Infinity-Norm Sphere-Decoding

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***Abstract*—The most promising approaches for efficient detection in multiple-input multiple-output (MIMO) wireless systems are based on sphere-decoding (SD). The conventional (and optimum) norm that is used to conduct the tree traversal step in SD is the l^2-norm. It was, however, recently shown that using the l^∞-norm instead significantly reduces the VLSI implementation complexity of SD at only a marginal performance loss. These savings are due to a reduction in the length of the critical path and the silicon area of the circuit, but also, as observed previously through simulation results, a consequence of a reduction in the *computational (algorithmic) complexity*. The aim of this paper is an analytical performance and computational complexity analysis of l^∞-norm SD. For i.i.d. Rayleigh fading MIMO channels, we show that l^∞-norm SD achieves full diversity order with an asymptotic SNR gap, compared to l^2-norm SD, that increases at most linearly in the number of receive antennas. Moreover, we provide a closed-form expression for the computational complexity of l^∞-norm SD.**

I. Introduction

The most promising approaches for efficient detection in multiple-input multiple-output (MIMO) systems are based on sphere-decoding (SD) [1]–[4], which amounts to triangularizing the channel matrix and performing a weighted tree search subject to a sphere constraint. While conducting tree traversal using the l^2-norm (referred to as SD-l^2) is optimum, it was observed in [5] that performing tree traversal based on the l^∞-norm instead (referred to as SD-l^∞) results in significantly reduced VLSI implementation complexity at only a marginal performance loss. The results in [5] indicate area-timing products for SD-l^∞ that are up to a factor of 5 lower than those for SD-l^2. These remarkable savings are due to a reduction in the length of the critical path and the silicon area of the circuit, but also, as observed through simulation results in [5], a consequence of a reduction in the *computational (algorithmic) complexity* in terms of the number of nodes visited in the tree search. SD-l^∞ therefore appears to be a promising approach to near-optimum MIMO detection at low hardware complexity.

Contributions: The goal of this paper is to deepen the understanding of SD-l^∞ through an analytical performance and complexity[1] analysis for i.i.d. Rayleigh fading MIMO channels. Our main contributions are as follows:

- We prove that SD-l^∞ achieves full diversity order.
- We show that the gap in signal-to-noise ratio (SNR) incurred by SD-l^∞, compared to SD-l^2, increases at most linearly in the number of receive antennas.
- We derive a closed-form expression for the complexity of SD-l^∞.

Notation: We write $A_{i,j}$ for the entry in the ith row and jth column of the matrix $\mathbf{A}$ and x_i for the ith entry of the vector $\mathbf{x}$. For unitary $\mathbf{A}$, we have $\mathbf{A}^H\mathbf{A} = \mathbf{A}\mathbf{A}^H = \mathbf{I}$, where ^{H} denotes conjugate transposition and $\mathbf{I}$ is the identity matrix. The l^2- and the l^∞-norm of a vector $\mathbf{x} \in \mathbb{C}^M$ are defined as $\|\mathbf{x}\|_2 = \sqrt{|x_1|^2 + \cdots + |x_M|^2}$ and $\|\mathbf{x}\|_\infty = \max\{|x_1|, \ldots, |x_M|\}$, respectively. We will also need the $l^{\widetilde{\infty}}$-norm defined as $\|\mathbf{x}\|_{\widetilde{\infty}} = \max\{|\mathrm{Re}\{x_1\}|, |\mathrm{Im}\{x_1\}|, \ldots, |\mathrm{Im}\{x_M\}|\}$. We note that the l^2-norm is invariant with respect to (w.r.t.) unitary transformations, i.e., $\|\mathbf{x}\|_2 = \|\mathbf{A}\mathbf{x}\|_2$ if $\mathbf{A}$ is unitary. $\mathbb{E}\{\cdot\}$ stands for the expectation operator and $\Phi_x(s) = \mathbb{E}\{e^{sx}\}$ refers to the moment generating function (MGF) of the random variable (RV) x. We say that a RV x is χ-distributed with a degrees of freedom, i.e., $x \sim \chi_a$, if its probability density function (pdf) is given by $f_x(t) = \frac{2^{1-a/2}}{\Gamma(a/2)} t^{a-1} e^{-\frac{t^2}{2}}$, for $t \geq 0$, $f_x(t) = 0$, for $t < 0$, [6], where $\Gamma(a) = \int_0^\infty y^{a-1}e^{-y}dy$ refers to the Gamma function. The cumulative distribution function (cdf) of a χ_a-distributed RV x is given by $P[x \leq t] = \gamma_{a/2}(t^2/2)$. Here, $\gamma_a(t) = \frac{1}{\Gamma(a)}\int_0^t y^{a-1}e^{-y}dy$ denotes the (regularized) lower incomplete Gamma function. If the RV x is χ_a-distributed, x^2 is χ_a^2-distributed. Furthermore, $x \sim \mathcal{CN}(0, \sigma_x^2)$ denotes that the RV x is circularly symmetric complex Gaussian distributed with variance σ_x^2. The "little o" notation $g(x) = o(f(x))$, $x \to x_0$, stands for $\lim_{x\to x_0} g(x)/f(x) = 0$, and $g(x) \overset{a}{\sim} f(x)$, $x \to x_0$, means that $\lim_{x\to x_0} g(x)/f(x) = 1$. Finally, by $g(x) \lesssim f(x)$, $x \to x_0$, for positive functions $g(x)$ and $f(x)$, we denote $\lim_{x\to x_0} g(x)/f(x) \leq 1$.

