Entropy and Minimax Risk of Hypoelliptic Pseudodifferential Operators

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Abstract

We characterize the entropy and minimax risk of a broad class of compact pseudodifferential operators. Under suitable decay and regularity conditions on the symbol, we combine a Weyl-type asymptotic relation between the eigenvalue-counting function and the phase-space volume of the symbol with a general correspondence between spectral quantities, entropy, and minimax risk for compact operators. This approach yields explicit asymptotic formulae for both entropy and minimax risk directly in terms of the symbol. As an application, we derive sharp entropy and minimax risk asymptotics for unit balls in Sobolev spaces on $\mathbb{R}^d$, thereby extending Pinsker's theorem for Sobolev classes beyond the bounded-domain setting, and showing that the sharp asymptotic constants are determined by phase-space geometry rather than domain geometry.

H. Bölcskei dedicates this paper to Prof. Thomas Kailath on the occasion of his 90th birthday.

1 Introduction

Metric entropy provides a natural measure of the massiveness of classes of mathematical objects—such as functions, dynamical systems, or statistical estimators—by quantifying the number of bits required to approximate their elements uniformly to a given precision. This concept plays a central role in a wide range of mathematical fields, including approximation theory [16, 17, 41–43], harmonic analysis [20–22, 29], high-dimensional statistics and probability [59, 60], and machine learning [3, 25, 34]. The theory of metric entropy itself has also attracted renewed interest in recent years [2–6]; this line of work goes beyond purely scaling-based considerations and develops a more quantitative framework for complexity estimation.

Minimax risk in non-parametric estimation provides a complementary notion of complexity, quantifying the intrinsic difficulty of recovering elements of a class from noisy observations; see, e.g., [37, 47, 58]. While metric entropy and minimax risk are closely related, existing characterizations typically treat them through separate analytical frameworks and under restrictive structural assumptions. The present paper develops a unified framework for characterizing the entropy and minimax risk of a large class of compact pseudodifferential operators. Explicit asymptotic characterizations are derived in terms of the associated operator symbols. These quantities characterize operator compactness by describing, respectively, the effective size of the operator image of the unit ball and the intrinsic difficulty of recovering elements of this image from noisy measurements [2, 4].

We build on the well-known observation that images of pseudodifferential operators are often localized in phase space, with localization properties governed by the associated

operator symbols; see, for example, the discussion of uncertainty principles and phase-space concentration in [26]. This observation motivates characterizing the metric entropy and minimax risk of operator images as a natural and effective way to study the complexity of sets of phase-space-localized signals. Such a viewpoint was adopted by the authors in [4, Section 3.1], where the metric entropy of signal classes subject to time–frequency concentration constraints is analyzed via the ellipsoidal structure of the image of the unit ball under the Landau–Pollak–Slepian operator. This approach yields improvements over the previously best-known entropy characterizations for both the Landau–Pollak–Slepian operator and Sobolev balls.

The central contribution of the present paper is to expand this localization-based methodology by dispensing with explicit ellipsoidal structure and developing a symbol-based framework that applies to broad classes of compact pseudodifferential operators, thereby enabling a unified analysis of entropy and minimax risk under general phase-space localization conditions encoded by symbol decay and regularity.

For compact pseudodifferential operators with sufficiently regular symbols, phase-space localization at the symbol level is reflected in the spectral properties of the operator; see, e.g., [26] for the underlying microlocal principle. A spectral formulation is therefore natural. In the analysis of entropy and minimax risk for compact linear operators, it suffices to restrict attention to operators that are positive and self-adjoint. This follows from the fact that any compact operator admits a polar decomposition, and that metric entropy and minimax risk depend only on the geometry of the image of the unit ball, which is preserved under the isometry induced by the polar decomposition (see, e.g., [4], [52], or [13, Chapter 3.4]). Positive self-adjoint compact operators T admit a spectral decomposition; the corresponding nonzero eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}^*}$, listed in non-increasing order and counted with multiplicity, are nonnegative and tend to zero. The spectral properties of T are typically characterized through the eigenvalue-counting function

$$M_T(\lambda) \coloneqq \#\{n \in \mathbb{N}^* \mid \lambda_n \geq \lambda\}, \quad \text{for all } \lambda > 0. \tag{1}$$

It is well established that the entropy of a positive self-adjoint compact linear operator is governed by its spectral properties; see, e.g., [4, 12, 14, 24, 39, 52]. Expressing the operator in its eigenbasis turns these spectral properties into a geometric description of the image of the unit ball under the operator. In particular, this image is an ellipsoid whose semi-axes are given by the eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}^*}$, see [4]. As shown in [2, Theorems 2, 4, and 5], this ellipsoidal structure yields sharp asymptotic characterizations for both the entropy H_T and the minimax risk R_T of T via the type-τ (for $\tau \geq 1$) integrals

$$I_\tau(\varepsilon) \coloneqq \int_\varepsilon^\infty \frac{M_T(\lambda)}{\lambda^\tau}\, d\lambda, \quad \varepsilon > 0. \tag{2}$$

Specifically, the minimax risk R_T is asymptotically determined by the type-2 and type-3 integrals according to

$$R_T(\kappa) \sim \kappa^2 \varepsilon_\kappa I_2(\varepsilon_\kappa), \quad \text{as } \kappa \to 0, \tag{3}$$

where the critical radius ε_κ is defined as the unique solution of

$$\kappa^2 \left(2I_3(\varepsilon_\kappa) - \frac{I_2(\varepsilon_\kappa)}{\varepsilon_\kappa} \right) = 1, \quad \text{for all } \kappa > 0. \tag{4}$$

Moreover, under a mild regularity condition on the eigenvalue-counting function M_T, the entropy H_T is asymptotically equivalent to the type-1 integral, that is,

$$H_T(\varepsilon) \sim I_1(\varepsilon), \quad \text{as } \varepsilon \to 0. \tag{5}$$

The relations (3)–(5) reduce the characterization of the asymptotic behavior of the entropy and minimax risk of a compact operator to that of its asymptotic spectral distribution. This reduction has already proved effective in [4, Section 3.2] and [2, Section 5], where sharp asymptotic characterizations of metric entropy and minimax risk

for unit balls in Sobolev spaces on bounded domains were derived. In that setting, the operator T is related to the inverse of the Laplacian, and the required asymptotics of the eigenvalue-counting function follow from the Weyl law (see, e.g., [54, Chapter 9.5]) and its Riesz-means counterpart (see [28]). The goal of the present paper is to extend this spectral-reduction strategy to operator classes for which the asymptotic behavior of the eigenvalue-counting function can be derived from symbol-level information, notably compact pseudodifferential operators. We keep the discussion in this section informal; Section 2 defines metric entropy and minimax risk formally, in (12) and (16) respectively.

An important consequence of our results is a conceptual extension of Pinsker's theorem [51] to unbounded domains. Pinsker's original result and its subsequent extensions are formulated in settings where compactness arises from working on bounded domains, leading to compact embeddings of Sobolev balls into L^2. We show that after restoring compactness on $\mathbb{R}^d$ through spatial confinement, the same minimax principle continues to govern statistical estimation. In this setting, the sharp asymptotic constants are determined by phase-space geometry rather than by domain geometry, indicating that Pinsker's theorem reflects a structural feature of nonparametric estimation rather than an artifact of bounded domains.

Pseudodifferential operators are linear operators acting on the Schwartz space

$$\mathcal{S}(\mathbb{R}^d) = \left\{ f \in C^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} \left| x^\alpha \partial^\beta f(x) \right| < \infty \text{ for all } \alpha, \beta \in \mathbb{N}^d \right\},$$

and are of the form

$$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\, \hat{f}(\omega)\, e^{2i\pi x\cdot\omega}\, d\omega, \tag{6}$$

where $\sigma \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is the associated symbol, $x \cdot \omega$ denotes the Euclidean inner product in $\mathbb{R}^d$, and $\hat{f}$ is the Fourier transform of f, defined in (116). Under appropriate regularity assumptions on σ, such operators can be extended compactly to $L^2(\mathbb{R}^d)$ and admit a well-developed spectral theory. Pseudodifferential operators were originally introduced in mathematical physics in connection with quantization; see, e.g., [61, 62]. They subsequently became central tools in harmonic analysis (see [55] and [27, Chapter 2]), time-frequency analysis (see [30, Chapter 14]), electrical engineering in the context of linear-time-varying systems [63, 64], and wavelet theory ([46, Chapter 2.10]). In pure mathematics, pseudodifferential operators play a fundamental role in partial differential equations and microlocal analysis [32, 33, 35, 53, 56, 57], and they form a key ingredient in the proof of the Atiyah-Singer index theorem (see, e.g., [40]).

The mapping (6) that associates a linear operator to a symbol is known as the Kohn–Nirenberg quantization. In addition to (6), two other classical quantization schemes are commonly used, namely the Weyl quantization and the right quantization; these are reviewed in Appendix A. An important aspect of our analysis is that the results do not depend on the particular choice of quantization; this is formalized in Theorem 1. Accordingly, we often write H_σ, M_σ, and R_σ for the entropy, eigenvalue-counting function, and minimax risk of T_σ, omitting explicit reference to the quantization when it is irrelevant.

Many properties of pseudodifferential operators relevant to spectral asymptotics are naturally expressed at the level of the symbol. In particular, for symbols that are positive and decay at infinity, a central quantity in our analysis is the phase-space volume above level $\lambda > 0$, defined by

$$V_\sigma(\lambda) := \int_{\mathbb{R}^{2d}} \mathbb{1}_{\{\sigma(x,\omega) > \lambda\}}\, dx\, d\omega. \tag{7}$$

The function V_σ records the phase-space distribution of the symbol. Under the regularity conditions specified in the main results below, the behavior of $V_\sigma(\lambda)$ as $\lambda \to 0$ quantifies the decay of σ. As recalled in Theorem 8, this gives rise to a Weyl-type asymptotic for the spectral distribution of T_σ, namely

$$M_\sigma(\lambda) \sim V_\sigma(\lambda), \quad \lambda \to 0. \tag{8}$$

Relation (8) serves as the link between symbol-level quantities and the general entropy and minimax-risk characterizations in (3)–(5). Combining these results leads to the main conclusions of the paper, stated in Theorem 1 and Corollaries 2–3. In particular, the entropy satisfies

$$H_\sigma(\varepsilon) \sim \int_{\mathbb{R}^{2d}} \ln_+\left(\frac{\sigma(x,\omega)}{\varepsilon}\right) dx\, d\omega, \quad \varepsilon \to 0, \tag{9}$$

and the minimax risk has the asymptotic behavior

$$R_\sigma(\kappa) \sim \kappa^2 \int_{\mathbb{R}^{2d}} \left(1 - \frac{\varepsilon_\kappa}{\sigma(x,\omega)}\right)_+ dx\, d\omega, \quad \kappa \to 0, \tag{10}$$

where the critical radius ε_κ is determined by

$$\kappa^2 \int_{\mathbb{R}^{2d}} \frac{1}{\sigma(x,\omega)\,\varepsilon_\kappa}\left(1 - \frac{\varepsilon_\kappa}{\sigma(x,\omega)}\right)_+ dx\, d\omega = 1, \quad \kappa > 0. \tag{11}$$

To put (8) into context, we review the classical Weyl-type theory for hypoelliptic operators in Appendix A. In Section 2, we adapt these ideas to the compact symbol-decay setting considered here and use them to establish the entropy–volume relation of Theorem 1. Section 2 also contains the principal conceptual contributions of the paper, namely the asymptotic formulae (9)–(11) together with their interpretation. Finally, Section 3 applies these results to Sobolev spaces on $\mathbb{R}^d$, complementing the corresponding results for bounded domains obtained in [4, Section 3.2] and establishing Pinsker's theorem in the unbounded setting.

Notation. We write $\mathbb{N}$ for the set of natural numbers including zero, $\mathbb{N}^*$ for the set of natural numbers excluding zero, $\mathbb{R}$ for the real numbers, $\mathbb{R}_+^*$ for the positive real numbers, and $\mathbb{C}$ for the complex numbers. $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. For $d \in \mathbb{N}^*$, we denote by ω_d the volume of the unit ball in $\mathbb{R}^d$, and by $\|z\|_2$ the Euclidean norm of $z \in \mathbb{R}^d$.

$L^2(\mathbb{R}^d)$, $C^\infty(\mathbb{R}^d)$, and $C_0^\infty(\mathbb{R}^d)$ designate, respectively, the Lebesgue space of square-integrable functions, the space of infinitely differentiable functions on $\mathbb{R}^d$, and the space of compactly-supported infinitely differentiable functions on $\mathbb{R}^d$. When relevant, we will disambiguate the complex- and real-valued functions in the notation, for instance by writing $L^2(\mathbb{R}^d, \mathbb{R})$ or $L^2(\mathbb{R}^d, \mathbb{C})$, and likewise for C^∞, and C_0^∞. For $f \in C^\infty(\mathbb{R}^d)$, we write ∇f for its gradient and $\partial_i f$ for its partial derivative with respect to the i-th coordinate, where $i \in \{1, \dots, d\}$. A multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ defines the partial derivative $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$, with order $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

When comparing the asymptotic behavior of functions $f, g \colon \mathbb{R}_+^* \to \mathbb{R}_+^*$ as $x \to 0$, we write $f(x) = o_{x\to 0}(g(x))$ if $\lim_{x\to 0} f(x)/g(x) = 0$, and $f(x) = O_{x\to 0}(g(x))$ if there exists a constant $C > 0$ such that $\limsup_{x\to 0} f(x)/g(x) \le C$. Further, $f(x) \sim g(x)$ as $x \to 0$ if $\lim_{x\to 0} f(x)/g(x) = 1$. We write $f(x) \lesssim g(x)$ as $x \to 0$ if there exist constants $C > 0$ and $x_0 > 0$ such that $f(x) \le C\, g(x)$ for all $x \in (0, x_0)$, and $f(x) \gtrsim g(x)$ as $x \to 0$ if $g(x) \lesssim f(x)$. Moreover, $f(x) \asymp g(x)$ as $x \to 0$ if there exist constants $0 < c \le C < \infty$ and $x_0 > 0$ so that $c\, g(x) \le f(x) \le C\, g(x)$, $x \in (0, x_0)$. These relations are used analogously as $x \to \infty$, with $(0, x_0)$ replaced by (x_0, ∞), in particular for sequences as $n \to \infty$, and for functions of $z \in \mathbb{R}^d$ as $\|z\|_2 \to \infty$. We further say that f is regularly varying at zero with index $\nu \in \mathbb{R}$ if $\lim_{x\to 0} f(tx)/f(x) = t^\nu$ for every $t > 0$.

Finally, $\ln(\cdot)$ denotes the natural logarithm, $\ln_+(x) = \max\{0, \ln(x)\}$ for $x \ge 0$ (with the convention $\ln_+(0) = 0$), $(x)_+ = \max\{0, x\}$ for $x \in \mathbb{R}$, and $\mathbb{1}_X(\cdot)$ stands for the indicator function of the set X.

2 Entropy and Minimax Risk of Compact Hypoelliptic Pseudodifferential Operators

We recall the definition of metric entropy for compact sets and the associated notion of entropy for compact linear operators. Let $\mathcal{H}$ be a separable real Hilbert space and let $\mathcal{K} \subset \mathcal{H}$ be a compact set. For $\varepsilon > 0$, an ε-covering of $\mathcal{K}$ is a finite set $\{x_1, \dots, x_N\} \subset \mathcal{H}$ with the property that, for every $x \in \mathcal{K}$, there exists an index $i \in \{1, \dots, N\}$ such that

$$\|x - x_i\|_{\mathcal{H}} \leq \varepsilon.$$

The ε-covering number $N(\varepsilon; \mathcal{K}, \|\cdot\|_{\mathcal{H}})$ is defined as the cardinality of a smallest such ε-covering; the metric entropy $H(\varepsilon; \mathcal{K}, \|\cdot\|_{\mathcal{H}})$ of $\mathcal{K}$ is the natural logarithm of the ε-covering number.

Given a compact linear operator $T\colon \mathcal{H} \to \mathcal{H}$, the entropy of T is defined as the metric entropy of the closure of the image of the unit ball $\mathcal{B}_{\mathcal{H}} \subset \mathcal{H}$ under T, namely

$$H_T(\varepsilon) := \ln N\Big(\varepsilon; \overline{T(\mathcal{B}_{\mathcal{H}})}, \|\cdot\|_{\mathcal{H}}\Big). \tag{12}$$

Since T is compact, the set $\overline{T(\mathcal{B}_{\mathcal{H}})}$ is compact in $\mathcal{H}$, and the right-hand side of (12) is therefore finite for every $\varepsilon > 0$.

The symbols treated in the classical literature on spectral asymptotics for pseudodifferential operators are predominantly drawn from the class $\mathrm{H}\Gamma^{+}(\mathbb{R}^{2d})$ of positive-order hypoelliptic symbols; see Theorem 7 and the review of the standard spectral theory for hypoelliptic pseudodifferential operators in Appendix A. In this regime, the associated operators are necessarily non-compact. Consequently, their entropy is not well defined and their minimax risk does not converge to zero. By contrast, operators whose symbols lie in $\mathrm{H}\Gamma^{-}(\mathbb{R}^{2d})$ can be extended to compact operators on $L^2(\mathbb{R}^d, \mathbb{C})$; see, for example, [53, Theorem 24.4]. The results developed here build on spectral asymptotics for such compact hypoelliptic operators, recalled in Theorem 8.

