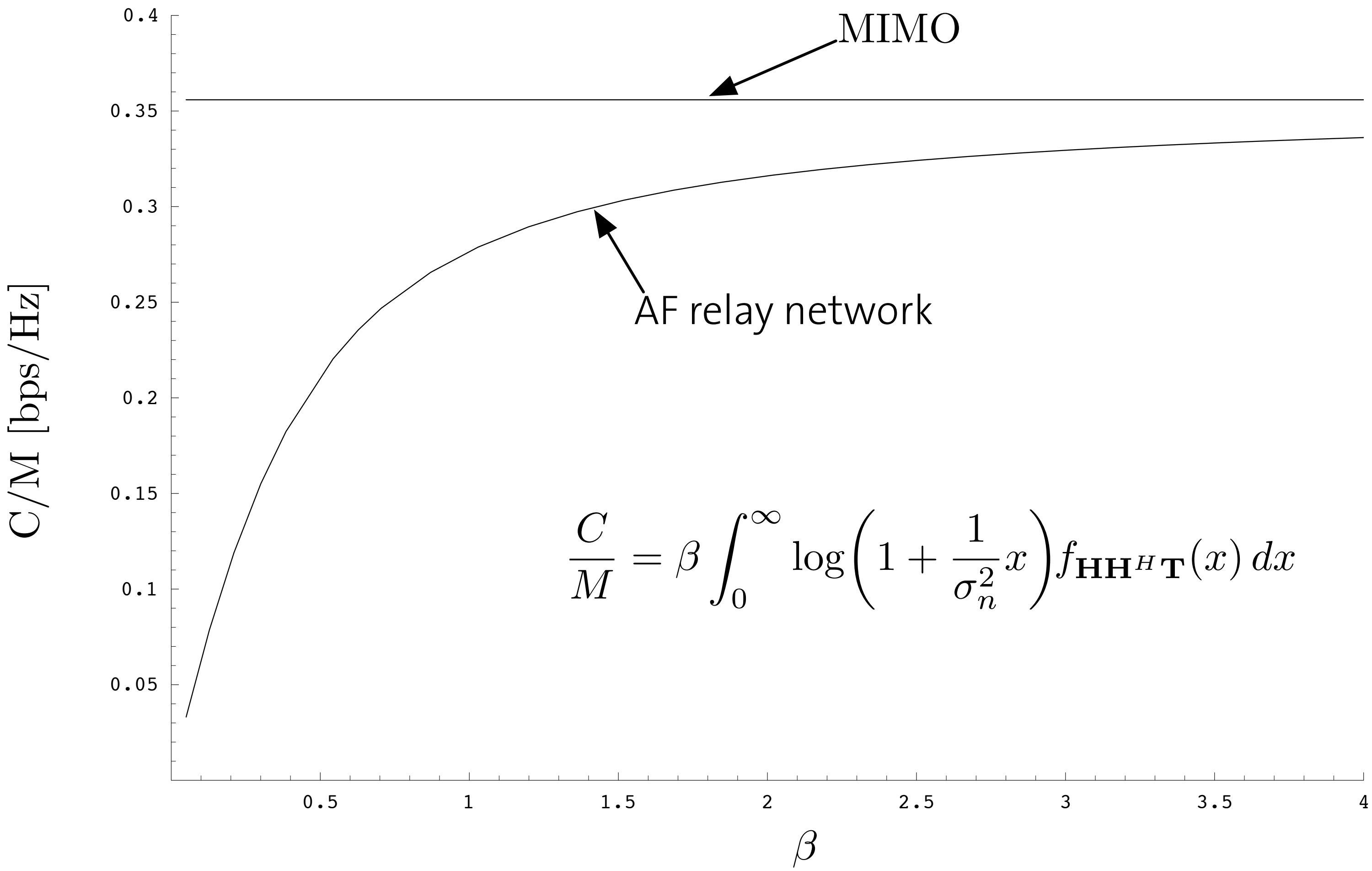
- $\Rightarrow m = m_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(z)$ satisfies the following equation of order 4:

$$d^2z^2m^4 + 2d^2z^2m^3 + (d^2z^2 + 2d^2z - z)m^2 + (2d^2z - 1)m + d^2 = 0$$

- Only one of the roots satisfies the initial equation

- Asymptotic PDF $f_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(x)$ can be computed **analytically** using the Stieltjes inversion formula

Asymptotic Capacity for $d = 1$ as Function of $\beta = K/M$



$$\frac{C}{M} = \beta \int_0^\infty \log\left(1 + \frac{1}{\sigma_n^2} x\right) f_{\mathbf{H}\mathbf{H}^H\mathbf{T}}(x)\, dx$$