A. System Model

We consider an $N \times M$ MIMO system with M transmit antennas and $N \geq M$ receive antennas. The corresponding complex-baseband input-output relation is given by

$$\mathbf{r} = \mathbf{H}\mathbf{d}' + \mathbf{w}$$

where $\mathbf{d}' = (d_1' \cdots d_M')^T$ denotes the transmitted data vector, $\mathbf{H}$ is the $N \times M$ channel matrix, $\mathbf{r} = (r_1 \cdots r_N)^T$ is the received vector, and $\mathbf{w} = (w_1 \cdots w_N)^T$ denotes the additive

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[1]In the remainder of the paper, the term "complexity" shall always refer to computational complexity.

noise vector. The symbols d'_m, drawn from a finite alphabet $\mathcal{A}$, have zero-mean and unit variance. Furthermore, we assume that the $H_{n,m}$ are i.i.d. $\mathcal{CN}(0, 1/M)$ and the w_n are i.i.d. $\mathcal{CN}(0, \sigma^2)$. The SNR per receive antenna is $\rho = 1/\sigma^2$.

B. Sphere-Decoding

We now briefly review SD based on the l^2-norm [1]–[4] and (suboptimum) SD based on the l^∞-norm [5].

1) SD based on the l^2-norm: SD-l^2 performs maximum-likelihood (ML) detection by finding

$$\widehat{\mathbf{d}}_{\text{ML}} = \arg\min_{\mathbf{d}\in\mathcal{A}^M} \|\mathbf{r} - \mathbf{H}\mathbf{d}\|_2^2 \tag{1}$$

through a tree search subject to a *sphere constraint* (SC), which amounts to considering only those data vectors $\mathbf{d}$ that satisfy $\|\mathbf{r} - \mathbf{H}\mathbf{d}\|_2^2 \leq C_2^2$. Here, the sphere radius C_2 has to be chosen sufficiently large for the search sphere to contain at least one data vector. The SC is cast into a weighted tree search problem by first performing a QR decomposition of $\mathbf{H}$ resulting in

$$\mathbf{H} = \mathbf{Q}\begin{bmatrix}\mathbf{R}\\ \mathbf{0}\end{bmatrix}$$

where $\mathbf{Q}$ is an $N\times N$ unitary matrix, $\mathbf{R}$ is an $M\times M$ upper triangular matrix, and $\mathbf{0}$ denotes an all-zeros matrix of size $(N-M)\times M$. Then, the SC can equivalently be written as

$$\|\mathbf{z}(\mathbf{d})\|_2^2 \leq C_2^2 \tag{2}$$

where

$$\mathbf{z}(\mathbf{d}) = \mathbf{y} - \begin{bmatrix}\mathbf{R}\\ \mathbf{0}\end{bmatrix}\mathbf{d} \qquad \text{with} \quad \mathbf{y} = \mathbf{Q}^H\mathbf{r} = \begin{bmatrix}\mathbf{R}\\ \mathbf{0}\end{bmatrix}\mathbf{d}' + \mathbf{n}. \tag{3}$$

Here, $\mathbf{n} = \mathbf{Q}^H\mathbf{w}$ is again i.i.d. $\mathcal{CN}(0, \sigma^2)$. The subvectors $\mathbf{d}_k \triangleq (d_{M-k+1} \cdots d_M)^T \in \mathcal{A}^k$, $k = 1, \ldots, M$, can be arranged in a tree with the root above level $k = 1$ and corresponding leaves at level $k = M$; a specific $\mathbf{d}_k$ is associated with a node in this tree at level k. The metric $\|\mathbf{z}(\mathbf{d})\|_2^2$ can then be computed recursively with $\|\mathbf{z}_k(\mathbf{d}_k)\|_2^2 = \|\mathbf{z}_{k-1}(\mathbf{d}_{k-1})\|_2^2 + |\Delta_k(\mathbf{d}_k)|^2$, $k = 1, \ldots, M$, where

$$\Delta_k(\mathbf{d}_k) = y_{M-k+1} - \sum_{i=M-k+1}^{M} R_{M-k+1,i}\, d_i \tag{4}$$

and $\mathbf{z}_k(\mathbf{d}_k)$ contains the last $k + N - M$ elements of $\mathbf{z}(\mathbf{d})$ in (3). Thanks to the upper triangular structure of $\mathbf{R}$, $\mathbf{z}_k(\mathbf{d}_k)$ depends only on $\mathbf{d}_k$. Thus, a necessary condition for $\mathbf{d}$ to satisfy the SC is that any associated $\mathbf{d}_k$ satisfies the *partial SC* (PSC) $\|\mathbf{z}_k(\mathbf{d}_k)\|_2^2 \leq C_2^2$. Consequently, we can find all data vectors satisfying the SC (2) through a weighted tree search. The tree is traversed starting at level $k = 1$. If the PSC is violated by $\mathbf{d}_k$, the node associated with $\mathbf{d}_k$ and all its children are pruned from the tree. The ML solution (1) is found by choosing, among all surviving leaf nodes $\mathbf{d} = \mathbf{d}_M$, the one with minimum $\|\mathbf{z}(\mathbf{d})\|_2$.