The volume function (7) is defined through the superlevel sets of the symbol σ and hence require it to be real-valued. While standard pseudodifferential calculus, recalled in Appendix A, is developed for operators between spaces of complex-valued functions, the characterization of entropy and minimax risk we draw from [2] pertains to ellipsoids in the real sequence space $\ell^2(\mathbb{N}^*)$, and the function classes we consider consist of real-valued functions. We hence restrict our attention to pseudodifferential operators that map real-valued functions to real-valued functions and so act on $L^2(\mathbb{R}^d, \mathbb{R})$; we say that such operators preserve the real structure. It is shown in Appendix A that a pseudodifferential operator with real-valued symbol preserves the real structure if its symbol is frequency-even, that is, $\sigma(x, -\omega) = \sigma(x, \omega)$ for all $(x, \omega) \in \mathbb{R}^{2d}$.

The asymptotic formulae derived below are obtained under the assumption that the volume function V_σ is regularly varying at zero with negative index. This assumption excludes highly irregular decay behavior while encompassing the polynomial decay rates that arise naturally in hypoelliptic pseudodifferential calculus, as well as mild logarithmic perturbations thereof. Its role is to ensure sufficient regularity of the associated eigenvalue-counting function to allow for a precise evaluation of the leading-order asymptotics in the integral characterizations of entropy and minimax risk.

More concretely, regular variation of the volume function V_σ implies condition (RC) in [2, Theorem 2], allowing one to pass from spectral information to the entropy and minimax-risk asymptotics in (9)–(11). We emphasize that condition (RC) is never verified directly in our arguments, and enters only through the standing regular-variation assumption on V_σ. Background on regular variation and a detailed proof of the implication 'V_σ regularly varying $\Rightarrow$ condition (RC)' can be found in [9] and in [2, Appendix C]. Regular variation of V_σ neither implies nor is implied by hypoellipticity of σ, as the examples in Appendix A show.

Theorem 1. *Let $\sigma \in \mathrm{H}\Gamma^{-}(\mathbb{R}^{2d})$ be a real-valued frequency-even symbol, and assume that the associated volume function V_σ is regularly varying at zero with negative index. Then,*

$$H_\sigma(\varepsilon) \sim \int_\varepsilon^\infty \frac{V_\sigma(\lambda)}{\lambda}\, d\lambda, \qquad \varepsilon \to 0, \tag{13}$$

where H_σ denotes the entropy of the operator associated with σ, independently of whether σ is quantized using the left, right, or Weyl quantization. Moreover, if σ is strictly positive and the operator $T_{\sigma^{-1}}$ is invertible, then

$$H_{T^{-1}_{\sigma^{-1}}}(\varepsilon) \sim H_\sigma(\varepsilon), \qquad \varepsilon \to 0. \tag{14}$$

Proof. See Section 4.1. □

The proof of Theorem 1 is carried out within the standard hypoelliptic pseudodifferential operator framework summarized in Appendix A, following [53]. The asymptotics in (13) readily extend to operators on $L^2(\mathbb{R}^d, \mathbb{C})$, with the right-hand side of (13) multiplied by two, since a complex ellipsoid with positive semi-axes $\mu_1, \mu_2, \dots$ can be viewed as a real ellipsoid with semi-axes $\mu_1, \mu_1, \mu_2, \mu_2, \dots$.

The following heuristic motivates the use of (8) in the derivation of the entropy and minimax-risk asymptotics (9) and (10). Guided by the uncertainty principle, one expects the eigenfunctions of T_σ to be essentially localized in phase space on regions of unit volume. Within such a localized region centered at a point $(x_0, \omega_0) \in \mathbb{R}^{2d}$, the symbol σ may be regarded as approximately constant, with value $\sigma(x_0, \omega_0)$.

Under this approximation, the number of eigenvalues exceeding a level $\lambda > 0$ is expected to be proportional to the number of disjoint unit-volume regions of phase space centered at points (x, ω) for which $\sigma(x, \omega) > \lambda$. This leads naturally to the volume-based relation (8). A related heuristic for elliptic operators is discussed by Fefferman in [26]. Of course, the argument above is purely heuristic; a central contribution of the present work is to identify precise conditions under which the insertion of (8) into the entropy and minimax-risk integrals can be rigorously justified in the compact, symbol-decay setting considered here.

The volume principle underlying (8) is reminiscent of the approximate diagonalization results in underspread operator theory [8, 23, 31, 38, 44, 45, 48], in that both frameworks interpret the Weyl symbol as an effective time–frequency transfer function and relate spectral quantities to its pointwise values through a phase-space localization principle. In that setting, one expects entropy formulae of the same log-integral form as (9), paralleling Shannon-type expressions for frequency-selective channels. Developing a rigorous analogue of Theorem 1 for underspread operators is an interesting direction for future work.

We now present a reformulation of Theorem 1 that makes the connection between entropy and the symbol explicit. Again, the symbol σ in Corollary 2 may be taken to be the left, right, or Weyl symbol of the associated operator.

Corollary 2. *Let $\sigma \in \mathrm{H}\Gamma^{-}(\mathbb{R}^{2d})$ be a real-valued strictly positive frequency-even symbol and assume that the associated volume function V_σ is regularly varying at zero. Then,*

$$H_\sigma(\varepsilon) \sim \int_{\mathbb{R}^{2d}} \ln_+\!\left(\frac{\sigma(x,\omega)}{\varepsilon}\right) dx\, d\omega, \qquad \varepsilon \to 0. \tag{15}$$

Proof. See Section 4.2. □

Recall that, after normalization by a factor $\ln(2)$, the metric entropy of a compact set admits an information-theoretic interpretation as the number of bits required to uniformly encode the set with accuracy ε. From this perspective, relation (15) suggests the following interpretation. The phase space $\mathbb{R}^{2d}$ may be partitioned into regions of unit volume,

centered at points (x, ω), each contributing to the total entropy the number of bits required to encode the local symbol value $\sigma(x, \omega)$ to within ε. At the heuristic level, this contribution is given by $\lceil \log_+(\sigma(x, \omega)/\varepsilon)/\ln(2) \rceil$, which is equivalent in the limit $\varepsilon \to 0$ to $\log_+(\sigma(x, \omega)/\varepsilon)/\ln(2)$. Corollary 2 shows that this phase-space heuristic is in fact correct and yields a sharp characterization of the leading-order term of the entropy. Cells on which $\sigma(x, \omega) \leq \varepsilon$ contribute nothing, so that the entropy aggregates the contributions of the active cells alone.

Minimax risk has been studied extensively in nonparametric estimation for function classes and compact subsets of Hilbert spaces, in particular through the analysis of the spectral decay of associated compact operators, such as covariance operators, embedding operators, or diagonal operators arising from basis expansions; see, for example, [36, 51, 58]. In these settings, the function class is typically represented implicitly as the image of a unit ball under a compact linear operator, and minimax risk is characterized in terms of the eigenvalues or singular values of that operator.

In contrast, the present framework treats minimax risk as a functional of the operator itself and aims to characterize its asymptotic behavior directly at the symbol level. This operator-centric perspective makes it possible to link minimax risk explicitly to phase-space geometry and spectral asymptotics, in parallel with the corresponding entropy analysis developed above.

We make this precise in two steps, defining the minimax risk in the Gaussian white-noise model first for a compact set $\mathcal{K} \subset \mathcal{H}$ and then for a compact operator. Let $\{u_j\}_{j \in \mathbb{N}^*}$ be an arbitrary orthonormal basis of $\mathcal{H}$, and suppose that, for an unknown $x \in \mathcal{K}$, one observes the sequence $y = \{y_j\}_{j \in \mathbb{N}^*}$ with $y_j = \langle x, u_j \rangle + \kappa \xi_j$, for all $j \in \mathbb{N}^*$, where $\kappa > 0$ is the noise level and $\{\xi_j\}_{j \in \mathbb{N}}$ are independent standard normal random variables. The minimax risk of estimating x from y is

$$R(\kappa; \mathcal{K}) := \inf_{\hat{x}_\kappa} \sup_{x \in \mathcal{K}} \mathbb{E}_{y \sim x}\Big[\big\| \hat{x}_\kappa(y) - x \big\|_{\mathcal{H}}^2 \Big], \tag{16}$$

where the infimum is over all measurable estimators $\hat{x}_\kappa \colon \mathbb{R}^{\mathbb{N}^*} \to \mathcal{H}$, with $\mathbb{R}^{\mathbb{N}^*}$ endowed with the product σ-algebra, and $\mathbb{E}_{y \sim x}$ denotes expectation with respect to the law of y for fixed x. The risk does not depend on the choice of the basis, as the law of the noise is invariant under orthogonal transformations. The minimax risk R_T of a compact linear operator $T \colon \mathcal{H} \to \mathcal{H}$ is now defined as in (16) with $\mathcal{K} = \overline{T(\mathcal{B}_{\mathcal{H}})}$, and we write R_σ for R_{T_σ}.

The asymptotic behavior of the minimax risk then follows from Theorem 1 via the general correspondence between metric entropy and minimax risk developed in [2].

Corollary 3. *Let $\sigma \in \mathrm{H}\Gamma^-(\mathbb{R}^{2d})$ be a real-valued strictly positive frequency-even symbol, let T_σ be the Weyl quantization of σ, and assume that the associated volume function V_σ is regularly varying at zero. Then, the minimax risk satisfies*

$$R_\sigma(\kappa) \sim \kappa^2 \int_{\mathbb{R}^{2d}} \left(1 - \frac{\varepsilon_\kappa}{\sigma(x, \omega)} \right)_+ dx \, d\omega, \qquad \kappa \to 0, \tag{17}$$

where the critical radius ε_κ is determined (implicitly) by

$$\kappa^2 \int_{\mathbb{R}^{2d}} \frac{1}{\sigma(x, \omega)\, \varepsilon_\kappa} \left(1 - \frac{\varepsilon_\kappa}{\sigma(x, \omega)} \right)_+ dx \, d\omega = 1, \qquad \kappa > 0. \tag{18}$$

Proof. See Section 4.3. □

3 Applications to Sobolev Spaces

Throughout this section, T_σ denotes the Weyl quantization of σ. We now apply the results developed in the preceding sections to characterize the metric entropy of unit balls

in Sobolev spaces on $\mathbb{R}^d$. This extends the corresponding analysis for Sobolev spaces on bounded domains presented in [4, Section 3.2]. In contrast to the bounded-domain setting, the unit ball of a classical Sobolev space on $\mathbb{R}^d$—as defined, for example, in [56, Definition 3.1] or [11, Section 9.1]—is not compact with respect to the $L^2(\mathbb{R}^d, \mathbb{R})$ metric due to translation invariance (cf. [1, Example 6.11]). As a result, its metric entropy is infinite for every $\varepsilon \in (0, 1)$.

Compactness-restoring variants of Sobolev spaces on $\mathbb{R}^d$, such as those involving confining potentials or weights, are classical and have been studied extensively from the perspectives of functional analysis and spectral theory.

By modifying the Sobolev structure so as to restore compactness while preserving the underlying notion of regularity, one obtains function classes that fall within the scope of the present theory. Two such modifications are particularly natural and widely used in practice: Sobolev spaces with a confining potential (Theorem 4) and weighted Sobolev spaces (Theorem 5). In both cases, the resulting spaces admit compact embeddings into $L^2(\mathbb{R}^d, \mathbb{R})$. Throughout this section, the metric entropy of a set is taken with respect to $\|\cdot\|_{L^2(\mathbb{R}^d,\mathbb{R})}$, and we accordingly write $H(\varepsilon; \mathcal{K})$ for $H(\varepsilon; \mathcal{K}, \|\cdot\|_{L^2(\mathbb{R}^d,\mathbb{R})})$.

We begin with Sobolev spaces equipped with a confining potential. Fix $s > 0$, let $u \in C^\infty(\mathbb{R}^d, \mathbb{R})$ be a potential function, and consider the symbol

$$\sigma_{\mathcal{S}}(x, \omega) = \big(1 + u(x) + (2\pi\|\omega\|_2)^2\big)^{s/2}, \tag{19}$$

and the corresponding power of the Schrödinger operator

$$T_{\mathcal{S}} = (I + U - \Delta)^{s/2}, \tag{20}$$

where I is the identity operator, U acts by pointwise multiplication with $u(\cdot)$, and $-\Delta$ is the Laplace operator whose symbol is given by $(2\pi\|\omega\|_2)^2$. For $s = 2$, $T_{\mathcal{S}}$ is the Schrödinger operator itself and $\sigma_{\mathcal{S}}$ its Weyl symbol. As a concrete and widely studied example, we work with the quadratic potential (cf. [53, Section 25.3]), that is, $u(x) = c\,\|x\|_2^2$, for all $x \in \mathbb{R}^d$, with a fixed constant $c > 0$.

The Sobolev space of order s induced by $T_{\mathcal{S}}$ is defined as

$$\mathcal{H}_{\mathcal{S}}^{(s)} \coloneqq \overline{C_0^\infty(\mathbb{R}^d, \mathbb{R})}^{\|\cdot\|_{\mathcal{S}}^{(s)}}, \qquad \|\varphi\|_{\mathcal{S}}^{(s)} \coloneqq \|T_{\mathcal{S}}\varphi\|_{L^2(\mathbb{R}^d,\mathbb{R})}, \quad \varphi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}),$$

where the closure is taken with respect to the topology induced by $\|\cdot\|_{\mathcal{S}}^{(s)}$, so that $\mathcal{H}_{\mathcal{S}}^{(s)}$ embeds continuously into $L^2(\mathbb{R}^d, \mathbb{R})$. In this sense, $\mathcal{H}_{\mathcal{S}}^{(s)}$ may be viewed as a Sobolev space of order s with a confining potential, encoding s degrees of smoothness together with spatial localization. The corresponding unit ball is

$$\mathcal{B}_{\mathcal{S}}^{(s)} \coloneqq \big\{f \in \mathcal{H}_{\mathcal{S}}^{(s)} : \|T_{\mathcal{S}} f\|_{L^2(\mathbb{R}^d,\mathbb{R})} \le 1\big\}. \tag{21}$$

We are now in a position to characterize the metric entropy of $\mathcal{B}_{\mathcal{S}}^{(s)}$.

Theorem 4. *Let $d \in \mathbb{N}^*$ and $s, c > 0$. The metric entropy of $\mathcal{B}_{\mathcal{S}}^{(s)}$ obeys*

$$H\Big(\varepsilon; \mathcal{B}_{\mathcal{S}}^{(s)}\Big) \sim \frac{s\,\omega_{2d}}{2d(2\pi\sqrt{c})^d}\,\varepsilon^{-2d/s}, \qquad \varepsilon \to 0, \tag{22}$$

and the corresponding minimax risk admits the asymptotic expansion

$$R\Big(\kappa; \mathcal{B}_{\mathcal{S}}^{(s)}\Big) \sim \frac{d+s}{d}\left(\frac{d\,s\,\omega_{2d}\,\kappa^2}{(2\pi\sqrt{c})^d(d+s)(2d+s)}\right)^{\frac{s}{d+s}}, \qquad \kappa \to 0. \tag{23}$$

Proof. See Section 4.4. □

Theorem 4, in particular the minimax-risk asymptotic (23), extends Pinsker's theorem to Sobolev spaces on $\mathbb{R}^d$ with a confining potential. Pinsker's original result was established in [51]; general reviews can be found in [37, 47, 58], and an extension to arbitrary bounded domains is provided in [2, Theorem 9]. The result obtained here shows that Pinsker's principle is not confined to bounded domains: after restoring compactness through spatial confinement, the minimax risk remains governed by the same structural mechanism identified by Pinsker, with the sharp asymptotic constant now determined by phase-space volume rather than boundary effects. This indicates that Pinsker's theorem reflects an underlying geometric principle of statistical estimation, rather than an artifact of bounded domains or specific boundary conditions.

As a second application, we consider weighted Sobolev spaces, defined following [32, Section 2.3]. Fix $s, r, c > 0$, set $u(x) = c\|x\|_2^2$ for $x \in \mathbb{R}^d$, and let

$$T_{\mathcal{W}} \coloneqq (I - \Delta)^{s/2}(I + U)^{r/2},$$

where I is the identity operator, U acts by pointwise multiplication with $u(\cdot)$, and $-\Delta$ is the Laplacian operator. As before, $T_{\mathcal{W}}$ is interpreted as a pseudodifferential operator. By the Weyl composition formula [53, Theorem 23.6], the Weyl symbol of $T_{\mathcal{W}}$ agrees, up to lower-order terms, with

$$\sigma_{\mathcal{W}}(x, \omega) = \big(1 + (2\pi\|\omega\|_2)^2\big)^{s/2}\big(1 + c\|x\|_2^2\big)^{r/2}. \tag{24}$$

The weighted Sobolev space of regularity s and weight exponent r induced by $T_{\mathcal{W}}$ is defined as

$$\mathcal{H}_{\mathcal{W}}^{(s,r)} \coloneqq \overline{C_0^\infty(\mathbb{R}^d, \mathbb{R})}^{\|\cdot\|_{\mathcal{W}}^{(s,r)}}, \qquad \|\varphi\|_{\mathcal{W}}^{(s,r)} \coloneqq \|T_{\mathcal{W}}\varphi\|_{L^2(\mathbb{R}^d,\mathbb{R})}, \quad \varphi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}),$$

where the closure is taken with respect to the topology induced by $\|\cdot\|_{\mathcal{W}}^{(s,r)}$, so that $\mathcal{H}_{\mathcal{W}}^{(s,r)}$ embeds continuously into $L^2(\mathbb{R}^d, \mathbb{R})$. In this sense, $\mathcal{H}_{\mathcal{W}}^{(s,r)}$ may be viewed as a weighted Sobolev space of regularity s, where the weight exponent r enforces spatial localization. The corresponding unit ball is

$$\mathcal{B}_{\mathcal{W}}^{(s,r)} \coloneqq \big\{f \in \mathcal{H}_{\mathcal{W}}^{(s,r)} : \|T_{\mathcal{W}} f\|_{L^2(\mathbb{R}^d,\mathbb{R})} \le 1\big\}. \tag{25}$$

We are now ready to characterize the metric entropy of $\mathcal{B}_{\mathcal{W}}^{(s,r)}$.