2) SD based on the l^∞-norm: We define SD-l^∞ as the algorithm obtained by replacing the SC (2) by the *box constraint* (BC) $\|\mathbf{z}(\mathbf{d})\|_\infty \leq C_\infty$. The metric $\|\mathbf{z}(\mathbf{d})\|_\infty$ can again be computed recursively according to $\|\mathbf{z}_k(\mathbf{d}_k)\|_\infty = \max\big\{\|\mathbf{z}_{k-1}(\mathbf{d}_{k-1})\|_\infty,\, |\Delta_k(\mathbf{d}_k)|\big\}$, where $\Delta_k(\mathbf{d}_k)$ is defined in (4). Consequently, the PSC is replaced by the *partial box constraint* (PBC)

$$\|\mathbf{z}_k(\mathbf{d}_k)\|_\infty \leq C_\infty. \tag{5}$$

If the PBC is violated by $\mathbf{d}_k$, the node associated with $\mathbf{d}_k$ and all its children are pruned from the tree. The l^∞-optimal solution is obtained by choosing, among all surviving leaf nodes $\mathbf{d} = \mathbf{d}_M$, the one with minimum $\|\mathbf{z}(\mathbf{d})\|_\infty$, i.e.,

$$\widehat{\mathbf{d}}_\infty = \arg\min_{\mathbf{d}\in\mathcal{A}^M} \|\mathbf{z}(\mathbf{d})\|_\infty. \tag{6}$$

Slightly abusing terminology, we call the side length C_∞ of the search box the "*radius*" associated with SD-l^∞. Like in the SD-l^2 case, C_∞ has to be chosen large enough to ensure that at least one data vector is found by the algorithm.

Discussion: The SD-$l^{\widetilde{\infty}}$ implementation reported in [5] is actually based on the $l^{\widetilde{\infty}}$-norm (see Section "Notation") rather than the l^∞-norm. Here, the essential aspect is that the computation of the $l^{\widetilde{\infty}}$-norm, as opposed to the l^∞- and l^2-norm, does not require squaring operations, which is the main cause for the critical path length and circuit area reduction in VLSI implementations (see [5, Fig. 2]). Nevertheless, in the following, for the sake of simplicity of exposition, we shall analyze SD-l^∞ based on the conventional l^∞-norm. This already captures the fundamental aspects (w.r.t. performance and complexity) of SD using the $l^{\widetilde{\infty}}$-norm (referred to as SD-$l^{\widetilde{\infty}}$). The modifications of the SD-l^∞ results to account for the use of the $l^{\widetilde{\infty}}$-norm are briefly described in Section IV.

Finally, we emphasize that SD-l^∞ (SD-$l^{\widetilde{\infty}}$) as defined above does *not* correspond to l^∞-norm ($l^{\widetilde{\infty}}$-norm) decoding on the "full" channel matrix $\mathbf{H}$ since $\|\mathbf{r} - \mathbf{H}\mathbf{d}\|_\infty \neq \|\mathbf{z}(\mathbf{d})\|_\infty$, in general. This is in stark contrast to l^2-norm decoding, where $\|\mathbf{r} - \mathbf{H}\mathbf{d}\|_2 = \|\mathbf{z}(\mathbf{d})\|_2$.

II. Error Probability of SD-l^∞

In this section, we show that SD-l^∞ achieves the same diversity order as ML (i.e., SD-l^2) detection and we quantify the SNR loss incurred by SD-l^∞.

A. Diversity Order and SNR Gap

Denoting the error probability as a function of SNR ρ as $\mathrm{P}(\rho)$, the associated SNR exponent δ is defined as $\delta = -\lim_{\rho\to\infty}(\log \mathrm{P}(\rho)/\log\rho)$ [7], [8]. Equivalently, we can write $\mathrm{P}(\rho) = (K\rho)^{-\delta} + o(\rho^{-\delta})$, $\rho \to \infty$, with some constant $K > 0$. If $\mathrm{P}_1(\rho)$ and $\mathrm{P}_2(\rho)$ have the same SNR exponent, we define an asymptotic SNR gap α via $\mathrm{P}_1(\rho) \overset{\alpha}{\sim} \mathrm{P}_2(\alpha\rho)$, $\rho \to \infty$. In the following, we first focus on the behavior of the pairwise error probability (PEP) and then analyze the total error probability. The following considerations correspond to multiplexing gain $r = 0$ in the framework of [8]. Note that even for $r = 0$ conventional suboptimum detection schemes like linear equalization-based or V-BLAST detectors are unable to achieve the full diversity order of N and just realize a diversity order of $N-M+1$ [8]–[10].

1) Pairwise Error Probability: Assume that $\mathbf{d}'$ was transmitted. The probability of erroneously deciding in favor of some other vector $\mathbf{d} \neq \mathbf{d}'$ is denoted as $\mathrm{P}_{\mathbf{d}'\to\mathbf{d},\mathrm{ML}}(\rho)$ in the SD-l^2 case and $\mathrm{P}_{\mathbf{d}'\to\mathbf{d},\infty}(\rho)$ in the SD-l^∞ case.

From (6) it follows that

$$\mathrm{P}_{\mathbf{d}'\to\mathbf{d},\infty}(\rho) \leq \mathrm{P}\Big[\|\mathbf{z}(\mathbf{d})\|_\infty \leq \|\mathbf{z}(\mathbf{d}')\|_\infty\Big] = \mathrm{P}\left[\left\|\begin{bmatrix}\mathbf{R}\\ \mathbf{0}\end{bmatrix}\mathbf{b}+\mathbf{n}\right\|_\infty \leq \|\mathbf{n}\|_\infty\right]. \quad (7)$$

Note that for SD-l^∞, unlike for SD-l^2, the event $\|\mathbf{z}(\mathbf{d})\|_\infty = \|\mathbf{z}(\mathbf{d}')\|_\infty$ can, in general, occur with non-zero probability. Declaring an error in this case certainly yields an upper bound on $\mathrm{P}_{\mathbf{d}'\to\mathbf{d},\infty}(\rho)$. Next, we apply the upper and lower bounds $\frac{1}{N}\|\mathbf{x}\|_2^2 \leq \|\mathbf{x}\|_\infty^2 \leq \|\mathbf{x}\|_2^2$, $\mathbf{x} \in \mathbb{C}^N$, to (7) and exploit the invariance of the l^2-norm to unitary transformations to get

$$\mathrm{P}_{\mathbf{d}'\to\mathbf{d},\infty}(\rho) \leq \mathrm{P}\left[\frac{1}{\sqrt{N}}\left\|\begin{bmatrix}\mathbf{R}\\ \mathbf{0}\end{bmatrix}\mathbf{b}+\mathbf{n}\right\|_2 \leq \|\mathbf{n}\|_2\right] = \mathrm{P}\left[\frac{1}{\sqrt{N}}\|\mathbf{H}\mathbf{b}+\mathbf{w}\|_2 \leq \|\mathbf{w}\|_2\right].$$

Applying the inverse triangle inequality $\|\mathbf{H}\mathbf{b}+\mathbf{w}\|_2 \geq \big|\|\mathbf{H}\mathbf{b}\|_2 - \|\mathbf{w}\|_2\big|$ and noting that $|x| \geq x$, for all $x \in \mathbb{R}$, we further obtain

$$\mathrm{P}_{\mathbf{d}'\to\mathbf{d},\infty}(\rho) \leq \mathrm{P}\left[\|\mathbf{w}\|_2 \geq \frac{1}{\sqrt{N}+1}\|\mathbf{H}\mathbf{b}\|_2\right]. \quad (8)$$

With $\frac{\sqrt{2}}{\sigma}\|\mathbf{w}\|_2 \sim \chi_{2N}$, conditioning on $\mathbf{H}$, and applying the Chernoff upper bound yields

$$\mathrm{P}\left[\|\mathbf{w}\|_2 \geq \frac{1}{\sqrt{N}+1}\|\mathbf{H}\mathbf{b}\|_2 \,\Big|\, \mathbf{H}\right] \leq \Phi_{\chi^2_{2N}}(s)\, e^{-s\rho\frac{2\|\mathbf{H}\mathbf{b}\|_2^2}{(\sqrt{N}+1)^2}} \quad (9)$$

for $0 \leq s < 1/2$ and with $\Phi_{\chi^2_{2N}}(s) = (1-2s)^{-N}$ denoting the MGF of a χ^2_{2N}-distributed RV. Averaging (9) over $\mathbf{H}$ and using the fact that $s = 1/4$ minimizes the resulting right hand side then results in

$$\mathrm{P}_{\mathbf{d}'\to\mathbf{d},\infty}(\rho) \leq 2^N\left(1+\rho\,\frac{\|\mathbf{b}\|_2^2}{2\big(\sqrt{N}+1\big)^2 M}\right)^{-N} \triangleq \mathrm{UB}_\infty(\rho) \quad (10)$$

where we used $2M\|\mathbf{H}\mathbf{b}\|_2^2/\|\mathbf{b}\|_2^2 \sim \chi^2_{2N}$. From (10) we can immediately conclude that the SNR exponent of $\mathrm{P}_{\mathbf{d}'\to\mathbf{d},\infty}(\rho)$ equals N for any non-zero $\mathbf{b}$, as is also the case for ML detection. There is, however, an SNR gap between $\mathrm{P}_{\mathbf{d}'\to\mathbf{d},\infty}(\rho)$ and $\mathrm{P}_{\mathbf{d}'\to\mathbf{d},\mathrm{ML}}(\rho)$, which can be quantified as follows. We start by evaluating [11, Eq. (20)] for the case at hand to get

$$\mathrm{P}_{\mathbf{d}'\to\mathbf{d},\mathrm{ML}}(\rho) \geq \frac{1}{2}\frac{1}{4^N}\binom{2N}{N}\left(1+\rho\,\frac{\|\mathbf{b}\|_2^2}{4M}\right)^{-N} \triangleq \mathrm{LB}_{\mathrm{ML}}(\rho).$$

The asymptotic SNR gap between $\mathrm{UB}_\infty(\rho)$ and $\mathrm{LB}_{\mathrm{ML}}(\rho)$, denoted as β, i.e., $\mathrm{UB}_\infty(\rho) \overset{a}{\sim} \mathrm{LB}_{\mathrm{ML}}(\rho/\beta)$, $\rho \to \infty$, is obtained as

$$\beta = 4\left(\sqrt{N}+1\right)^2\left[\frac{1}{2}\binom{2N}{N}\right]^{-\frac{1}{N}}. \quad (11)$$

We can thus conclude that the asymptotic SNR gap between the PEP of SD-l^∞ and SD-l^2 is upper-bounded by β, or, equivalently, we have

$$\mathrm{P}_{\mathbf{d}'\to\mathbf{d},\infty}(\rho) \lesssim \mathrm{P}_{\mathbf{d}'\to\mathbf{d},\mathrm{ML}}(\rho/\beta). \quad (12)$$