Theorem 5. *Let $d \in \mathbb{N}^*$ and $s, r, c > 0$. The metric entropy of $\mathcal{B}_{\mathcal{W}}^{(s,r)}$ obeys*

$$H\Big(\varepsilon; \mathcal{B}_{\mathcal{W}}^{(s,r)}\Big) \sim \frac{\omega_d^2}{(2\pi\sqrt{c})^d} \begin{cases} \Xi_{r,s,d}\, \varepsilon^{-\frac{d}{\min\{r,s\}}}, & \text{if } r \neq s, \\ \varepsilon^{-\frac{d}{s}} \ln\big(\varepsilon^{-1}\big), & \text{if } r = s, \end{cases} \qquad \varepsilon \to 0,$$

where, for $r \neq s$,

$$\Xi_{r,s,d} \coloneqq \frac{\min\{r,s\}\,\Gamma\Big(\frac{d\,|s-r|}{2\min\{r,s\}}\Big)\Gamma\big(\frac{d}{2}\big)}{2\Gamma\Big(\frac{d\max\{r,s\}}{2\min\{r,s\}}\Big)}, \tag{26}$$

and Γ denotes Euler's Gamma function.

Proof. See Section 4.5. □

The expression (26) for the constant appearing in Theorem 5 can often be simplified by exploiting standard identities for the Gamma function [7]. In particular, when the dimension is even, say $d = 2k$ with $k \in \mathbb{N}^*$, the Gamma factors reduce to factorials and one obtains

$$\Xi_{r,s,2k} = \frac{(k-1)!\,\min\{r,s\}^{k+1}}{2\prod_{j=1}^{k}\big(k\max\{r,s\} - j\min\{r,s\}\big)}, \qquad r \neq s.$$

For example, in dimension $d = 2$ this yields

$$\Xi_{r,s,2} = \frac{\min\{r,s\}^2}{2\,(\max\{r,s\} - \min\{r,s\})}.$$

Theorem 5 also delivers sharp entropy asymptotics for compact embeddings between weighted Sobolev spaces defined through the operators $T_{\mathcal{W}}$. In particular, for parameters $s_1 > s_2$ and $r_1 > r_2$, consider the inclusion map

$$I\colon \mathcal{H}_{\mathcal{W}}^{(s_1,r_1)} \longrightarrow \mathcal{H}_{\mathcal{W}}^{(s_2,r_2)},$$

which is compact by Rellich's theorem (see [32, Theorem 2.17 and Exercise 2.12]). The special case treated in Theorem 5 corresponds to the choice $s_2 = r_2 = 0$. The same analysis as in the proof of Theorem 5 applies to the embedding operator I with the parameters (r, s) replaced by $(r_1 - r_2, s_1 - s_2)$. Consequently,

$$H_I(\varepsilon) \sim \frac{\omega_d^2}{(2\pi\sqrt{c})^d} \begin{cases} \Xi_{r_1-r_2,s_1-s_2,d}\, \varepsilon^{-\frac{d}{\min\{r_1-r_2,s_1-s_2\}}}, & \text{if } r_1 + s_2 \neq r_2 + s_1, \\ \varepsilon^{-\frac{d}{s_1-s_2}} \ln\bigl(\varepsilon^{-1}\bigr), & \text{if } r_1 + s_2 = r_2 + s_1, \end{cases} \qquad \varepsilon \to 0,$$

where the constant $\Xi_{r,s,d}$ is defined as in Theorem 5. The order of the entropy numbers for such embeddings has been studied in a more general setting—for instance for weighted Besov and Triebel–Lizorkin spaces [24, Theorem 4.3.2]. However, these results do not identify the leading constants. The present approach provides these constants explicitly in the weighted Sobolev setting.

4 Proofs

4.1 Proof of Theorem 1

We first assume that the symbol σ is strictly positive and refer to the discussion following Lemma 6 for the general case. Next recall that, by definition of $\mathrm{H}\Gamma^-(\mathbb{R}^{2d})$ (see (121)), there exist $\rho \in (0,1]$ and real numbers m_+, m_- with $m_+ \geq m_- > 0$, all of which we fix throughout the proof, such that $\sigma \in \mathrm{H}\Gamma_\rho^{-m_+,-m_-}(\mathbb{R}^{2d})$. Moreover, by [53, Lemma 25.1, 1)a)], $\sigma \in \mathrm{H}\Gamma_\rho^{-m_+,-m_-}(\mathbb{R}^{2d})$ implies that $\sigma^{-1} \in \mathrm{H}\Gamma_\rho^{m_-,m_+}(\mathbb{R}^{2d})$.

To derive the entropy asymptotics, we first reduce the Weyl quantization T_σ to a positive self-adjoint operator. By the polar decomposition [18, VIII.3.11], there exists a partial isometry U such that $T_\sigma = U|T_\sigma|$. Since U acts isometrically on the range of $|T_\sigma|$, it maps the image of the unit ball under $|T_\sigma|$ onto the image of the unit ball under T_σ. Consequently,

$$H_{T_\sigma}(\varepsilon) = H_{|T_\sigma|}(\varepsilon), \qquad \varepsilon > 0. \tag{27}$$

Our strategy is to invoke the spectral asymptotics of Dauge and Robert stated in Theorem 8. To this end, we choose the weight functions

$$\phi(z) = \varphi(z) = \bigl(1 + \|z\|_2^2\bigr)^{\rho/2}, \qquad w(z) = \sigma(z), \qquad z \in \mathbb{R}^{2d}. \tag{28}$$

Once it is verified that this choice satisfies conditions *(H1)*, *(H2)*, *(W)*, and *(N)*, and that $\sigma \in \Gamma(\mathbb{R}^{2d}; w, \phi, \varphi)$, Theorem 8 yields

$$M_{|T_\sigma|}(\lambda) \sim V_\sigma(\lambda), \qquad \lambda \to 0. \tag{29}$$

Since $w = \sigma$, the assumption $V_w(\lambda) = O(V_\sigma(\lambda))$ required in Theorem 8 holds trivially. Because V_σ is assumed to be regularly varying at zero, it follows that $M_{|T_\sigma|}$ is regularly varying at zero as well, with the same index; $\sigma \in \mathrm{H}\Gamma_\rho^{-m_+,-m_-}(\mathbb{R}^{2d})$ further implies that this index of regular variation belongs to $[-2d/m_-, -2d/m_+]$, and is in particular negative.

We mention that, although Theorem 8 is stated for pseudodifferential operators on $L^2(\mathbb{R}^d,\mathbb{C})$, its spectral asymptotics apply without change to T_σ acting on $L^2(\mathbb{R}^d,\mathbb{R})$. Indeed, T_σ is the Weyl quantization of a real-valued symbol and hence self-adjoint on $L^2(\mathbb{R}^d,\mathbb{C})$, with real eigenvalues, and σ is frequency-even, so that T_σ preserves the real structure. Consequently, for a non-zero eigenvalue λ, writing $f = f_1 + if_2$ with f_1 and f_2 real-valued and comparing real and imaginary parts shows that $T_\sigma f = \lambda f$ holds if and only if $T_\sigma f_1 = \lambda f_1$ and $T_\sigma f_2 = \lambda f_2$. The eigenspace of λ in $L^2(\mathbb{R}^d,\mathbb{C})$ therefore consists exactly of the functions $f = f_1 + if_2$ with f_1 and f_2 in the eigenspace of λ in $L^2(\mathbb{R}^d,\mathbb{R})$, which is finite-dimensional as T_σ is compact. An orthonormal basis of the eigenspace of λ in $L^2(\mathbb{R}^d,\mathbb{R})$ is an orthonormal basis of the eigenspace of λ in $L^2(\mathbb{R}^d,\mathbb{C})$, so that the eigenspaces have the same dimension and the eigenvalue-counting function of T_σ on $L^2(\mathbb{R}^d,\mathbb{R})$ and $L^2(\mathbb{R}^d,\mathbb{C})$ coincide. As T_σ is self-adjoint, the eigenvalues of $|T_\sigma|$ are the absolute values of those of T_σ, counted with multiplicity. The eigenvalue-counting functions of $|T_\sigma|$ on $L^2(\mathbb{R}^d,\mathbb{R})$ and $L^2(\mathbb{R}^d,\mathbb{C})$ therefore coincide as well.

Since $|T_\sigma|$ is a positive compact operator, it admits an orthonormal eigenbasis $\{u_n\}_{n\in\mathbb{N}^*}$ with corresponding eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}^*}$ listed in nonincreasing order and converging to zero. In this basis, the image of the unit ball under $|T_\sigma|$ is the ellipsoid

$$\mathcal{E}_\lambda = \Big\{ x = \sum_{n=1}^{\infty} x_n u_n \mid \sum_{n=1}^{\infty} \frac{x_n^2}{\lambda_n^2} \le 1 \Big\}.$$

The semi-axis-counting function of the ellipsoid $\mathcal{E}_\lambda$ coincides with the eigenvalue-counting function $M_{|T_\sigma|}(\lambda)$ of $|T_\sigma|$. Hence, the entropy $H_{|T_\sigma|}(\varepsilon)$ is precisely the metric entropy of the ellipsoid $\mathcal{E}_\lambda$. Since $M_{|T_\sigma|}(\lambda) \sim V_\sigma(\lambda)$ as $\lambda \to 0$ and V_σ is regularly varying at zero, the regularity condition (RC) required in [2, Theorem 2] is satisfied (cf. [2, Appendix C.1]). We may therefore apply [2, Theorem 2] to conclude that

$$H_{|T_\sigma|}(\varepsilon) \sim \int_\varepsilon^\infty \frac{M_{|T_\sigma|}(\lambda)}{\lambda}\, d\lambda, \qquad \varepsilon \to 0.$$

Combining this with (27) and (29) yields

$$H_{T_\sigma}(\varepsilon) = H_{|T_\sigma|}(\varepsilon) \sim \int_\varepsilon^\infty \frac{V_\sigma(\lambda)}{\lambda}\, d\lambda, \qquad \varepsilon \to 0, \tag{30}$$

which is precisely (13). It remains to verify conditions *(H1)–(N)* for the choice (28).

Verifying (H1). To establish that ϕ^{-1} and φ^{-1} are (ϕ,φ)-continuous, we start from the elementary inequality

$$1 + (t_1 + t_2)^2 \le \big((1 + t_1^2)^{1/2} + t_2\big)^2, \qquad t_1, t_2 \ge 0,$$

which follows by direct expansion. Applying this with $t_1 = \|z\|_2$ and $t_2 = \|z'\|_2$, and using the triangle inequality $\|z + z'\|_2 \le \|z\|_2 + \|z'\|_2$, we obtain

$$\big(1 + \|z + z'\|_2^2\big)^{1/2} \le \big(1 + \|z\|_2^2\big)^{1/2} + \|z'\|_2, \qquad z, z' \in \mathbb{R}^{2d}. \tag{31}$$

Replacing z by $z + z'$ and z' by $-z'$ in (31), yields the corresponding lower bound

$$\big(1 + \|z + z'\|_2^2\big)^{1/2} \ge \big(1 + \|z\|_2^2\big)^{1/2} - \|z'\|_2, \qquad z, z' \in \mathbb{R}^{2d}. \tag{32}$$

Fix $c > 0$ and take $x, x', \omega, \omega' \in \mathbb{R}^d$ such that

$$\frac{\|x'\|_2}{\varphi(x,\omega)} + \frac{\|\omega'\|_2}{\phi(x,\omega)} \le c.$$

Setting $z = (x,\omega)$ and $z' = (x',\omega')$, and using $\phi(z) = \varphi(z) = (1 + \|z\|_2^2)^{\rho/2}$, we obtain

$$\|z'\|_2 \le \|x'\|_2 + \|\omega'\|_2 \le c\,(1 + \|z\|_2^2)^{\rho/2} \le c\,(1 + \|z\|_2^2)^{1/2},$$

since $\rho \le 1$. Combining this with (31)–(32) yields

$$1 - c \le \frac{(1 + \|z + z'\|_2^2)^{1/2}}{(1 + \|z\|_2^2)^{1/2}} \le 1 + c.$$

Raising both sides to the power ρ and choosing $c = 1/2$ shows that

$$C^{-1} \le \frac{\phi^{-1}(z + z')}{\phi^{-1}(z)} \le C, \qquad C = 2^{\rho},$$

which proves that ϕ^{-1} is (ϕ, φ)-continuous. Since $\phi = \varphi$ by (28), we have $\phi^{-1} = \varphi^{-1}$. Therefore, the same estimate holds for φ^{-1}, and both functions are (ϕ, φ)-continuous.

We next verify that ϕ^{-1} and φ^{-1} are $(1, 1)$-temperate. By Peetre's inequality (see [57, (2.21)]),

$$(1 + \|z\|_2^2)^{1/2} \le \sqrt{2}\,(1 + \|z'\|_2^2)^{1/2}(1 + \|z + z'\|_2^2)^{1/2}, \qquad z, z' \in \mathbb{R}^{2d}. \tag{33}$$

Rewriting (33) gives

$$(1 + \|z + z'\|_2^2)^{-\rho/2} \le 2^{\rho/2}\,(1 + \|z\|_2^2)^{-\rho/2}(1 + \|z'\|_2^2)^{\rho/2}. \tag{34}$$

For $z = (x, \omega)$ and $z' = (x', \omega')$, using $1 + \|z'\|_2^2 \le (1 + \|x'\|_2 + \|\omega'\|_2)^2$, we obtain

$$\phi^{-1}(x + x', \omega + \omega') \le 2^{\rho/2}\,\phi^{-1}(x, \omega)\,\big(1 + \|x'\|_2 + \|\omega'\|_2\big)^{\rho}.$$

Since $\phi = \varphi$ by (28), we have $\phi^{-1} = \varphi^{-1}$. Hence, the same bound holds for φ^{-1}, and both functions are $(1, 1)$-temperate.

Finally, since $\rho > 0$, we have $\phi^{-1}(z) = \varphi^{-1}(z) \le 1$ for all $z \in \mathbb{R}^{2d}$, and hence both functions are bounded on $\mathbb{R}^{2d}$. Together with the (ϕ, φ)-continuity and $(1, 1)$-temperateness established above, this verifies condition *(H1)*.

Verifying (H2). To verify condition *(H2)*, we start from the elementary bound $2t \le 1 + t^2$, for $t \ge 0$. Applying this bound first with $t = \|x\|_2$ and then with $t = \|\omega\|_2$, where $x, \omega \in \mathbb{R}^d$, and adding the resulting inequalities, yields

$$1 + \|x\|_2 + \|\omega\|_2 \le 2\big(1 + \|x\|_2^2 + \|\omega\|_2^2\big), \qquad x, \omega \in \mathbb{R}^d.$$

Recalling that $\phi(x, \omega) = \varphi(x, \omega) = (1 + \|x\|_2^2 + \|\omega\|_2^2)^{\rho/2}$, this bound can be rewritten as

$$1 + \|x\|_2 + \|\omega\|_2 \le 2\big(\phi(x, \omega)\,\varphi(x, \omega)\big)^{1/\rho}, \qquad x, \omega \in \mathbb{R}^d.$$

This is exactly condition (127) with $C = 2^{-\rho}$ and $\zeta = \rho$.

Verifying (W). By hypoellipticity of σ (recall (120)), there exists $R > 0$ such that, for every multi-index $\alpha \in \mathbb{N}^{2d}$, the bound

$$\big|\partial^{\alpha}\sigma(z)\big| \le C_{\alpha}\,\sigma(z)\big(1 + \|z\|_2^2\big)^{-\frac{\rho|\alpha|}{2}} \tag{35}$$

holds for all $z \in \mathbb{R}^{2d}$ with $\|z\|_2 \ge R$, for some constant $C_{\alpha} > 0$ depending on α. For fixed α, the function $z \mapsto \sigma(z)\big(1 + \|z\|_2^2\big)^{-\frac{\rho|\alpha|}{2}}$ is continuous and strictly positive, while $z \mapsto \partial^{\alpha}\sigma(z)$ is continuous since $\sigma \in C^{\infty}(\mathbb{R}^{2d})$. As the closed ball $\{z \in \mathbb{R}^{2d} : \|z\|_2 \le R\}$ is compact, there exists a constant $C'_{\alpha} > 0$ such that

$$\big|\partial^{\alpha}\sigma(z)\big| \le C'_{\alpha}\,\sigma(z)\big(1 + \|z\|_2^2\big)^{-\frac{\rho|\alpha|}{2}}, \qquad \|z\|_2 \le R. \tag{36}$$

Combining (35) and (36), and recalling that $w = \sigma$ and $\phi = \varphi = (1 + \|\cdot\|_2^2)^{\rho/2}$, we obtain, for all $x, \omega \in \mathbb{R}^d$ and all multi-indices $\alpha, \beta \in \mathbb{N}^d$,

$$\big|\partial_x^{\alpha}\partial_{\omega}^{\beta} w(x, \omega)\big| \le C_{\alpha,\beta}\,w(x, \omega)\big(1 + \|(x, \omega)\|_2^2\big)^{-\frac{\rho(|\alpha|+|\beta|)}{2}} \tag{37}$$

$$= C_{\alpha,\beta}\frac{w(x, \omega)}{\varphi(x, \omega)^{|\alpha|}\,\phi(x, \omega)^{|\beta|}}. \tag{38}$$

This establishes condition (126) and hence shows that $w \in \Gamma(\mathbb{R}^{2d}; w, \phi, \varphi)$, as required for *(W)*.