2) Total Error Probability: In the following, we consider the total error probability $\mathrm{P}_\mathcal{E}(\rho) \triangleq \mathrm{P}\big[\mathbf{d}' \neq \widehat{\mathbf{d}}\big]$. We start by noting that

$$\mathrm{P}_\mathcal{E}(\rho) = |\mathcal{A}|^{-M}\sum_{\forall \mathbf{d}'}\mathrm{P}_{\mathcal{E}|\mathbf{d}'}(\rho) \quad (13)$$

assuming equally likely transmitted data vectors $\mathbf{d}'$. Here, $\mathrm{P}_{\mathcal{E}|\mathbf{d}'}(\rho)$ refers to the total error probability conditioned on $\mathbf{d}'$ being transmitted, which can be bounded as

$$\mathrm{P}_{\mathbf{d}'\to\text{any}\,\mathbf{d}}(\rho) \leq \mathrm{P}_{\mathcal{E}|\mathbf{d}'}(\rho) \leq \sum_{\forall \mathbf{d}\neq\mathbf{d}'}\mathrm{P}_{\mathbf{d}'\to\mathbf{d}}(\rho). \quad (14)$$

It follows that

$$\mathrm{P}_\mathcal{E}(\rho) \leq |\mathcal{A}|^{-M}\sum_{\forall \mathbf{d}'}\sum_{\forall \mathbf{d}\neq\mathbf{d}'}\mathrm{P}_{\mathbf{d}'\to\mathbf{d}}(\rho). \quad (15)$$

As the SNR exponent of $\mathrm{P}_{\mathbf{d}'\to\mathbf{d},\infty}(\rho)$ equals N for all $\mathbf{d}'$ and $\mathbf{d} \neq \mathbf{d}'$ (cf. (10)), we can conclude that SD-l^∞ (like ML detection) *achieves full diversity order* N. The corresponding asymptotic SNR gap is obtained as follows. The total error probabilities of SD-l^∞ and SD-l^2 are referred to as $\mathrm{P}_{\mathcal{E}_\infty}(\rho)$ and $\mathrm{P}_{\mathcal{E}_{\mathrm{ML}}}(\rho)$, respectively. With (12)-(15), we get

$$\begin{aligned}\mathrm{P}_{\mathcal{E}_\infty}(\rho) &\leq |\mathcal{A}|^{-M}\sum_{\forall \mathbf{d}'}\sum_{\forall \mathbf{d}\neq\mathbf{d}'}\mathrm{P}_{\mathbf{d}'\to\mathbf{d},\infty}(\rho)\\ &\lesssim |\mathcal{A}|^{-M}\sum_{\forall \mathbf{d}'}\sum_{\forall \mathbf{d}\neq\mathbf{d}'}\mathrm{P}_{\mathbf{d}'\to\mathbf{d},\mathrm{ML}}(\rho/\beta)\\ &\leq |\mathcal{A}|^{-M}\sum_{\forall \mathbf{d}'}\sum_{\forall \mathbf{d}\neq\mathbf{d}'}\mathrm{P}_{\mathcal{E}_{\mathrm{ML}}|\mathbf{d}'}(\rho/\beta)\\ &\leq |\mathcal{A}|^{M}\mathrm{P}_{\mathcal{E}_{\mathrm{ML}}}(\rho/\beta). \end{aligned} \quad (16)$$

Since $\mathrm{P}_{\mathcal{E}_{\mathrm{ML}}}(\rho)$ has SNR exponent N, we can furthermore write $\mathrm{P}_{\mathcal{E}_{\mathrm{ML}}}(\rho) = (K_{\mathrm{ML}}\,\rho)^{-N} + o(\rho^{-N})$, $\rho \to \infty$, with some constant $K_{\mathrm{ML}} > 0$. With $\mathrm{P}_{\mathcal{E}_\infty}(\rho) \lesssim |\mathcal{A}|^M\mathrm{P}_{\mathcal{E}_{\mathrm{ML}}}(\rho/\beta)$ from (16) and $N \geq M$, this yields $\mathrm{P}_{\mathcal{E}_\infty}(\rho) \lesssim \mathrm{P}_{\mathcal{E}_{\mathrm{ML}}}(\rho/(|\mathcal{A}|\beta))$, which establishes that the asymptotic SNR gap incurred by SD-l^∞ is upper-bounded by $|\mathcal{A}|\beta$ with β specified in (11). Furthermore, using $\binom{m}{l} \geq \left(\frac{m}{l}\right)^l$, we have $\binom{2N}{N} \geq 2^N$, which, when used in (11), implies that $\beta \leq 4\left(\sqrt{N}+1\right)^2 \leq 16N$. Thus, the asymptotic SNR gap between the total error probabilities is upper-bounded by $16|\mathcal{A}|N$. Indeed, SD-l^∞ performs much better, in absolute terms, than this simple upper bound suggests (see Section II-B). However, the value of this result resides in demonstrating that the asymptotic SNR gap incurred by SD-l^∞ scales at most linearly in the number of receive antennas.

B. Simulation Results

We next compare the error-rate performance of SD-l^∞ to that of SD-l^2 (ML) detection by means of simulation results. Fig. 1 shows the corresponding error probabilities as a function of SNR ρ for a 2×2, 4×4, and 8×8 MIMO system,

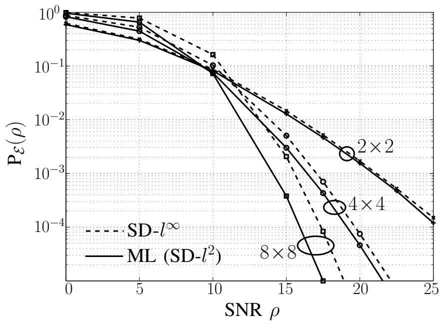

Fig. 1. *Uncoded error probability as a function of SNR ρ for SD-l^∞ and SD-l^2 (ML) detection for a 2×2, 4×4, and 8×8 MIMO system, respectively.*

respectively. In all cases, statistically independent and equally likely 4-QAM symbols were used. We can observe that SD-l^∞ achieves full diversity order and shows near-ML performance with a performance loss that increases for increasing $M = N$.

III. Complexity of SD-l^∞

In this section, we analyze the complexity of SD-l^∞ by deriving an analytic expression for the average number of nodes visited in the tree search. A node $\mathbf{d}_k$ is visited if and only if its corresponding PBC (5) is satisfied. We consider a fixed C_∞ and average w.r.t. channel, noise, and data realizations.

A. Basic Approach

Our methodology is similar to that used in [12], [13] for SD-l^2. The key difference lies in the computation of the partial metric distributions as detailed in Section III-B.

For a given C_∞, a simple counting argument yields the number of nodes $S_{\infty,k}$ visited at tree level k as $S_{\infty,k} = \sum_{\forall \mathbf{d}_k} I(\mathbf{z}_k(\mathbf{d}_k))$, where

$$I(\mathbf{z}_k(\mathbf{d}_k)) \triangleq \begin{cases} 1, & \text{if } \|\mathbf{z}_k(\mathbf{d}_k)\|_\infty \le C_\infty \\ 0, & \text{otherwise.} \end{cases}$$

First, we note that $\mathbb{E}\{I(\mathbf{z}_k(\mathbf{d}_k))\} = \mathrm{P}\big[\|\mathbf{z}_k(\mathbf{d}_k)\|_\infty \le C_\infty\big]$, where the expectation is w.r.t. the channel $\mathbf{R}$, noise $\mathbf{n}$, and data $\mathbf{d}'$. Consequently, we have

$$\mathbb{E}\{S_{\infty,k}\} = \sum_{\forall \mathbf{d}_k} \mathrm{P}\big[\|\mathbf{z}_k(\mathbf{d}_k)\|_\infty \le C_\infty\big] \tag{17}$$

with the average number of visited nodes $\mathbb{E}\{S_\infty\} = \sum_{k=1}^{M} \mathbb{E}\{S_{\infty,k}\}$. Next, we condition on the transmitted data subvector $\mathbf{d}'_k \in \mathcal{A}^k$ and write $\mathrm{P}\big[\|\mathbf{z}_k(\mathbf{d}_k)\|_\infty \le C_\infty \,|\, \mathbf{d}'_k\big] = \mathrm{P}\big[\|\mathbf{z}_k(\mathbf{b}_k)\|_\infty \le C_\infty\big]$ with

$$\mathbf{z}_k(\mathbf{b}_k) = \begin{bmatrix} \mathbf{R}_k \\ \mathbf{0} \end{bmatrix} \mathbf{b}_k + \begin{bmatrix} \mathbf{n}_k \\ \mathbf{n}_L \end{bmatrix} \tag{18}$$

where $\mathbf{b}_k \triangleq \mathbf{d}'_k - \mathbf{d}_k$ is an error subvector, $\mathbf{R}_k$ denotes the $k \times k$ upper triangular submatrix of $\mathbf{R}$ associated with $\mathbf{b}_k$, $\mathbf{n}_k \triangleq (n_{M-k+1} \cdots n_M)^T$, and $\mathbf{n}_L \triangleq (n_{M+1} \cdots n_N)^T$. Consequently, (17) can be written as

$$\mathbb{E}\{S_{\infty,k}\} = \frac{1}{|\mathcal{A}|^k} \sum_{\forall \mathbf{b}_k} \mathrm{P}\big[\|\mathbf{z}_k(\mathbf{b}_k)\|_\infty \le C_\infty\big]. \tag{19}$$

Here, we assumed equally likely transmitted data subvectors $\mathbf{d}'_k$ for all tree levels $k = 1, \ldots, M$, which holds, e.g., for statistically independent and equally likely data symbols.

B. Computation of the Partial Metric Distributions

From (19) we can see that the computation of $\mathbb{E}\{S_{\infty,k}\}$ requires knowledge of the distributions of the partial metrics $\|\mathbf{z}_k(\mathbf{b}_k)\|_\infty$. For SD-$l^2$ this problem was considered in [12] and it was shown that $\|\mathbf{z}_k(\mathbf{b}_k)\|_2$ is a χ-distributed RV, which leads to an expression for $\mathrm{P}\big[\|\mathbf{z}_k(\mathbf{b}_k)\|_2 \le C_2\big]$ in terms of an incomplete Gamma function. The derivation in [12] relies heavily on the fact that the l^2-norm is invariant w.r.t. unitary transformations. Consequently, this approach does not carry over to the l^∞-case considered here. Instead, we follow a direct approach as detailed below.