We show that the weight $w = \sigma$ is (ϕ, φ)-temperate by a case distinction on z'. Fix $z, z' \in \mathbb{R}^{2d}$.

Case 1: $\|z'\|_2 \leq \|z\|_2^\rho / 2$. Define the function

$$g\colon [0,1] \to \mathbb{R}, \qquad g(t) \coloneqq \ln\big(\sigma(z + tz')\big).$$

Then

$$g'(t) = \frac{\nabla\sigma(z + tz') \cdot z'}{\sigma(z + tz')}, \qquad t \in [0,1]. \tag{39}$$

By hypoellipticity of σ, applying (35) and (36) with $|\alpha| = 1$ yields bounds on each first-order partial derivative of σ. Combining the componentwise bounds corresponding to $|\alpha| = 1$, there exists a constant $K > 0$ such that

$$\|\nabla\sigma(z + tz')\|_2 \leq K\, \sigma(z + tz')\big(1 + \|z + tz'\|_2^2\big)^{-\rho/2}, \qquad t \in [0,1]. \tag{40}$$

Substituting this bound into (39) and applying the Cauchy-Schwarz inequality yields

$$|g'(t)| \leq K\, \|z'\|_2 \left(1 + \|z + tz'\|_2^2\right)^{-\rho/2}, \qquad t \in [0,1]. \tag{41}$$

Under the restriction of this case, $\|z'\|_2 \leq \|z\|_2^\rho / 2$, and assuming $\|z\|_2 \geq 1$, we estimate

$$\|z + tz'\|_2 \geq \|z\|_2 - t\|z'\|_2 \geq \|z\|_2 - \tfrac{1}{2}\|z\|_2^\rho \geq \tfrac{1}{2}\|z\|_2, \qquad t \in [0,1],$$

where the last inequality uses $\rho \leq 1$. Inserting this bound into (41) shows that the right-hand side is uniformly bounded for $t \in [0,1]$. Hence, there exists a constant $K_0 > 0$ such that

$$|g'(t)| \leq K_0, \qquad t \in [0,1]. \tag{42}$$

If $\|z\|_2 < 1$, then the assumption of this case implies $\|z'\|_2 \leq \frac{1}{2}$. Consequently, for all $t \in [0,1]$, the points $z + tz'$ remain in a fixed compact subset of $\mathbb{R}^{2d}$. Since $\sigma \in C^\infty(\mathbb{R}^{2d})$ is strictly positive, the function $\nabla\sigma/\sigma$ is continuous and therefore bounded on this set. It follows that $|g'(t)|$ is uniformly bounded for all $t \in [0,1]$, which yields (42). Integrating (42) over the interval $[0,1]$ gives

$$\left|\ln\left(\frac{\sigma(z + z')}{\sigma(z)}\right)\right| \leq K_0, \tag{43}$$

and therefore

$$\sigma(z + z') \leq e^{K_0}\, \sigma(z).$$

Writing $z = (x, \omega)$ and $z' = (x', \omega')$, and recalling that $w = \sigma$, we hence get

$$w(x + x', \omega + \omega') \leq e^{K_0}\, w(x, \omega) \leq e^{K_0}\, w(x, \omega)\left[1 + \|x'\|_2\, \phi(x, \omega) + \|\omega'\|_2\, \varphi(x, \omega)\right],$$

which is exactly (125) with $C = e^{K_0}$ and $\zeta = 1$.

Case 2: $\|z'\|_2 \geq \|z\|_2^\rho / 2$. By hypoellipticity, there exist constants $R > 0$ and $C_1 > 0$ with

$$\sigma(z) \geq C_1 \left(1 + \|z\|_2^2\right)^{-m_+/2}, \qquad \|z\|_2 \geq R.$$

Since σ is continuous and strictly positive on $\mathbb{R}^{2d}$, it attains a strictly positive minimum on the closed ball $B_R = \{z \in \mathbb{R}^{2d} : \|z\|_2 \leq R\}$. Hence, there exists $C_2 > 0$ satisfying

$$\sigma(z) \geq C_2 \left(1 + \|z\|_2^2\right)^{-m_+/2}, \qquad \|z\|_2 \leq R.$$

These estimates imply that there exists a constant $K_1 > 0$ such that

$$\sigma(z) \geq K_1 \left(1 + \|z\|_2^2\right)^{-m_+/2}, \qquad z \in \mathbb{R}^{2d}.$$

Using this lower bound in conjunction with the corresponding upper bound from (120), we obtain

$$\frac{\sigma(z+z')}{\sigma(z)} \le K_2 \frac{\big(1+\|z+z'\|_2^2\big)^{-m_-/2}}{\big(1+\|z\|_2^2\big)^{-m_+/2}},$$

for some constant $K_2 > 0$. Applying Peetre's inequality (33) to the numerator gives

$$\frac{\sigma(z+z')}{\sigma(z)} \le K_3 \big(1+\|z'\|_2^2\big)^{m_-/2} \big(1+\|z\|_2^2\big)^{(m_+-m_-)/2},$$

for some constant $K_3 > 0$. Using the assumption of this case, $\|z'\|_2 \ge \|z\|_2^\rho/2$, we can bound $\big(1+\|z\|_2^2\big)^{(m_+-m_-)/2} \le C\,\big(1+\|z'\|_2^2\big)^{(m_+-m_-)/(2\rho)}$, which leads to

$$\sigma(z+z') \le K_4\,\sigma(z)\big(1+\|z'\|_2^2\big)^{\frac{m_-}{2}+\frac{m_+-m_-}{2\rho}}, \tag{44}$$

for some constant $K_4 > 0$. Writing $z = (x,\omega)$ and $z' = (x',\omega')$, we note that

$$1+\|z'\|_2^2 = 1+\|x'\|_2^2+\|\omega'\|_2^2 \le \big(1+\|x'\|_2+\|\omega'\|_2\big)^2 \le \big(1+\|x'\|_2\,\phi(x,\omega)+\|\omega'\|_2\,\varphi(x,\omega)\big)^2.$$

Recalling that $w = \sigma$, this implies

$$w(x+x',\omega+\omega') \le K_4\,w(x,\omega)\big[1+\|x'\|_2\,\phi(x,\omega)+\|\omega'\|_2\,\varphi(x,\omega)\big]^{\,m_-+\frac{m_+-m_-}{\rho}},$$

which is precisely the (ϕ,φ)-temperateness condition (125) with $C = K_4$ and $\zeta = m_- + (m_+ - m_-)/\rho$.

Finally, taking z' small enough for Case 1 to apply and for (124) to be satisfied, the (ϕ,φ)-continuity of w is established after exponentiating (43).

Verifying (N). By hypoellipticity of $\sigma \in \mathrm{H}\Gamma_\rho^{-m_+,-m_-}(\mathbb{R}^{2d})$ (recall (120)), there exist constants $C_1, C_2, R > 0$ such that

$$C_1\big(1+\|z\|_2^2\big)^{-m_+/2} \le \sigma(z) \le C_2\big(1+\|z\|_2^2\big)^{-m_-/2}, \qquad z \in \mathbb{R}^{2d} \text{ with } \|z\|_2 \ge R. \tag{45}$$

With the choice $w = \sigma$, the bounds in (45) imply

$$C_2^{-\frac{2\rho}{m_-}}\,w(z)^{\frac{2\rho}{m_-}} \le (1+\|z\|_2^2)^{-\rho} \le C_1^{-\frac{2\rho}{m_+}}\,w(z)^{\frac{2\rho}{m_+}}, \qquad z \in \mathbb{R}^{2d} \text{ with } \|z\|_2 \ge R. \tag{46}$$

Since $w = \sigma$ is continuous and strictly positive, its restriction to the closed ball $\{z \in \mathbb{R}^{2d} : \|z\|_2 \le R\}$ attains a positive minimum and a finite maximum. Define

$$C_- := \min_{\|z\|_2 \le R} w(z), \qquad C_+ := \max_{\|z\|_2 \le R} w(z).$$

Then, for all $z \in \mathbb{R}^{2d}$ with $\|z\|_2 \le R$,

$$\big(1+R^2\big)^{-\rho} C_+^{-\frac{2\rho}{m_-}}\,w(z)^{\frac{2\rho}{m_-}} \le \big(1+R^2\big)^{-\rho} \le \big(1+\|z\|_2^2\big)^{-\rho} \le 1 \le C_-^{-\frac{2\rho}{m_+}}\,w(z)^{\frac{2\rho}{m_+}}. \tag{47}$$

Combining (46) and (47), and setting

$$K' := \min\Big\{C_2^{-\frac{2\rho}{m_-}}, (1+R^2)^{-\rho} C_+^{-\frac{2\rho}{m_-}}\Big\}, \qquad K := \max\Big\{C_1^{-\frac{2\rho}{m_+}}, C_-^{-\frac{2\rho}{m_+}}\Big\},$$

$\gamma' = 2\rho/m_-$, and $\gamma = 2\rho/m_+$, we obtain

$$K' w(z)^{\gamma'} \le \phi^{-1}(z)\,\varphi^{-1}(z) \le K w(z)^{\gamma}, \qquad z \in \mathbb{R}^{2d},$$

which is exactly condition (128).

This completes the verification of conditions *(H1)–(N)* and hence establishes (13) for the Weyl quantization. We next show that the same entropy asymptotics hold for the left and right quantizations. To this end, we first state a technical lemma, which is proved at the end of this section.

Lemma 6. *Let $\rho_1 \in (0,1]$ and $m > 0$. Let T_1 and T_2 be compact linear operators on $L^2(\mathbb{R}^d, \mathbb{C})$, both preserving the real structure, such that $T_1 = T_2(I + K) + K_{-\infty}$, for some operators $K \in \Psi^{-m}_{\rho_1}(\mathbb{R}^d)$ and $K_{-\infty} \in \Psi^{-\infty}(\mathbb{R}^d)$, both preserving the real structure. Assume further that there exists a real-valued strictly positive symbol $\sigma \in \mathrm{H}\Gamma^{-}(\mathbb{R}^{2d})$ whose associated volume function V_σ is regularly varying at zero, and that either T_1 or T_2 is the Weyl quantization of σ. Then*

$$H_{T_1}(\varepsilon) \sim H_{T_2}(\varepsilon), \qquad \varepsilon \to 0. \tag{48}$$

Lemma 6 allows us to remove the strict positivity assumption on the symbol σ. Indeed, by hypoellipticity, the symbol σ has a fixed sign outside a sufficiently large ball in phase space. By regular variation of the volume function with negative index, this sign must be positive. One may therefore modify σ inside this ball by adding a compactly supported frequency-even smooth perturbation so as to make it strictly positive everywhere. The operator thus obtained differs from T_σ only by an additive smoothing operator $K_{-\infty} \in \Psi^{-\infty}(\mathbb{R}^d)$, and one gets the desired result upon application of Lemma 6.

Invariance under change of quantization. We establish invariance of the entropy asymptotics under a change of quantization by treating explicitly the Weyl–left case; the Weyl–right case follows by the same arguments. Let T_1 and T_2 denote the left and Weyl quantizations of the symbol σ, respectively, and let σ_1 be the Weyl symbol of T_1. Since T_1 is the left quantization of a symbol $\sigma \in \mathrm{H}\Gamma^{-m_+,-m_-}_{\rho}(\mathbb{R}^{2d})$, Proposition 25.1 in [53] implies that the corresponding Weyl symbol σ_1 also belongs to $\mathrm{H}\Gamma^{-m_+,-m_-}_{\rho}(\mathbb{R}^{2d})$.

Since T_2 is the Weyl quantization of a hypoelliptic symbol in $\mathrm{H}\Gamma^{-m_+,-m_-}_{\rho}(\mathbb{R}^{2d})$, Theorem 25.1 in [53] guarantees the existence of a two-sided parametrix B_2 with symbol $b_2 \in \mathrm{H}\Gamma^{m_-,m_+}_{\rho}(\mathbb{R}^{2d})$ such that

$$T_2 B_2 = I + S, \qquad S \in \Psi^{-\infty}(\mathbb{R}^d). \tag{49}$$

Note that, upon replacing B_2 by $(B_2 + B_2^* + CB_2C + CB_2^*C)/4$, where C is the complex-conjugation operator $(Cf)(x) = \overline{f(x)}$ and $\cdot^*$ the adjoint, the symbol b_2 can be assumed, without loss of generality, to be frequency-even and real.

We may further assume, again without loss of generality, that the symbol b_2 is nowhere vanishing on $\mathbb{R}^{2d}$. Indeed, by hypoellipticity and the lower bound in (120), there exists $R > 0$ such that $b_2(z) \neq 0$ for all $z \in \mathbb{R}^{2d}$ with $\|z\|_2 \geq R$. If b_2 vanishes at some point in $\{z \in \mathbb{R}^{2d} \mid \|z\|_2 < R\}$, we can construct a frequency-even real-valued function $b_2' \in C^\infty(\mathbb{R}^{2d})$, supported in $\{z \in \mathbb{R}^{2d} \mid \|z\|_2 \leq 2R\}$, such that

$$b_2'' \coloneqq b_2 + b_2' \in \mathrm{H}\Gamma^{m_-,m_+}_{\rho}(\mathbb{R}^{2d})$$

and $b_2''(z) \neq 0$ for all $z \in \mathbb{R}^{2d}$. Let B_2' and B_2'' denote the Weyl quantizations of b_2' and b_2'', respectively. Replacing B_2 by B_2'' in (49) yields

$$T_2 B_2'' = I + S + T_2 B_2',$$

where $T_2 B_2' \in \Psi^{-\infty}(\mathbb{R}^d)$ since b_2' is compactly supported. Absorbing this term into the remainder, we obtain a representation of the form

$$T_2 B_2'' = I + S'', \qquad S'' \in \Psi^{-\infty}(\mathbb{R}^d),$$

with a parametrix whose symbol does not vanish on $\mathbb{R}^{2d}$.

Composing (49) on the right with $(T_1 - T_2)$ yields

$$T_2 B_2 (T_1 - T_2) = (T_1 - T_2) + S(T_1 - T_2).$$

Because $S \in \Psi^{-\infty}(\mathbb{R}^d)$ and $T_1 - T_2$ is a pseudodifferential operator, and since $\Psi^{-\infty}(\mathbb{R}^d)$ is a two-sided ideal in the pseudodifferential calculus, the composition $S(T_1 - T_2)$ belongs to $\Psi^{-\infty}(\mathbb{R}^d)$; see, for example, [32, Theorem 4.22] or [53, Theorem 23.6]. Letting

$$S' \coloneqq -S(T_1 - T_2) \in \Psi^{-\infty}(\mathbb{R}^d),$$

we may therefore rewrite the above identity as

$$T_2 B_2 (T_1 - T_2) = (T_1 - T_2) - S'.$$

Rearranging terms yields

$$T_1 = T_2\big(I + B_2(T_1 - T_2)\big) + S'. \tag{50}$$

With $K \coloneqq B_2(T_1 - T_2)$ and $K_{-\infty} = S'$, we see that (50) is of the form required in Lemma 6. Since B_2 and $T_1 - T_2$ preserve the real structure, K and $K_{-\infty}$ do as well. Therefore, once it is shown that $K \in \Psi_{\rho_1}^{-m}(\mathbb{R}^d)$ for some $\rho_1 > 0$ and $m > 0$, the lemma applies and yields

$$H_{T_1}(\varepsilon) \sim H_{T_2}(\varepsilon), \qquad \varepsilon \to 0. \tag{51}$$

We next prove that in fact $K \in \Psi_\rho^{-2\rho}(\mathbb{R}^d)$. This will be done by first showing that $\sigma^{-1}(\sigma_1 - \sigma) \in \Gamma_\rho^{-2\rho}(\mathbb{R}^{2d})$. Recall that σ_1 denotes the Weyl symbol of the left quantization T_1. By [53, Theorem 23.3], for every $N \in \mathbb{N}^*$ there exists a remainder $r_N \in \Gamma_\rho^{-m_- - 2(N+1)\rho}(\mathbb{R}^{2d})$ such that

$$\sigma_1(x,\omega) - \sigma(x,\omega) = \sum_{\substack{\alpha \in \mathbb{N}^d \\ 1 \le |\alpha| \le N}} \frac{1}{\alpha!}\left(\frac{i}{4\pi}\right)^{|\alpha|} \partial_\omega^\alpha \partial_x^\alpha \sigma(x,\omega) + r_N(x,\omega), \qquad x,\omega \in \mathbb{R}^d. \tag{52}$$

Since σ is strictly positive, division by σ is well-defined, and (52) yields

$$\frac{\sigma_1 - \sigma}{\sigma} = \sum_{\substack{\alpha \in \mathbb{N}^d \\ 1 \le |\alpha| \le N}} \frac{1}{\alpha!}\left(\frac{i}{4\pi}\right)^{|\alpha|} \frac{\partial_\omega^\alpha \partial_x^\alpha \sigma}{\sigma} + \frac{r_N}{\sigma}. \tag{53}$$

We treat the two terms on the right-hand side of (53) separately. First, since $\sigma^{-1} \in \mathrm{H}\Gamma_\rho^{m_-,m_+}(\mathbb{R}^{2d})$, application of [53, Lemma 25.1] allows us to conclude that

$$\frac{r_N}{\sigma} \in \Gamma_\rho^{m_+ - m_- - 2(N+1)\rho}(\mathbb{R}^{2d}).$$

Choosing N sufficiently large ensures

$$\frac{r_N}{\sigma} \in \Gamma_\rho^{-2\rho}(\mathbb{R}^{2d}). \tag{54}$$

Second, consider the derivative terms appearing in the finite sum in (53). For each multi-index $\alpha \neq 0$, the Leibniz rule together with the derivative bounds in (120) imply that

$$\sigma^{-1}\, \partial_\omega^\alpha \partial_x^\alpha \sigma \in \Gamma_\rho^{-2\rho|\alpha|}(\mathbb{R}^{2d}).$$

Since $|\alpha| \ge 1$, we have $-2\rho|\alpha| \le -2\rho$, and hence

$$\sigma^{-1}\, \partial_\omega^\alpha \partial_x^\alpha \sigma \in \Gamma_\rho^{-2\rho}(\mathbb{R}^{2d}). \tag{55}$$

Combining (54) and (55) in (53) yields

$$\sigma^{-1}(\sigma_1 - \sigma) \in \Gamma_\rho^{-2\rho}(\mathbb{R}^{2d}),$$

as claimed.