1) Distribution of $\|\mathbf{z}_k(\mathbf{b}_k)\|_\infty$: Since the nonzero entries in $\mathbf{R}$ are statistically independent [14, Lemma 2.1], the elements of $\mathbf{z}_k(\mathbf{b}_k)$ (conditioned on $\mathbf{b}_k$) are statistically independent as well. We thus have $\mathrm{P}\big[\|\mathbf{z}_k(\mathbf{b}_k)\|_\infty \le C_\infty\big] = \prod_{i=1}^{k+L} \mathrm{P}\Big[\big|[\mathbf{z}_k(\mathbf{b}_k)]_i\big| \le C_\infty\Big]$. For the bottom $L \triangleq N - M$ elements of $\mathbf{z}_k(\mathbf{b}_k)$ (18), $i = k+1, \ldots, k+L$, we have

$$\mathrm{P}\Big[\big|[\mathbf{z}_k(\mathbf{b}_k)]_i\big| \le C_\infty\Big] = 1 - e^{-\frac{C_\infty^2}{\sigma^2}}$$

which yields

$$\mathrm{P}\big[\|\mathbf{z}_k(\mathbf{b}_k)\|_\infty \le C_\infty\big] = \Big(1 - e^{-\frac{C_\infty^2}{\sigma^2}}\Big)^L \prod_{m=1}^{k} \mathrm{P}\Big[\big|[\mathbf{z}(\mathbf{b})]_{M-m+1}\big| \le C_\infty\Big]. \tag{20}$$

2) Distribution of $\big|[\mathbf{z}(\mathbf{b})]_{M-m+1}\big|$: An analytic expression for $\mathrm{P}\Big[\big|[\mathbf{z}(\mathbf{b})]_{M-m+1}\big| \le C_\infty\Big]$ can be obtained via direct integration using the fact that the nonzero entries of $\mathbf{R}$ are statistically independent with $R_{i,i} \sim \chi_{2(N-i+1)}/\sqrt{2M}$ and $R_{i,j} \sim \mathcal{CN}(0, 1/M)$, for $i = 1, \ldots, M$, $j > i$ [14, Lemma 2.1]. We skip the details of the somewhat lengthy derivation and refer the interested reader to [15], where it is shown that the distribution $\mathrm{P}\Big[\big|[\mathbf{z}(\mathbf{b})]_{M-m+1}\big| \le C_\infty\Big]$ is a binomial mixture of χ-distributions with degrees of freedom reaching from 2 up to $2(m+L)$. More specifically, we have

$$\mathrm{P}\Big[\big|[\mathbf{z}(\mathbf{b})]_{M-m+1}\big| \le C_\infty\Big] = \sum_{l=0}^{m+L-1} B_l(\mathbf{b}_m)\, \gamma_{m+L-l}\bigg(\frac{C_\infty^2}{\|\mathbf{b}_m\|_2^2/M + \sigma^2}\bigg) \tag{21}$$

with the coefficients $B_l(\mathbf{b}_m)$ given by the binomial probabilities

$$B_l(\mathbf{b}_m) = \binom{m+L-1}{l} (p(\mathbf{b}_m))^l \, (1-p(\mathbf{b}_m))^{m+L-1-l}$$

with parameter $p(\mathbf{b}_m) = (\|\mathbf{b}_{m-1}\|_2^2 + M\sigma^2)/(\|\mathbf{b}_m\|_2^2 + M\sigma^2)$. The pdf of the RV $\big|[\mathbf{z}(\mathbf{b})]_{M-m+1}\big|^2$ associated with the distribution (21) was found, in a different form, in [16] using an alternative derivation.

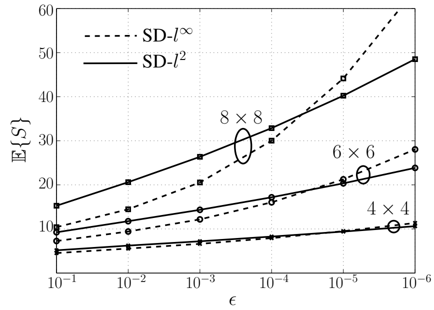

Fig. 2. *Expected number of visited nodes $\mathbb{E}\{S\}$ versus ϵ (see text) for SD-l^∞ and SD-l^2 for a 4×4, 6×6, and 8×8 MIMO system, respectively, with 4-QAM symbol alphabet and SNR $\rho = 15$dB.*

C. Final Complexity Expressions

Inserting (21) into (20) and using (19), we get

$$\mathbb{E}\{S_{\infty,k}\} = \frac{1}{|\mathcal{A}|^k}\left(1 - e^{-\frac{C_\infty^2}{\sigma^2}}\right)^L \sum_{\forall \mathbf{b}_k} \prod_{m=1}^{k} \sum_{l=0}^{m+L-1} B_l(\mathbf{b}_m)\, \gamma_{m+L-l}\left(\frac{C_\infty^2}{\|\mathbf{b}_m\|_2^2/M + \sigma^2}\right). \quad (22)$$

In comparison, for SD-l^2 it was found in [12] that the expected number of visited nodes $\mathbb{E}\{S_{2,k}\}$ at tree level k is given by

$$\mathbb{E}\{S_{2,k}\} = \frac{1}{|\mathcal{A}|^k} \sum_{\forall \mathbf{b}_k} \gamma_{k+L}\left(\frac{C_2^2}{\|\mathbf{b}_k\|_2^2/M + \sigma^2}\right). \quad (23)$$

Discussion and Numerical Results: Comparing (22) with (23) indicates that conducting tree traversal based on the l^∞-norm instead of the l^2-norm, will, in general, have an impact on the complexity of SD. For the following numerical results, we choose the radii C_∞ and C_2 such that SD-l^∞ and SD-l^2 find the transmitted data vector with the same (high) probability of $1-\epsilon$ (for more details see [15], or [12] for SD-l^2). Fig. 2 shows corresponding values of $\mathbb{E}\{S\}$ as a function of ϵ based on (22) and (23) for a 4×4, 6×6, and 8×8 MIMO system, respectively, at an SNR of $\rho = 15$dB. In all cases, statistically independent and equally likely 4-QAM symbols were used. In general, the complexity of SD-l^∞ can be lower or higher than that of SD-l^2. In practical systems, however, one usually operates at ϵ-values, where SD-l^∞ exhibits lower complexity than SD-l^2 [15]. From Fig. 2 we can furthermore infer that the complexity savings of SD-l^∞ tend to be more pronounced for increasing $M = N$. Finally, an asymptotic, in $M = N$, analysis of (22) reveals that the complexity of SD-l^∞ increases exponentially in the system size $M = N$ [15] (as is also the case for SD-l^2, see [17], [16]).