Returning to the parametrix identity $T_2 B_2 = I + S$, let s denote the Weyl symbol of the operator S. By the composition formula for Weyl quantization [53, Theorem 23.6]

and the hypoellipticity of both b_2 and σ, for every $M \in \mathbb{N}$ there exist constants $C_{\alpha,\beta} \in \mathbb{C}$ and a remainder $r'_M \in \Gamma_\rho^{m_+ + m_- + 2(M+1)\rho}(\mathbb{R}^{2d})$ such that

$$1 + s(z) = \sum_{\substack{\alpha,\beta \in \mathbb{N}^{2d} \\ 0 \le |\alpha|+|\beta| \le M}} C_{\alpha,\beta}\, \partial_z^\alpha b_2(z)\, \partial_z^\beta \sigma(z) + r'_M(z) \tag{56}$$

$$= b_2(z)\,\sigma(z) \left(1 + \sum_{\substack{\alpha,\beta \in \mathbb{N}^{2d} \\ 1 \le |\alpha|+|\beta| \le M}} C_{\alpha,\beta}\, \frac{\partial_z^\alpha b_2(z)}{b_2(z)}\, \frac{\partial_z^\beta \sigma(z)}{\sigma(z)} + \frac{r'_M(z)}{b_2(z)\,\sigma(z)} \right), \tag{57}$$

for all $z \in \mathbb{R}^{2d}$. Recall that $\sigma(z)$ is non-vanishing for all $z \in \mathbb{R}^{2d}$ by assumption, and that $b_2(z)$ has been chosen to be non-vanishing on $\mathbb{R}^{2d}$ without loss of generality.

Choosing M sufficiently large ensures that $(r'_M / b_2\sigma) \in \Gamma_\rho^{-2\rho}(\mathbb{R}^{2d})$. Moreover, by the Leibniz rule and hypoellipticity,

$$b_2^{-1}\, \partial^\alpha b_2 \in \Gamma_\rho^{-\rho|\alpha|}(\mathbb{R}^{2d}), \qquad \sigma^{-1}\, \partial^\beta \sigma \in \Gamma_\rho^{-\rho|\beta|}(\mathbb{R}^{2d}),$$

for all multi-indices α, β. Consequently, the expression in parentheses in (57) is of the form $1 + r'$ with $r' \in \Gamma_\rho^{-2\rho}(\mathbb{R}^{2d})$. Since $r'(z) \to 0$ as $\|z\|_2 \to \infty$, there exists $R > 0$ such that

$$|r'(z)| \le \tfrac{1}{2}, \qquad \text{for all } z \in \mathbb{R}^{2d} \text{ with } \|z\|_2 \ge R.$$

We may now modify r' on $\{z \in \mathbb{R}^{2d} \mid \|z\|_2 < R\}$ as follows. Let $\chi \in C_c^\infty(\mathbb{R}^{2d})$ satisfy

$$\chi(z) = 1 \quad \text{for } z \in \{z \in \mathbb{R}^{2d} \mid \|z\|_2 \le R\},$$

and

$$\chi(z) = 0 \quad \text{for } z \in \{z \in \mathbb{R}^{2d} \mid \|z\|_2 \ge 2R\},$$

and let $G\colon \mathbb{C} \to \mathbb{C}$ be a smooth truncation of the identity satisfying

$$G(a) = a \quad \text{for } |a| \le \tfrac{1}{2}, \qquad |G(a)| \le \tfrac{1}{2} \quad \text{for all } a \in \mathbb{C}.$$

Define

$$\widetilde{r}'(z) \coloneqq (1 - \chi(z))\, r'(z) + \chi(z)\, G(r'(z)), \qquad z \in \mathbb{R}^{2d}.$$

Then

$$\widetilde{r}' - r' = \chi\big(G(r') - r'\big) \in C_c^\infty(\mathbb{R}^{2d}),$$

and hence

$$\widetilde{r}' \in \Gamma_\rho^{-2\rho}(\mathbb{R}^{2d}),$$

since $r' \in \Gamma_\rho^{-2\rho}(\mathbb{R}^{2d})$ and $C_c^\infty(\mathbb{R}^{2d}) \subset \Gamma^{-\infty}(\mathbb{R}^{2d}) \subset \Gamma_\rho^{-2\rho}(\mathbb{R}^{2d})$. Redefining

$$1 + \widetilde{s} \coloneqq b_2\, \sigma(1 + \widetilde{r}'),$$

we get

$$\widetilde{s} - s = b_2\, \sigma(\widetilde{r}' - r') \in \Gamma^{-\infty}(\mathbb{R}^{2d}),$$

since $\widetilde{r}' - r' \in C_c^\infty(\mathbb{R}^{2d})$. Hence $\widetilde{s} \in \Gamma^{-\infty}(\mathbb{R}^{2d})$, while $\widetilde{r}' \in \Gamma_\rho^{-2\rho}(\mathbb{R}^{2d})$ by construction, and the identity

$$1 + \widetilde{s} = b_2\, \sigma(1 + \widetilde{r}')$$

retains the form of (56)–(57). Replacing (s, r') by $(\widetilde{s}, \widetilde{r}')$ in (56)–(57), we thus preserve all structural properties required in what follows—namely the identity, the symbol class of r', and the property of s—and we may therefore assume, without loss of generality,

$$|r'(z)| \le \tfrac{1}{2}, \qquad z \in \mathbb{R}^{2d}.$$

It follows that

$$b_2(z) = \sigma(z)^{-1}\,\frac{1+s(z)}{1+r'(z)}, \qquad z \in \mathbb{R}^{2d}.$$

Moreover,

$$(1+r')^{-1} = 1+q, \qquad q = -\frac{r'}{1+r'}.$$

Writing $q = r'\,H(r')$ with $H(a) \coloneqq -1/(1+a)$, we obtain $H(r') \in \Gamma^0_\rho(\mathbb{R}^{2d})$ and hence

$$q \in \Gamma^{-2\rho}_\rho(\mathbb{R}^{2d}).$$

Therefore,

$$\frac{1+s}{1+r'} = (1+s)(1+q) = 1+\tilde{r}, \qquad \tilde{r} \in \Gamma^{-2\rho}_\rho(\mathbb{R}^{2d}),$$

and consequently

$$b_2 = \sigma^{-1}(1+\tilde{r}), \qquad \tilde{r} \in \Gamma^{-2\rho}_\rho(\mathbb{R}^{2d}). \tag{58}$$

Applying the same arguments to the operator $B_2(T_1 - T_2)$ shows that its Weyl symbol is of the form

$$b_2(\sigma_1 - \sigma)(1+r''), \tag{59}$$

for some $r'' \in \Gamma^{-2\rho}_\rho(\mathbb{R}^{2d})$. Using (58) in (59) along with $\sigma^{-1}(\sigma_1 - \sigma) \in \Gamma^{-2\rho}_\rho(\mathbb{R}^{2d})$, we obtain

$$b_2(\sigma_1 - \sigma)(1+r'') = \sigma^{-1}(\sigma_1 - \sigma)(1+\tilde{r})(1+r'') \in \Gamma^{-2\rho}_\rho(\mathbb{R}^{2d})$$

and hence

$$B_2(T_1 - T_2) \in \Psi^{-2\rho}_\rho(\mathbb{R}^d).$$

This establishes the required property for the operator $K = B_2(T_1 - T_2)$, and thus completes the proof of the invariance under the change of quantization stated in (51).

Invertibility formula. By [53, Theorem 25.4], under strict positivity of the symbol $\sigma \in \mathrm{H}\Gamma^-(\mathbb{R}^{2d})$, the inverse of the operator $T_{\sigma^{-1}}$ is also a hypoelliptic operator with symbol in $\mathrm{H}\Gamma^-(\mathbb{R}^{2d})$. We now prove the second statement of Theorem 1, namely

$$H_{T^{-1}_{\sigma^{-1}}}(\varepsilon) \sim H_\sigma(\varepsilon), \qquad \varepsilon \to 0.$$

To this end, we establish a parametrix relation between T_σ and $T_{\sigma^{-1}}$, showing that their composition differs from the identity by a compact operator of strictly negative order. Using the assumed invertibility of $T_{\sigma^{-1}}$, this relation allows us to compare the entropy of T_σ with that of $T^{-1}_{\sigma^{-1}}$. Applying the Weyl composition formula [53, Theorem 23.6] to σ^{-1} and σ and following a similar line of arguments as the one leading to (58), shows that

$$T_{\sigma^{-1}} T_\sigma = I + R, \tag{60}$$

with $R \in \Psi^{-2\rho}_\rho(\mathbb{R}^d)$.

Since $T_{\sigma^{-1}}$ is invertible by assumption, (60) can be rewritten as

$$T_\sigma = T^{-1}_{\sigma^{-1}}(I+R). \tag{61}$$

By [53, Theorem 24.4], the operator $R \in \Psi^{-2\rho}_\rho(\mathbb{R}^d)$ extends to a compact operator on $L^2(\mathbb{R}^d)$. Identity (61) is therefore of the form required in Lemma 6, with $T_1 = T_\sigma$, $T_2 = T^{-1}_{\sigma^{-1}}$, $K = R$, and $K_{-\infty} = 0$. Due to σ being frequency-even, all these operators preserve the real structure, and applying the lemma yields

$$H_{T^{-1}_{\sigma^{-1}}}(\varepsilon) \sim H_\sigma(\varepsilon), \qquad \varepsilon \to 0,$$

which is precisely (14).

Proof of Lemma 6. Let k denote the Weyl symbol of K. Since $K \in \Psi^{-m}_{\rho_1}(\mathbb{R}^d)$ for some $m > 0$, we have $k \in \Gamma^{-m}_{\rho_1}(\mathbb{R}^{2d})$, hence k is of strictly negative order. The constant symbol 1 belongs to $\mathrm{H}\Gamma^{0,0}_{\rho_1}(\mathbb{R}^{2d})$. Applying Shubin's stability result for hypoelliptic symbols under lower-order additive perturbations [53, Lemma 25.1 (1c)] yields

$$1 + k \in \mathrm{H}\Gamma^{0,0}_{\rho_1}(\mathbb{R}^{2d}).$$

Consequently, by [53, Theorem 25.1], the operator $I + K$ admits a right parametrix, that is, there exists a pseudodifferential operator $P \in \Psi^0_{\rho_1}(\mathbb{R}^d)$ such that

$$(I + K)P = I + R_{-\infty}, \tag{62}$$

for some residual operator $R_{-\infty} \in \Psi^{-\infty}(\mathbb{R}^d)$. By the Weyl composition formula [53, Theorem 23.6], the Weyl symbol of $(I + K)P$ has the form

$$(1 + k)p + q,$$

where p is the Weyl symbol of P and $q \in \Gamma^{-2\rho_1}_{\rho_1}(\mathbb{R}^{2d})$. On the other hand, by (62), the Weyl symbol of $(I + K)P$ is $1 + r_{-\infty}$, where $r_{-\infty}$ is the symbol of $R_{-\infty}$. Using that $k \in \Gamma^{-m}_{\rho_1}(\mathbb{R}^{2d})$ and comparing expressions, it follows that there exists a symbol

$$r_1 \in \Gamma^{-\gamma}_{\rho_1}(\mathbb{R}^{2d}), \qquad \gamma := \min\{2\rho_1, m\},$$

such that

$$p = 1 + r_1.$$

Equivalently, in operator form,

$$P = I + R_1, \qquad R_1 \in \Psi^{-\gamma}_{\rho_1}(\mathbb{R}^d).$$

We now multiply the identity

$$T_1 = T_2(I + K) + K_{-\infty}$$

from the right by P and use (62) to obtain

$$\begin{aligned} T_1P &= T_2(I + K)P + K_{-\infty}P \\ &= T_2(I + R_{-\infty}) + K_{-\infty}P \\ &= T_2 + T_2R_{-\infty} + K_{-\infty}P. \end{aligned}$$

Rearranging terms yields

$$T_2 = T_1(I + R_1) + R'_{-\infty}, \tag{63}$$

where

$$R'_{-\infty} := -T_2R_{-\infty} - K_{-\infty}P \in \Psi^{-\infty}(\mathbb{R}^d),$$

since $\Psi^{-\infty}(\mathbb{R}^d)$ is a two-sided ideal in the pseudodifferential calculus. Equation (63) provides a representation of T_2 of the same structural form as the hypothesis of Lemma 6, namely

$$T_2 = T_1(I + R_1) + R'_{-\infty},$$

with $R_1 \in \Psi^{-\gamma}_{\rho_1}(\mathbb{R}^d)$ for some $\gamma > 0$ and $R'_{-\infty} \in \Psi^{-\infty}(\mathbb{R}^d)$. With an argument similar to the one following (49), upon replacement by $(R_1 + CR_1C)/2$ and $(R'_{-\infty} + CR'_{-\infty}C)/2$, respectively, R_1 and $R'_{-\infty}$ can be taken to preserve the real structure. Although the order γ may differ from the original exponent m, the lemma only requires the perturbation to be of strictly negative order. This symmetry explains why Lemma 6 only requires that either T_1 or T_2 be the Weyl quantization of σ.

We now turn to the comparison of the entropy numbers of T_1 and T_2. Recall that the entropy numbers $\{e_n(T)\}_{n\in\mathbb{N}}$ of a compact operator $T\colon L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ are defined by

$$e_n(T) \coloneqq \inf\bigl\{\varepsilon>0 \,\big|\, H_T(\varepsilon)\le n\bigr\}, \qquad n\in\mathbb{N}. \tag{64}$$

We refer to [13, 24, 43, 49, 50] for background on entropy numbers and their basic properties. Starting from the representation (63), and using the submultiplicativity and subadditivity of entropy numbers (see, for example, [43, Section 15.7] or [24, Section 1.3.1, Lemma 1]), we obtain

$$e_{n+5\lfloor n^{1/2}\rfloor}(T_2) \le e_n(T_1)\, e_{\lfloor n^{1/2}\rfloor}(I+R_1) + e_{4\lfloor n^{1/2}\rfloor}(R'_{-\infty}), \tag{65}$$

together with

$$e_{\lfloor n^{1/2}\rfloor}(I+R_1) \le e_0(I) + e_{\lfloor n^{1/2}\rfloor}(R_1), \qquad n\in\mathbb{N}. \tag{66}$$

Since $e_0(I)=\|I\|_{2\to 2}=1$ [43, Section 15.7], and R_1 by virtue of being compact satisfies

$$\lim_{j\to\infty} e_j(R_1)=0,$$

it follows, by (66), that

$$e_{\lfloor n^{1/2}\rfloor}(I+R_1)\le 1+o(1), \qquad n\to\infty. \tag{67}$$

We now estimate the term $e_{4\lfloor n^{1/2}\rfloor}\bigl(R'_{-\infty}\bigr)$ appearing in (65). Since $R'_{-\infty}\in\Psi^{-\infty}(\mathbb{R}^d)$, it has a Schwartz kernel (cf. the discussion following [53, Definition 23.4]) so that singular values decay faster than any polynomial (see, e.g., [15, Theorem 5.1 (5)]). Therefore, Carl's inequality [14, Corollary 1] guarantees that, for every $q>0$,

$$e_{4\lfloor n^{1/2}\rfloor}\bigl(R'_{-\infty}\bigr) = o\bigl(n^{-q}\bigr), \qquad n\to\infty. \tag{68}$$

From now on, we assume without loss of generality that $T_1=T_\sigma$, where T_σ is the Weyl quantization of the hypoelliptic symbol σ. If instead $T_2=T_\sigma$, the same argument applies after interchanging the roles of T_1 and T_2 from the outset.

An argument identical to that used to derive (68) shows that the entropy numbers of the operator $K_{-\infty}$ in the parametrix relation

$$T_1 = T_2(I+K)+K_{-\infty} \tag{69}$$

also decay faster than any polynomial rate. More precisely, for every $q>0$,

$$e_{4\lfloor n^{1/2}\rfloor}(K_{-\infty}) = o\bigl(n^{-q}\bigr), \qquad n\to\infty. \tag{70}$$

We next derive two-sided polynomial bounds for the entropy numbers of $T_1=T_\sigma$. Since $\sigma\in \mathrm{H}\Gamma^-_\rho(\mathbb{R}^{2d})$, there exist exponents $m_+\ge m_->0$ such that the two-sided hypoellipticity bounds (120) hold. Consequently, there exist constants $c_1,c_2>0$ and $\lambda_0>0$ so that, for all $\lambda\in(0,\lambda_0)$,

$$c_1\lambda^{-2d/m_+}\le V_\sigma(\lambda)\le c_2\lambda^{-2d/m_-}.$$

Since (13) has already been established for the Weyl quantization, this yields

$$H_{T_1}(\varepsilon)\sim\int_\varepsilon^\infty \frac{V_\sigma(\lambda)}{\lambda}\,d\lambda, \qquad \varepsilon\to 0,$$

and therefore

$$c_3\,\varepsilon^{-2d/m_+}\le H_{T_1}(\varepsilon)\le c_4\,\varepsilon^{-2d/m_-}, \qquad \varepsilon\in(0,\varepsilon_0),$$

for suitable constants $c_3,c_4,\varepsilon_0>0$. By the definition of the entropy numbers, these bounds imply the existence of constants $c_5,c_6>0$ such that

$$c_5\,n^{-m_+/(2d)}\lesssim e_n(T_1)\lesssim c_6\,n^{-m_-/(2d)}, \qquad n\to\infty. \tag{71}$$

Choose $q > m_+/(2d)$. Then (68) and (71) imply

$$e_{4\lfloor n^{1/2}\rfloor}\big(R'_{-\infty}\big) = o\big(e_n(T_1)\big), \qquad n \to \infty.$$

Combining this with (65) and (67), we obtain

$$e_{\,n+5\lfloor n^{1/2}\rfloor}(T_2) \le e_n(T_1)\big(1+o(1)\big), \qquad n \to \infty. \tag{72}$$

Based on (69), applying the same entropy-number calculus as before, we get

$$e_{\,n+5\lfloor n^{1/2}\rfloor}(T_1) \le e_n(T_2)\, e_{\lfloor n^{1/2}\rfloor}(I+K) + e_{4\lfloor n^{1/2}\rfloor}(K_{-\infty}). \tag{73}$$

Since K is compact,

$$e_{\lfloor n^{1/2}\rfloor}(I+K) \le 1 + o(1), \qquad n \to \infty.$$

Moreover, by (70),

$$e_{4\lfloor n^{1/2}\rfloor}(K_{-\infty}) = o\big(n^{-q}\big),$$

for every $q > 0$. Using the lower bound in (71) together with (73), we infer that

$$e_n(T_2) \gtrsim n^{-m_+/(2d)}, \qquad n \to \infty.$$

In particular,

$$e_{4\lfloor n^{1/2}\rfloor}(K_{-\infty}) = o\big(e_n(T_2)\big), \qquad n \to \infty.$$

Reinserting this into (73), we obtain

$$e_{\,n+5\lfloor n^{1/2}\rfloor}(T_1) \le e_n(T_2)\big(1+o(1)\big), \qquad n \to \infty. \tag{74}$$

The preceding argument was carried out under the assumption $T_1 = T_\sigma$. Since the parametrix relation assumed in Lemma 6 and the derived reverse parametrix relation are invariant under exchanging T_1 and T_2, the same reasoning applies when $T_2 = T_\sigma$. Consequently, the estimates (72) and (74) hold irrespective of which of the two operators is the Weyl quantization of σ.

We now convert the entropy-number relations (72) and (74) into corresponding asymptotic relations for the entropy functions. Fix $\varepsilon > 0$ and choose $n_\varepsilon \in \mathbb{N}$ such that

$$n_\varepsilon + 5\lfloor n_\varepsilon^{1/2}\rfloor < H_{T_2}(\varepsilon) \le n_\varepsilon + 1 + 5\lfloor (n_\varepsilon+1)^{1/2}\rfloor. \tag{75}$$

By the definition of entropy numbers this implies

$$e_{\,n_\varepsilon+5\lfloor n_\varepsilon^{1/2}\rfloor}(T_2) \ge \varepsilon.$$

Combining this with (72), we obtain

$$\varepsilon \le e_{\,n_\varepsilon+5\lfloor n_\varepsilon^{1/2}\rfloor}(T_2) \le e_{n_\varepsilon}(T_1)\big(1+o(1)\big), \qquad \varepsilon \to 0.$$

Hence, there exists a function $\zeta : \mathbb{R}_+^* \to \mathbb{R}_+^*$ with $\zeta(\varepsilon) \to 1$ as $\varepsilon \to 0$ such that

$$\varepsilon\,\zeta(\varepsilon) < e_{n_\varepsilon}(T_1). \tag{76}$$

Applying again the definition of entropy numbers yields

$$H_{T_1}(\varepsilon\,\zeta(\varepsilon)) \ge n_\varepsilon = H_{T_2}(\varepsilon)\big(1+o_{\varepsilon\to 0}(1)\big). \tag{77}$$

To obtain an upper bound for H_{T_1} in terms of H_{T_2}, let $m_\varepsilon := \lceil H_{T_2}(\varepsilon)\rceil$. Then $m_\varepsilon \to \infty$ as $\varepsilon \to 0$, and by the definition of entropy numbers we have

$$e_{m_\varepsilon}(T_2) \le \varepsilon.$$

Applying (74) with $n = m_\varepsilon$ yields

$$e_{m_\varepsilon + 5\lfloor m_\varepsilon^{1/2} \rfloor}(T_1) \le \varepsilon\big(1 + o(1)\big), \qquad \varepsilon \to 0.$$

Consequently, there exists a function $\zeta' : \mathbb{R}_+^* \to \mathbb{R}_+^*$ with $\zeta'(\varepsilon) \to 1$ as $\varepsilon \to 0$ such that

$$H_{T_1}\big(\varepsilon\,\zeta'(\varepsilon)\big) \le m_\varepsilon + 5\lfloor m_\varepsilon^{1/2} \rfloor. \tag{78}$$

Since

$$m_\varepsilon = H_{T_2}(\varepsilon) + O(1), \qquad \varepsilon \to 0,$$

and $H_{T_2}(\varepsilon) \to \infty$, we obtain

$$m_\varepsilon + 5\lfloor m_\varepsilon^{1/2} \rfloor = H_{T_2}(\varepsilon)\big(1 + o_{\varepsilon\to 0}(1)\big).$$

Substituting this relation into (78) yields

$$H_{T_1}\big(\varepsilon\,\zeta'(\varepsilon)\big) \le H_{T_2}(\varepsilon)\big(1 + o_{\varepsilon\to 0}(1)\big). \tag{79}$$

From here on we assume, without loss of generality, that $T_1 = T_\sigma$. The case $T_2 = T_\sigma$ is treated in the same way after interchanging the roles of T_1 and T_2. Since (13) has already been established for the Weyl quantization, we obtain

$$\begin{aligned} H_{T_1}(\varepsilon\,\zeta(\varepsilon)) &\sim \int_{\varepsilon\zeta(\varepsilon)}^{\infty} \frac{V_\sigma(\lambda)}{\lambda}\, d\lambda \qquad (80)\\ &= \int_{\varepsilon}^{\infty} \frac{V_\sigma(\lambda)}{\lambda}\, d\lambda + \int_{\varepsilon\zeta(\varepsilon)}^{\varepsilon} \frac{V_\sigma(\lambda)}{\lambda}\, d\lambda \\ &\sim H_{T_1}(\varepsilon) + \int_{\varepsilon\zeta(\varepsilon)}^{\varepsilon} \frac{V_\sigma(\lambda)}{\lambda}\, d\lambda, \qquad (81) \end{aligned}$$

as $\varepsilon \to 0$, where the second integral is understood with its natural orientation. Using the monotonicity of V_σ, the remainder term in (80)–(81) satisfies

$$\left| \int_{\varepsilon\zeta(\varepsilon)}^{\varepsilon} \frac{V_\sigma(\lambda)}{\lambda}\, d\lambda \right| \le \frac{V_\sigma(\varepsilon\,\zeta^*(\varepsilon))}{\varepsilon\,\zeta^*(\varepsilon)} \big|\varepsilon - \varepsilon\,\zeta(\varepsilon)\big| = \frac{V_\sigma(\varepsilon\,\zeta^*(\varepsilon))}{\zeta^*(\varepsilon)} |1 - \zeta(\varepsilon)|, \tag{82}$$

for all $\varepsilon > 0$, where $\zeta^*(\varepsilon) = \min\{1, \zeta(\varepsilon)\}$. Since V_σ is regularly varying at zero, Karamata's theorem implies

$$V_\sigma(\varepsilon\,\zeta^*(\varepsilon)) = O_{\varepsilon\to 0}\left(\int_{\varepsilon\zeta^*(\varepsilon)}^{\infty} \frac{V_\sigma(\lambda)}{\lambda}\, d\lambda \right). \tag{83}$$

Because $\zeta(\varepsilon) \to 1$ as $\varepsilon \to 0$, it follows from (83) that

$$V_\sigma(\varepsilon\,\zeta^*(\varepsilon))|1 - \zeta(\varepsilon)| = o_{\varepsilon\to 0}\left(\int_{\varepsilon\zeta^*(\varepsilon)}^{\infty} \frac{V_\sigma(\lambda)}{\lambda}\, d\lambda \right). \tag{84}$$

Combining (80)–(81), (82), and (84) therefore yields

$$H_{T_1}(\varepsilon\,\zeta(\varepsilon)) \sim H_{T_1}(\varepsilon), \qquad \varepsilon \to 0. \tag{85}$$

The same argument, with ζ replaced by ζ', gives

$$H_{T_1}\big(\varepsilon\,\zeta'(\varepsilon)\big) \sim H_{T_1}(\varepsilon), \qquad \varepsilon \to 0. \tag{86}$$

Finally, combining (77), (79), (85), and (86), we conclude that

$$H_{T_1}(\varepsilon) \sim H_{T_2}(\varepsilon), \qquad \varepsilon \to 0,$$

which completes the proof of Lemma 6. □

4.2 Proof of Corollary 2

From the definition (7) of V_σ, we have

$$\begin{aligned}\int_\varepsilon^\infty \frac{V_\sigma(\lambda)}{\lambda}\,d\lambda &= \int_\varepsilon^\infty \int_{\mathbb{R}^{2d}} \frac{\mathbb{1}_{\{\sigma(x,\omega)>\lambda\}}}{\lambda}\,dx\,d\omega\,d\lambda \\ &= \int_0^\infty \int_{\mathbb{R}^{2d}} \frac{\mathbb{1}_{\{\sigma(x,\omega)>\lambda>\varepsilon\}}}{\lambda}\,dx\,d\omega\,d\lambda, \qquad \varepsilon>0.\end{aligned}$$

Since the integrand is nonnegative, Tonelli's theorem yields

$$\int_0^\infty \int_{\mathbb{R}^{2d}} \frac{\mathbb{1}_{\{\sigma(x,\omega)>\lambda>\varepsilon\}}}{\lambda}\,dx\,d\omega\,d\lambda = \int_{\mathbb{R}^{2d}} \int_0^\infty \frac{\mathbb{1}_{\{\sigma(x,\omega)>\lambda>\varepsilon\}}}{\lambda}\,d\lambda\,dx\,d\omega, \qquad \varepsilon>0.$$

For fixed $(x,\omega)\in\mathbb{R}^{2d}$, the inner integral can be computed explicitly:

$$\int_0^\infty \frac{\mathbb{1}_{\{\sigma(x,\omega)>\lambda>\varepsilon\}}}{\lambda}\,d\lambda = \ln_+\!\left(\frac{\sigma(x,\omega)}{\varepsilon}\right),$$

where $\ln_+(t)=\max\{\ln(t),0\}$. Consequently,

$$\int_\varepsilon^\infty \frac{V_\sigma(\lambda)}{\lambda}\,d\lambda = \int_{\mathbb{R}^{2d}} \ln_+\!\left(\frac{\sigma(x,\omega)}{\varepsilon}\right)dx\,d\omega, \qquad \varepsilon>0.$$

The claim now follows directly from Theorem 1.

4.3 Proof of Corollary 3

We combine the theory of regular variation (see, e.g., [9]) with Theorem 1 to derive the asymptotics of the type-τ integrals (2). Specifically, for $\tau=2$ and $\tau=3$, definition (2) gives

$$I_2(\varepsilon)=\int_\varepsilon^\infty M_\sigma(\lambda)\,\lambda^{-2}\,d\lambda, \qquad I_3(\varepsilon)=\int_\varepsilon^\infty M_\sigma(\lambda)\,\lambda^{-3}\,d\lambda.$$

By Theorem 1, we have

$$H_\sigma(\varepsilon)\sim\int_\varepsilon^\infty \frac{V_\sigma(\lambda)}{\lambda}\,d\lambda, \qquad \varepsilon\to 0,$$

for all quantizations. Since V_σ is regularly varying at zero, condition (RC) holds and [2, Theorem 2] yields

$$H_\sigma(\varepsilon)\sim I_1(\varepsilon),$$

which, in turn, implies

$$I_1(\varepsilon)\sim\int_\varepsilon^\infty \frac{V_\sigma(\lambda)}{\lambda}\,d\lambda, \qquad \varepsilon\to 0.$$

Using integration by parts, we obtain

$$I_2(\varepsilon)=\varepsilon^{-1}I_1(\varepsilon)-\int_\varepsilon^\infty \frac{I_1(\lambda)}{\lambda^2}\,d\lambda,$$

and

$$I_3(\varepsilon)=\varepsilon^{-2}I_1(\varepsilon)-2\int_\varepsilon^\infty \frac{I_1(\lambda)}{\lambda^3}\,d\lambda.$$

Since I_1 is regularly varying at zero, another application of Karamata's theorem implies

$$I_2(\varepsilon)\sim\int_\varepsilon^\infty \frac{V_\sigma(\lambda)}{\lambda^2}\,d\lambda, \qquad I_3(\varepsilon)\sim\int_\varepsilon^\infty \frac{V_\sigma(\lambda)}{\lambda^3}\,d\lambda, \qquad \varepsilon\to 0.$$

In particular, by Pinsker's asymptotic minimax theorem (cf. [37, Theorem 5.3] and the discussion preceding [2, Theorem 5]) and [2, Theorem 4],

$$R_\sigma(\kappa) \sim \kappa^2\, \varepsilon_\kappa \int_{\varepsilon_\kappa}^\infty \frac{V_\sigma(\lambda)}{\lambda^2}\, d\lambda, \qquad \kappa \to 0, \tag{87}$$

where the critical radius ε_κ is determined by

$$\kappa^2 \int_{\varepsilon_\kappa}^\infty V_\sigma(\lambda) \left(\frac{2}{\lambda^3} - \frac{1}{\lambda^2 \varepsilon_\kappa} \right) d\lambda = 1, \qquad \kappa > 0. \tag{88}$$

We next compute the integrals appearing in (87) and (88) explicitly. From the definition (7) of V_σ,

$$\begin{aligned} \int_\varepsilon^\infty \frac{V_\sigma(\lambda)}{\lambda^2}\, d\lambda &= \int_\varepsilon^\infty \int_{\mathbb{R}^{2d}} \frac{\mathbb{1}_{\{\sigma(x,\omega)>\lambda\}}}{\lambda^2}\, dx\, d\omega\, d\lambda \\ &= \int_0^\infty \int_{\mathbb{R}^{2d}} \frac{\mathbb{1}_{\{\sigma(x,\omega)>\lambda>\varepsilon\}}}{\lambda^2}\, dx\, d\omega\, d\lambda. \end{aligned}$$

Since the integrand is nonnegative, Tonelli's theorem allows us to exchange the order of integration, giving

$$\int_{\mathbb{R}^{2d}} \int_0^\infty \frac{\mathbb{1}_{\{\sigma(x,\omega)>\lambda>\varepsilon\}}}{\lambda^2}\, d\lambda\, dx\, d\omega.$$

For fixed $(x,\omega) \in \mathbb{R}^{2d}$, the inner integral can be evaluated explicitly:

$$\int_0^\infty \frac{\mathbb{1}_{\{\sigma(x,\omega)>\lambda>\varepsilon\}}}{\lambda^2}\, d\lambda = \left(\frac{1}{\varepsilon} - \frac{1}{\sigma(x,\omega)} \right)_+.$$

Consequently,

$$\int_\varepsilon^\infty \frac{V_\sigma(\lambda)}{\lambda^2}\, d\lambda = \int_{\mathbb{R}^{2d}} \left(\frac{1}{\varepsilon} - \frac{1}{\sigma(x,\omega)} \right)_+ dx\, d\omega. \tag{89}$$

Substituting (89) into (87) yields (17).

A completely analogous computation gives

$$\int_\varepsilon^\infty \frac{V_\sigma(\lambda)}{\lambda^3}\, d\lambda = \int_{\mathbb{R}^{2d}} \left(\frac{1}{2\varepsilon^2} - \frac{1}{2\sigma(x,\omega)^2} \right)_+ dx\, d\omega. \tag{90}$$

Combining (89) and (90), we obtain

$$\begin{aligned} \int_\varepsilon^\infty V_\sigma(\lambda) \left(\frac{2}{\lambda^3} - \frac{1}{\lambda^2 \varepsilon} \right) d\lambda &= \int_{\mathbb{R}^{2d}} \left[\left(\frac{1}{\varepsilon^2} - \frac{1}{\sigma(x,\omega)^2} \right)_+ - \left(\frac{1}{\varepsilon^2} - \frac{1}{\sigma(x,\omega)\varepsilon} \right)_+ \right] dx\, d\omega \\ &= \int_{\mathbb{R}^{2d}} \frac{1}{\sigma(x,\omega)} \left(\frac{1}{\varepsilon} - \frac{1}{\sigma(x,\omega)} \right)_+ dx\, d\omega. \end{aligned}$$

Inserting this expression into (88) yields (18), thereby completing the proof.