IV. SD based on the $\widetilde{l^\infty}$-norm

In the following, we briefly outline how the results obtained for SD-l^∞ carry over to SD-$\widetilde{l^\infty}$ as employed in [5]. Using

$$\frac{1}{2}\|\mathbf{x}\|_\infty^2 \le \|\mathbf{x}\|_{\widetilde{\infty}}^2 \le \|\mathbf{x}\|_\infty^2, \qquad \mathbf{x} \in \mathbb{C}^N \quad (24)$$

and essentially following the steps (7)-(9), an upper bound on the PEP of SD-$\widetilde{l^\infty}$ is obtained as

$$\mathrm{P}_{\mathbf{d}' \to \mathbf{d}, \widetilde{\infty}}(\rho) \le \mathrm{P}\left[\|\mathbf{w}\|_2 \ge \frac{1}{\sqrt{2N}+1}\|\mathbf{H}\mathbf{b}\|_2\right]$$

which differs from (8) only through the factor 2 multiplying N. Consequently, employing the same arguments as for SD-l^∞ in Section II, we can conclude that SD-$\widetilde{l^\infty}$ *achieves full diversity order N* with an asymptotic SNR gap that increases *at most linearly* in N.

Finally, one can use the direct integration approach leading to (22) and the bounds (24), to compute tight upper and lower bounds on the complexity of SD-$\widetilde{l^\infty}$. One can also show that SD-$\widetilde{l^\infty}$ exhibits exponential complexity in the problem size $M = N$. For detailed results on SD-$\widetilde{l^\infty}$, we refer to [15].

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References

[1] U. Fincke and M. Pohst, "Improved methods for calculating vectors of short length in a lattice, including a complexity analysis," *Math. Comp.*, vol. 44, pp. 463–471, Apr. 1985.
[2] E. Viterbo and E. Biglieri, "A universal decoding algorithm for lattice codes," in *GRETSI 14-ème Colloq.*, Juan-les-Pins, France, pp. 611–614, Sept. 1993.
[3] E. Agrell, T. Eriksson, A. Vardy, and K. Zeger, "Closest point search in lattices," *IEEE Trans. Inf. Theory*, vol. 48, no. 8, pp. 2201–2214, Aug. 2002.
[4] C. Studer, A. Burg, and H. Bölcskei, "Soft-output sphere decoding: Algorithms and VLSI implementation," *IEEE J. on Select. Areas in Comm.*, vol. 26, no. 2, pp. 290–300, Feb. 2008.
[5] A. Burg, M. Borgmann, M. Wenk, M. Zellweger, W. Fichtner, and H. Bölcskei, "VLSI implementation of MIMO detection using the sphere decoding algorithm," *IEEE J. of Solid-State Circuits*, vol. 40, no. 7, pp. 1566–1577, July 2005.
[6] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, 3rd ed. New York: McGraw-Hill, 1991.
[7] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communications: Performance criterion and code construction," *IEEE Trans. Inf. Theory*, vol. 44, no. 2, pp. 744–765, Mar. 1998.
[8] L. Zheng and D. Tse, "Diversity and multiplexing: A fundamental tradeoff in multiple antenna channels," *IEEE Trans. Inf. Theory*, vol. 49, no. 5, pp. 1073–1096, May 2003.
[9] S. Loyka and F. Gagnon, "Performance analysis of the V-BLAST algorithm: An analytical approach," *IEEE Trans. Wireless Comm.*, vol. 3, no. 4, pp. 1326–1337, July 2004.
[10] Y. Jiang, X. Zheng, and J. Li, "Asymptotic performance analysis of V-BLAST," in *Proc. IEEE Globecom 2005*, vol. 6, St. Louis, MO, pp. 3882–3886, Nov. 2005.
[11] H. Lu, Y. Wang, P. Kumar, and K. Chugg, "Remarks on space-time codes including a new lower bound and an improved code," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2752–2757, Oct. 2003.
[12] B. Hassibi and H. Vikalo, "On the sphere decoding algorithm I. Expected complexity," *IEEE Trans. Signal Processing*, vol. 53, no. 8, pp. 2806–2818, Aug. 2005.
[13] A. D. Murugan, H. El Gamal, M. O. Damen, and G. Caire, "A unified framework for tree search decoding: Rediscovering the sequential decoder," *IEEE Trans. Inform. Theory*, vol. 52, no. 3, pp. 933–953, Mar. 2006.
[14] A. M. Tulino and S. Verdú, *Random Matrix Theory and Wireless Communications*. Hanover, MA: Now Publishers Inc. 2004.
[15] D. Seethaler and H. Bölcskei, "Infinity-norm sphere decoding," *in preparation*.
[16] R. Gowaikar and B. Hassibi, "Statistical pruning for near-maximum likelihood decoding," *IEEE Trans. Signal Processing*, vol. 55, no. 6, pp. 2661–2675, June 2007.
[17] J. Jaldén and B. Ottersten, "On the complexity of sphere decoding in digital communications," *IEEE Trans. Signal Processing*, vol. 53, no. 4, pp. 1474–1484, Apr. 2005.