4.4 Proof of Theorem 4

By [53, Proposition 25.4], $\mathcal{B}_\mathcal{S}^{(s)}$ is compact, and therefore bounded, in $L^2(\mathbb{R}^d)$. We next show that $\ker T_\mathcal{S} = \{0\}$. If $T_\mathcal{S}\psi = 0$ for some nonzero $\psi \in L^2(\mathbb{R}^d)$, then $\mu\psi \in \mathcal{B}_\mathcal{S}^{(s)}$ for all $\mu \in \mathbb{R}$, which contradicts boundedness of $\mathcal{B}_\mathcal{S}^{(s)}$ in $L^2(\mathbb{R}^d)$. Since $T_\mathcal{S}$ is self-adjoint, $\ker T_\mathcal{S}^* = \ker T_\mathcal{S} = \{0\}$, and [53, Theorem 25.4] therefore guarantees the existence of an inverse $T_\mathcal{S}^{-1}$, which can be extended via [53, Theorem 24.4] to a compact operator on $L^2(\mathbb{R}^d)$. Setting $g = T_\mathcal{S} f$ in (21), we obtain

$$\mathcal{B}_\mathcal{S}^{(s)} = \{T_\mathcal{S}^{-1} g : \ \|g\|_{L^2(\mathbb{R}^d)} \le 1\} = T_\mathcal{S}^{-1} \mathcal{B}_2,$$

where $\mathcal{B}_2$ denotes the unit ball in $L^2(\mathbb{R}^d)$. Consequently,

$$H\Big(\varepsilon; \mathcal{B}_{\mathcal{S}}^{(s)}\Big) = H_{T_{\mathcal{S}}^{-1}}(\varepsilon), \qquad \varepsilon > 0. \tag{91}$$

To apply Theorem 1, we study the volume function

$$V_{\sigma_{\mathcal{S}}^{-1}}(\lambda) = \int_{\mathbb{R}^{2d}} \mathbb{1}_{\{(1+c\|x\|_2^2+(2\pi\|\omega\|_2)^2)^{-s/2}>\lambda\}}\, dx\, d\omega, \qquad \lambda \in (0,1). \tag{92}$$

Performing the change of variables $x \mapsto x'/\sqrt{c}$ and $\omega \mapsto \omega'/(2\pi)$ yields

$$V_{\sigma_{\mathcal{S}}^{-1}}(\lambda) = \frac{1}{(2\pi\sqrt{c})^d} \int_{\mathbb{R}^{2d}} \mathbb{1}_{\{(1+\|x'\|_2^2+\|\omega'\|_2^2)^{-s/2}>\lambda\}}\, dx'\, d\omega' \tag{93}$$

$$= \frac{1}{(2\pi\sqrt{c})^d} \int_{\mathbb{R}^{2d}} \mathbb{1}_{\{\|x'\|_2^2+\|\omega'\|_2^2<\lambda^{-2/s}-1\}}\, dx'\, d\omega'. \tag{94}$$

Hence $V_{\sigma_{\mathcal{S}}^{-1}}(\lambda)$ is proportional to the volume of a ball of radius $(\lambda^{-2/s}-1)^{1/2}$ in $\mathbb{R}^{2d}$, and therefore

$$V_{\sigma_{\mathcal{S}}^{-1}}(\lambda) = \frac{\omega_{2d}}{(2\pi\sqrt{c})^d}(\lambda^{-2/s}-1)^d \sim \frac{\omega_{2d}}{(2\pi\sqrt{c})^d}\lambda^{-2d/s}, \qquad \lambda \to 0.$$

A direct calculation further shows that

$$\int_{\varepsilon}^{\infty} \frac{V_{\sigma_{\mathcal{S}}^{-1}}(\lambda)}{\lambda}\, d\lambda \sim \frac{s\,\omega_{2d}}{2d(2\pi\sqrt{c})^d}\, \varepsilon^{-2d/s}, \qquad \varepsilon \to 0. \tag{95}$$

The symbol $\sigma_{\mathcal{S}}^{-1}$ satisfies the assumptions of Theorem 1. Applying successively (91), (14), (13), and (95), we obtain

$$\begin{aligned} H\Big(\varepsilon; \mathcal{B}_{\mathcal{S}}^{(s)}\Big) &= H_{T_{\mathcal{S}}^{-1}}(\varepsilon) \sim H_{\sigma_{\mathcal{S}}^{-1}}(\varepsilon) \\ &\sim \int_{\varepsilon}^{\infty} \frac{V_{\sigma_{\mathcal{S}}^{-1}}(\lambda)}{\lambda}\, d\lambda \sim \frac{s\,\omega_{2d}}{2d(2\pi\sqrt{c})^d}\, \varepsilon^{-2d/s}, \qquad \varepsilon \to 0, \end{aligned}$$

which establishes (22).

Finally, by [2, Appendix A, Table (iv)], metric entropy scaling of the form

$$H(\varepsilon) \sim \frac{\mathfrak{c}\,\varepsilon^{-\alpha}}{\alpha}, \qquad \varepsilon \to 0,$$

with $\mathfrak{c}, \alpha > 0$, is equivalent to minimax risk scaling

$$R(\kappa) \sim \frac{\alpha+2}{\alpha}\left(\frac{\mathfrak{c}\,\alpha\,\kappa^2}{(\alpha+1)(\alpha+2)}\right)^{\frac{2}{\alpha+2}}, \qquad \kappa \to 0.$$

In view of (22), for $\mathcal{B}_{\mathcal{S}}^{(s)}$ we may choose

$$\mathfrak{c} = \frac{\omega_{2d}}{(2\pi\sqrt{c})^d}, \qquad \alpha = \frac{2d}{s},$$

which yields

$$R\Big(\kappa; \mathcal{B}_{\mathcal{S}}^{(s)}\Big) \sim \frac{d+s}{d}\left(\frac{d\,s\,\omega_{2d}\,\kappa^2}{(2\pi\sqrt{c})^d(d+s)(2d+s)}\right)^{\frac{s}{d+s}}, \qquad \kappa \to 0.$$

This establishes (23) and completes the proof of Theorem 4.

4.5 Proof of Theorem 5

It follows from Rellich's compactness theorem (see [32, Theorem 2.17 and Exercise 2.12]) that $\mathcal{B}_{\mathcal{W}}^{(s,r)}$ is compact in $L^2(\mathbb{R}^d)$. Proceeding as in the proof of Theorem 4, with $\sigma_{\mathcal{W}}$ in place of $\sigma_{\mathcal{S}}$, we reduce the problem to determining the asymptotic behavior of

$$V_{\sigma_{\mathcal{W}}^{-1}}(\lambda) = \frac{1}{(2\pi\sqrt{c})^d} \int_{\mathbb{R}^{2d}} \mathbb{1}_{\{(1+\|\omega'\|_2^2)^{-s/2}(1+\|x'\|_2^2)^{-r/2}>\lambda\}}\, dx'\, d\omega', \qquad \lambda > 0. \tag{96}$$

We rewrite the integral in polar coordinates to get

$$\int_0^\infty \int_0^\infty \mathbb{1}_{\{(1+u_0^2)^{-s/2}(1+v_0^2)^{-r/2}>\lambda\}}\, s_{d-1}^2 (u_0 v_0)^{d-1}\, du_0\, dv_0,$$

where s_{d-1} denotes the surface measure of the unit sphere in $\mathbb{R}^d$. With the change of variables $u = u_0^d$ and $v = v_0^d$, we get

$$V_{\sigma_{\mathcal{W}}^{-1}}(\lambda) = \frac{\omega_d^2}{(2\pi\sqrt{c})^d} \int_0^\infty \int_0^\infty \mathbb{1}_{\{(1+u^{2/d})^{-s/2}(1+v^{2/d})^{-r/2}>\lambda\}}\, du\, dv, \tag{97}$$

using $\omega_d = s_{d-1}/d$.

Let us first assume that $r < s$. For $v \geq 0$, the inequality $(1+u^{2/d})^{-s/2}(1+v^{2/d})^{-r/2} > \lambda$ has solutions in u only if $\lambda(1+v^{2/d})^{r/2} < 1$. Solving this inequality for v gives $v < (\lambda^{-2/r}-1)^{d/2}$, and we therefore define

$$v_\lambda := (\lambda^{-2/r} - 1)^{d/2}. \tag{98}$$

Fix $v \in [0, v_\lambda)$. Then the condition $(1+u^{2/d})^{-s/2}(1+v^{2/d})^{-r/2} > \lambda$ is equivalent to $1 + u^{2/d} < \lambda^{-2/s}(1+v^{2/d})^{-r/s}$, and hence to

$$u < \left(\lambda^{-2/s}(1+v^{2/d})^{-r/s} - 1\right)^{d/2}.$$

Consequently, integrating first with respect to u gives

$$V_{\sigma_{\mathcal{W}}^{-1}}(\lambda) = \frac{\omega_d^2}{(2\pi\sqrt{c})^d} \int_0^{v_\lambda} \left(\lambda^{-2/s}(1+v^{2/d})^{-r/s} - 1\right)^{d/2} dv, \qquad \lambda > 0.$$

After the change of variables $t = v/v_\lambda$, the integral becomes

$$V_{\sigma_{\mathcal{W}}^{-1}}(\lambda) = \frac{\omega_d^2\, v_\lambda}{(2\pi\sqrt{c})^d} \int_0^1 u_\lambda(t)\, dt, \qquad \lambda > 0, \tag{99}$$

where

$$u_\lambda(t) := \left(\lambda^{-2/s}\left(1 + (tv_\lambda)^{2/d}\right)^{-r/s} - 1\right)^{d/2}, \qquad t \in [0,1). \tag{100}$$

Note that the integral (99) includes the endpoint $t = 1$, although the admissible region corresponds to $t \in [0,1)$. This is justified since the singleton $\{1\}$ has Lebesgue measure zero and u_λ extends continuously to $[0,1]$ with $u_\lambda(1) = 0$. From the definition of v_λ in (98) we have $v_\lambda \sim \lambda^{-d/r}$, as $\lambda \to 0$. Letting $\gamma := 2r/(ds)$ and using (100), we obtain the pointwise limit

$$u_\lambda(t) \longrightarrow \left(t^{-\gamma} - 1\right)^{d/2}, \qquad \lambda \to 0, \quad t \in (0,1).$$

Since $r < s$, we have $\gamma < 2/d$, and therefore the function $t \mapsto (t^{-\gamma}-1)^{d/2}$ is integrable on $(0,1)$. Moreover,

$$\left(1 + (tv_\lambda)^{2/d}\right)^{-r/s} \leq t^{-\gamma}\left(1 + v_\lambda^{2/d}\right)^{-r/s} = t^{-\gamma}\lambda^{2/s}, \qquad \lambda > 0,\ t \in (0,1),$$

which implies

$$u_\lambda(t) \le \left(t^{-\gamma}-1\right)^{d/2}.$$

Since $u_\lambda(t) \to (t^{-\gamma}-1)^{d/2}$ pointwise on $(0,1)$ as $\lambda \to 0$, and $|u_\lambda(t)| \le (t^{-\gamma}-1)^{d/2}$ with the latter integrable on $(0,1)$, the hypotheses of the dominated convergence theorem are satisfied. Consequently,

$$\int_0^1 u_\lambda(t)\,dt \;\longrightarrow\; \int_0^1 (t^{-\gamma}-1)^{d/2}\,dt, \qquad \lambda \to 0.$$

Using the representation of $V_{\sigma_{\mathcal{W}}^{-1}}(\lambda)$ derived above and applying the dominated convergence theorem, we obtain

$$V_{\sigma_{\mathcal{W}}^{-1}}(\lambda) = \frac{\omega_d^2\, v_\lambda}{(2\pi\sqrt{c})^d}\int_0^1 u_\lambda(t)\,dt \;\sim\; \frac{\omega_d^2\, v_\lambda}{(2\pi\sqrt{c})^d}\int_0^1 (t^{-\gamma}-1)^{d/2}dt, \qquad \lambda \to 0. \tag{101}$$

Changing variables according to $t_0 = t^\gamma$ in the integral on the right-hand side of (101), we obtain

$$\int_0^1 (t^{-\gamma}-1)^{d/2}dt = \frac{1}{\gamma}\int_0^1 t_0^{\frac{1}{\gamma}-1-\frac{d}{2}}(1-t_0)^{d/2}\,dt_0 = \frac{1}{\gamma}B\left(\frac{1}{\gamma}-\frac{d}{2},\frac{d}{2}+1\right), \tag{102}$$

where B denotes Euler's Beta function. Using the relation between the Beta and Gamma functions (see [7, Chapter 2]), we further get

$$\frac{1}{\gamma}B\left(\frac{1}{\gamma}-\frac{d}{2},\frac{d}{2}+1\right) = \frac{\Gamma\left(\frac{1}{\gamma}-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+1\right)}{\gamma\,\Gamma\left(\frac{1}{\gamma}+1\right)} = \frac{d\,\Gamma\left(\frac{d(s-r)}{2r}\right)\Gamma\left(\frac{d}{2}\right)}{2\,\Gamma\left(\frac{ds}{2r}\right)}. \tag{103}$$

Combining (98), (101), (102), and (103), we conclude that

$$V_{\sigma_{\mathcal{W}}^{-1}}(\lambda) \sim \frac{d\,\omega_d^2\,\Gamma\left(\frac{d(s-r)}{2r}\right)\Gamma\left(\frac{d}{2}\right)}{2(2\pi\sqrt{c})^d\,\Gamma\left(\frac{ds}{2r}\right)}\lambda^{-d/r}, \qquad \lambda \to 0, \tag{104}$$

in the case $r < s$.

By symmetry, the case $r > s$ is treated in the same way after interchanging the roles of r and s throughout the preceding argument. In particular, the auxiliary quantities v_λ, u_λ, and γ are redefined with r and s swapped, so that $\gamma = 2s/(dr)$. This yields

$$V_{\sigma_{\mathcal{W}}^{-1}}(\lambda) \sim \frac{d\,\omega_d^2\,\Gamma\left(\frac{d(r-s)}{2s}\right)\Gamma\left(\frac{d}{2}\right)}{2(2\pi\sqrt{c})^d\,\Gamma\left(\frac{dr}{2s}\right)}\lambda^{-d/s}, \quad \lambda \to 0. \tag{105}$$

For $r = s$, we first observe that

$$(1+u^{2/d})^{-s/2} \sim (1+u)^{-s/d}, \qquad u \to \infty.$$

Hence, for every fixed $\eta \in (0,1)$ there exists $R_\eta > 0$ such that

$$(1-\eta)(1+u^{2/d})^{-s/2} \le (1+u)^{-s/d} \le (1+\eta)(1+u^{2/d})^{-s/2}, \qquad u \ge R_\eta, \tag{106}$$

and the same estimates hold with u replaced by v. Now, whenever $u, v \ge R_\eta$, (106) and its version with u replaced by v imply that

$$\begin{aligned}(1-\eta)^2(1+u^{2/d})^{-s/2}(1+v^{2/d})^{-s/2} &\le (1+u)^{-s/d}(1+v)^{-s/d}\\ &\le (1+\eta)^2(1+u^{2/d})^{-s/2}(1+v^{2/d})^{-s/2}.\end{aligned}$$

Consequently, for $u, v \geq R_\eta$, we obtain the set inclusions

$$\{(1+u^{2/d})^{-s/2}(1+v^{2/d})^{-s/2} > \lambda\} \subseteq \{(1+u)^{-s/d}(1+v)^{-s/d} > (1-\eta)^2\lambda\} \tag{107}$$

and

$$\{(1+u)^{-s/d}(1+v)^{-s/d} > (1+\eta)^2\lambda\} \subseteq \{(1+u^{2/d})^{-s/2}(1+v^{2/d})^{-s/2} > \lambda\}. \tag{108}$$

We next decompose the double integral (97) defining $V_{\sigma_{\mathcal{W}}^{-1}}(\lambda)$ according to

$$V_{\sigma_{\mathcal{W}}^{-1}}(\lambda) = \frac{\omega_d^2}{(2\pi\sqrt{c})^d}\left(\int_0^{R_\eta}\int_0^{R_\eta} + 2\int_{R_\eta}^{\infty}\int_0^{R_\eta} + \int_{R_\eta}^{\infty}\int_{R_\eta}^{\infty}\right)\mathbb{1}_{\{(1+u^{2/d})^{-s/2}(1+v^{2/d})^{-s/2}>\lambda\}}\,du\,dv.$$

The first integral satisfies

$$\int_0^{R_\eta}\int_0^{R_\eta}\mathbb{1}_{\{(1+u^{2/d})^{-s/2}(1+v^{2/d})^{-s/2}>\lambda\}}\,du\,dv \longrightarrow R_\eta^2, \qquad \lambda \to 0. \tag{109}$$

Next, for the mixed region we have

$$\begin{aligned}\int_{R_\eta}^{\infty}\int_0^{R_\eta}\mathbb{1}_{\{(1+u^{2/d})^{-s/2}(1+v^{2/d})^{-s/2}>\lambda\}}\,du\,dv &\leq R_\eta\int_{R_\eta}^{\infty}\mathbb{1}_{\{(1+v^{2/d})^{-s/2}>\lambda\}}\,dv \\ &= O_{\lambda\to 0}\Big(\lambda^{-d/s}\Big).\end{aligned} \tag{110}$$

It remains to analyze the tail region

$$\int_{R_\eta}^{\infty}\int_{R_\eta}^{\infty}\mathbb{1}_{\{(1+u^{2/d})^{-s/2}(1+v^{2/d})^{-s/2}>\lambda\}}\,du\,dv.$$

To this end, we introduce the tail part of the corresponding comparison integral

$$\tilde{I}_\eta(\lambda) := \int_{R_\eta}^{\infty}\int_{R_\eta}^{\infty}\mathbb{1}_{\{(1+u)^{-s/d}(1+v)^{-s/d}>\lambda\}}\,du\,dv.$$

For λ sufficiently small, the admissible region in this tail integral is given by

$$R_\eta \leq v < \frac{\lambda^{-d/s}}{1+R_\eta} - 1, \qquad R_\eta \leq u < \frac{\lambda^{-d/s}}{1+v} - 1,$$

and therefore

$$\tilde{I}_\eta(\lambda) = \int_{R_\eta}^{\frac{\lambda^{-d/s}}{1+R_\eta}-1}\left(\frac{\lambda^{-d/s}}{1+v} - 1 - R_\eta\right)dv = \frac{d}{s}\lambda^{-d/s}\ln(\lambda^{-1}) + O_{\lambda\to 0}\Big(\lambda^{-d/s}\Big),$$

so that

$$\tilde{I}_\eta(\lambda) \sim \frac{d}{s}\lambda^{-d/s}\ln(\lambda^{-1}), \qquad \lambda \to 0. \tag{111}$$

We now use the set inclusions (107) and (108) to get

$$\tilde{I}_\eta\big((1+\eta)^2\lambda\big) \;\leq\; \int_{R_\eta}^{\infty}\int_{R_\eta}^{\infty}\mathbb{1}_{\{(1+u^{2/d})^{-s/2}(1+v^{2/d})^{-s/2}>\lambda\}}\,du\,dv$$

and

$$\int_{R_\eta}^{\infty}\int_{R_\eta}^{\infty}\mathbb{1}_{\{(1+u^{2/d})^{-s/2}(1+v^{2/d})^{-s/2}>\lambda\}}\,du\,dv \;\leq\; \tilde{I}_\eta\big((1-\eta)^2\lambda\big).$$

By (111), this implies

$$(1+\eta)^{-2d/s} \leq \liminf_{\lambda\to 0}\frac{\displaystyle\int_{R_\eta}^{\infty}\int_{R_\eta}^{\infty}\mathbb{1}_{\{(1+u^{2/d})^{-s/2}(1+v^{2/d})^{-s/2}>\lambda\}}\,du\,dv}{\frac{d}{s}\lambda^{-d/s}\ln(\lambda^{-1})} \tag{112}$$

and

$$\limsup_{\lambda\to 0} \frac{\int_{R_\eta}^{\infty}\int_{R_\eta}^{\infty} \mathbb{1}_{\{(1+u^{2/d})^{-s/2}(1+v^{2/d})^{-s/2}>\lambda\}}\, du\, dv}{\frac{d}{s}\lambda^{-d/s}\ln(\lambda^{-1})} \leq (1-\eta)^{-2d/s}. \tag{113}$$

Combining (112)–(113) with (109) and (110), and taking $\eta \in (0,1)$ arbitrarily small, we can conclude that

$$V_{\sigma_{\mathcal{W}}^{-1}}(\lambda) \sim \frac{d\,\omega_d^2}{s(2\pi\sqrt{c})^d}\lambda^{-d/s}\ln(\lambda^{-1}), \qquad \lambda \to 0. \tag{114}$$

We now apply Theorem 1 with $\sigma = \sigma_{\mathcal{W}}^{-1}$ to obtain

$$H_{\sigma_{\mathcal{W}}^{-1}}(\varepsilon) \sim \int_{\varepsilon}^{\infty} \frac{V_{\sigma_{\mathcal{W}}^{-1}}(\lambda)}{\lambda}\, d\lambda, \qquad \varepsilon \to 0. \tag{115}$$

Combining (104), (105), (114), and (115), we arrive at

$$H_{\sigma_{\mathcal{W}}^{-1}}(\varepsilon) \sim \frac{\omega_d^2}{(2\pi\sqrt{c})^d} \begin{cases} \Xi_{r,s,d}\,\varepsilon^{-d/\min\{r,s\}}, & \text{if } r \neq s, \\ \varepsilon^{-d/s}\ln(\varepsilon^{-1}), & \text{if } r = s, \end{cases} \qquad \varepsilon \to 0,$$

where

$$\Xi_{r,s,d} := \frac{\min\{r,s\}\,\Gamma\left(\frac{d|s-r|}{2\min\{r,s\}}\right)\Gamma\left(\frac{d}{2}\right)}{2\,\Gamma\left(\frac{d\max\{r,s\}}{2\min\{r,s\}}\right)}.$$

Finally,

$$H\left(\varepsilon; \mathcal{B}_{\mathcal{W}}^{(s,r)}\right) \sim H_{\sigma_{\mathcal{W}}^{-1}}(\varepsilon), \qquad \varepsilon \to 0,$$

which completes the proof.

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A Review of Spectral Theory for Pseudodifferential Operators

This appendix reviews the spectral-theoretic material required for the development of our results. We begin by recalling the Kohn–Nirenberg representation of a pseudodifferential operator. With the Fourier transform defined by

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(y)\, e^{-2\pi i\, y\cdot\omega}\, dy, \qquad \omega \in \mathbb{R}^d, \tag{116}$$

the operator T_σ associated with a symbol σ admits the integral representation

$$(T_\sigma f)(x) = \int_{\mathbb{R}^{2d}} \sigma(x,\omega)\, e^{2\pi i(x-y)\cdot\omega}\, f(y)\; dy\; d\omega, \qquad x \in \mathbb{R}^d. \tag{117}$$

Following standard practice (see, e.g., [53, Definition 23.1]), we assume throughout that the symbols under consideration satisfy appropriate regularity and growth conditions. Specifically, for $N \in \mathbb{N}^*$ (typically $N = 2d$ or $N = 3d$ below), $m \in \mathbb{R}$, and $\rho \in (0,1]$, we define the symbol class $\Gamma^m_\rho(\mathbb{R}^N)$ as the set of complex-valued functions $\sigma \in C^\infty(\mathbb{R}^N)$ such that, for every multi-index $\alpha \in \mathbb{N}^N$, there exists a constant $C_\alpha > 0$ with

$$\left|\partial^\alpha \sigma(z)\right| \le C_\alpha \left(1 + \|z\|_2^2\right)^{\frac{m-\rho|\alpha|}{2}}, \qquad z \in \mathbb{R}^N. \tag{118}$$

For later use, it is convenient to work with a slightly more general integral representation than (117), allowing the symbol to depend on the integration variable y as well. Specifically, we consider operators of the form

$$(T_{\sigma_a} f)(x) = \int_{\mathbb{R}^{2d}} \sigma_a(x,y,\omega)\, e^{2\pi i(x-y)\cdot\omega}\, f(y)\; dy\; d\omega, \qquad x \in \mathbb{R}^d, \tag{119}$$

where $\sigma_a \in \Gamma^m_\rho(\mathbb{R}^{3d})$ is referred to as the amplitude of the operator. The precise definition of an amplitude varies somewhat across the literature; see, for example, [53, Definition 23.3] and [57, Definition 2.1]. We follow [53, Definition 23.3] and let $\Pi^m_\rho(\mathbb{R}^{3d})$ denote the set of complex-valued functions $\sigma_a \in C^\infty(\mathbb{R}^{3d})$ for which there exists an $m' \in \mathbb{R}$ such that, for all multi-indices $\alpha, \beta, \gamma \in \mathbb{N}^d$, one can find a constant $C_{\alpha,\beta,\gamma} > 0$ with

$$|\partial_\omega^\alpha \partial_x^\beta \partial_y^\gamma \sigma_a(x,y,\omega)| \le C_{\alpha,\beta,\gamma} \big(1 + \|z\|_2^2\big)^{\frac{m-|\alpha+\beta+\gamma|}{2}} \big(1 + \|x-y\|_2^2\big)^{\frac{m'+|\alpha+\beta+\gamma|}{2}},$$

where $z \in (x, y, \omega) \in \mathbb{R}^{3d}$. Operators of the form (119) with amplitude in $\sigma \in \Pi^m_\rho(\mathbb{R}^{3d})$ define the class $\Psi^m_\rho(\mathbb{R}^d)$. Finally, we set

$$\Psi^{-\infty} := \bigcap_{m\in\mathbb{R}} \Psi^m_\rho(\mathbb{R}^d),$$

which can be shown to be independent of the parameter ρ; see the discussion following [53, Definition 23.4].

The amplitude formulation (119) provides a unified framework encompassing several standard quantization schemes. The Kohn–Nirenberg representation (117) is recovered when the amplitude is independent of the variable y, that is, when there exists a symbol $\sigma_L \in \Gamma^m_\rho(\mathbb{R}^{2d})$ such that

$$\sigma_a(x, y, \omega) = \sigma_L(x, \omega).$$

In this case, σ_L is referred to as the Kohn–Nirenberg symbol (or left symbol) of the operator T_{σ_a}. If $\sigma_W \in \Gamma^m_\rho(\mathbb{R}^{2d})$ and

$$\sigma_a(x, y, \omega) = \sigma_W\big(\tfrac{x+y}{2}, \omega\big),$$

then T_{σ_a} coincides with the Weyl quantization of the symbol σ_W, which is therefore called the Weyl symbol of the operator. Similarly, when the amplitude depends only on the second spatial variable,

$$\sigma_a(x, y, \omega) = \sigma_R(y, \omega), \qquad \sigma_R \in \Gamma^m_\rho(\mathbb{R}^{2d}),$$

the function σ_R is referred to as the right symbol of T_{σ_a}. Conversely, every operator $T_{\sigma_a} \in \Psi^m_\rho(\mathbb{R}^d)$ admits equivalent representations in terms of a left symbol, a Weyl symbol, and a right symbol; see [53, Theorem 23.1].

We recall that an operator T is said to preserve the real structure if it sends real-valued functions to real-valued functions, or, equivalently, if it commutes with complex conjugation. Furthermore,

$$\begin{aligned}\overline{(T_{\sigma_a}\bar{f})}(x) &= \int_{\mathbb{R}^{2d}} \overline{\sigma_a(x, y, \omega)}\, e^{-2\pi i(x-y)\cdot\omega}\, f(y)\; dy\; d\omega \\ &= \int_{\mathbb{R}^{2d}} \overline{\sigma_a(x, y, -\omega)}\, e^{2\pi i(x-y)\cdot\omega}\, f(y)\; dy\; d\omega, \qquad x \in \mathbb{R}^d,\end{aligned}$$

where the last step uses the change of variables $\omega \mapsto -\omega$. Therefore, for pseudodifferential operators with real-valued Weyl, right, or left symbol, evenness of this symbol in the frequency variable ω is sufficient for the operator to preserve the real structure.

A central theme in the spectral theory of pseudodifferential operators is that the asymptotic behavior of the eigenvalue-counting function can be characterized in terms of the associated symbol, independently of the chosen quantization (Kohn–Nirenberg, Weyl, or right). We highlight two results of this type. The first, stated as Theorem 7, is a classical Weyl-type result for hypoelliptic pseudodifferential operators and is included here for context. The second, Theorem 8, concerns compact pseudodifferential operators and provides the spectral characterization used in the proof of our main result.

To state Theorem 7, we first introduce the notion of a hypoelliptic symbol; see, for example, [53, Definition 25.1]. A complex-valued function $\sigma \in C^\infty(\mathbb{R}^{2d})$ is said to be hypoelliptic if there are constants $C_1, C_2, , R > 0$, a parameter $\rho \in (0, 1]$, and real numbers $m_+ \geq m_-$ such that, for every multi-index $\alpha \in \mathbb{N}^{2d}$, one can find a constant $C_\alpha > 0$ for which

$$C_1\big(1+\|z\|_2^2\big)^{\frac{m_-}{2}} \leq |\sigma(z)| \leq C_2\big(1+\|z\|_2^2\big)^{\frac{m_+}{2}}, \quad \text{and} \quad \frac{|\partial^\alpha \sigma(z)|}{|\sigma(z)|} \leq C_\alpha\big(1+\|z\|_2^2\big)^{-\frac{\rho|\alpha|}{2}}, \tag{120}$$

for all $z \in \mathbb{R}^{2d}$ with $\|z\|_2 \geq R$. We denote this class of symbols by $\mathrm{H}\Gamma^{m_-,m_+}_\rho(\mathbb{R}^{2d})$. Direct comparison of (120) with the estimate (118) yields that every hypoelliptic symbol is a

symbol in the sense of $\Gamma^{m_+}_\rho(\mathbb{R}^{2d})$, that is, $\mathrm{H}\Gamma^{m_-,m_+}_\rho(\mathbb{R}^{2d}) \subset \Gamma^{m_+}_\rho(\mathbb{R}^{2d})$. A hypoelliptic operator is a pseudodifferential operator whose symbol is hypoelliptic.

Depending on the asymptotic behavior of the symbol at infinity, hypoelliptic symbols exhibit qualitatively different spectral properties. We therefore distinguish between symbols that grow at infinity and symbols that decay at infinity.

Positive-order hypoelliptic symbols. A hypoelliptic symbol σ is said to be of positive order if it belongs to the class

$$\mathrm{H}\Gamma^{+}(\mathbb{R}^{2d}) \;\coloneqq\; \bigcup_{\rho\in(0,1]} \bigcup_{\substack{m_+,m_->0\\ m_+\geq m_-}} \mathrm{H}\Gamma^{m_-,m_+}_\rho(\mathbb{R}^{2d}).$$

That is, σ is of positive order if there exist parameters $\rho \in (0,1]$ and real numbers $m_+ \geq m_- > 0$ such that

$$\sigma \in \mathrm{H}\Gamma^{m_-,m_+}_\rho(\mathbb{R}^{2d}).$$

For symbols in this class, we have

$$|\sigma(z)| \to \infty, \qquad \text{as } \|z\|_2 \to \infty.$$

Negative-order hypoelliptic symbols. A hypoelliptic symbol σ is said to be of negative order if it belongs to the class

$$\mathrm{H}\Gamma^{-}(\mathbb{R}^{2d}) \;\coloneqq\; \bigcup_{\rho\in(0,1]} \bigcup_{\substack{m_+,m_->0\\ m_+\geq m_-}} \mathrm{H}\Gamma^{-m_+,-m_-}_\rho(\mathbb{R}^{2d}). \tag{121}$$

That is, σ is of negative order if there exist parameters $\rho \in (0,1]$ and real numbers $m_+ \geq m_- > 0$ such that

$$\sigma \in \mathrm{H}\Gamma^{-m_+,-m_-}_\rho(\mathbb{R}^{2d}).$$

For symbols in this class, we have

$$|\sigma(z)| \to 0, \qquad \text{as } \|z\|_2 \to \infty.$$

The classical spectral theory of hypoelliptic pseudodifferential operators predominantly addresses the positive-order case. This stands in contrast to the negative-order setting of the present paper, where the symbols of interest decay at infinity, so that the volume integral in (7) is finite. Nevertheless, in order to place our results in context and to recall the classical Weyl-type asymptotics available in the positive-order case, it is convenient to introduce counterparts of the eigenvalue-counting function (1) and of the volume function (7), which we denote by a superscript $\uparrow$. Specifically, for a real-valued Weyl symbol $\sigma_W \in \mathrm{H}\Gamma^{+}(\mathbb{R}^{2d})$, we define the upward eigenvalue-counting function of the pseudodifferential operator T_{σ_W} by

$$M^{\uparrow}_{\sigma_W}(\lambda) \;\coloneqq\; \#\big\{n \in \mathbb{N}^* : \lambda_n(T_{\sigma_W}) < \lambda\big\}, \qquad \lambda > 0,$$

where $\{\lambda_n(T_{\sigma_W})\}_{n\in\mathbb{N}^*}$ denotes the eigenvalues of T_{σ_W}. It follows from [53, Corollary 23.3 and Theorem 26.3] that T_{σ_W} has purely discrete spectrum. The upward volume associated with the same symbol σ_W is defined by

$$V^{\uparrow}_{\sigma_W}(\lambda) \;\coloneqq\; \int_{\mathbb{R}^{2d}} \mathbb{1}_{\{\sigma_W(x,\omega)<\lambda\}}\, dx\, d\omega, \qquad \lambda > 0. \tag{122}$$

With these notions in place, a classical Weyl-type result in the spectral theory of hypoelliptic pseudodifferential operators can be stated as follows.

Theorem 7 ([53, Theorem 30.1]). *Let $\sigma \in \mathrm{H}\Gamma^{+}(\mathbb{R}^{2d})$ be real-valued. Assume that there exist constants $R > 0$ and $C > 0$ such that $\sigma(z) > 0$ for all $z \in \mathbb{R}^{2d}$ with $\|z\|_2 \geq R$ and*

$$|z \cdot \nabla\sigma(z)| \geq C\,|\sigma(z)|, \qquad \text{for all } z \in \mathbb{R}^{2d} \text{ with } \|z\|_2 \geq R. \tag{123}$$

Then the upward eigenvalue-counting function $M_\sigma^{\uparrow}$ of the corresponding Weyl-quantized operator satisfies

$$M_\sigma^{\uparrow}(\lambda) \sim V_\sigma^{\uparrow}(\lambda), \qquad \text{as } \lambda \to \infty.$$

The second result we invoke is due to Dauge and Robert [19]. Their work concerns compact pseudodifferential operators of negative order and establishes a Weyl-type asymptotic formula for the eigenvalue-counting function, including a remainder estimate. Earlier results in this direction are due to Birman and Solomyak [10], who analyzed compact pseudodifferential operators with symbols that are homogeneous or quasi-homogeneous with respect to the phase variable. A typical example of such behavior is given by symbols σ for which there exists $\zeta < 0$ such that

$$\sigma(x, t\omega) = t^{\zeta}\sigma(x, \omega),$$

for all $t \geq 1$, $x \in \mathbb{R}^d$, and $\omega \in \mathbb{R}^d$ with $\|\omega\|_2$ sufficiently large. Dauge and Robert relax the requirement of exact homogeneity and work under more general decay assumptions at infinity, thereby extending Weyl-type spectral asymptotics to a broader class of compact pseudodifferential operators. This flexibility makes their result applicable to the negative-order symbol regime considered in the present paper.

Dauge–Robert formulate their results in terms of general weight functions $\phi, \varphi \in C^{\infty}(\mathbb{R}^{2d})$ taking values in $\mathbb{R}_+^*$; see [19, Definition 1.1]. In this framework, a function $w \in C^{\infty}(\mathbb{R}^{2d})$ taking values in $\mathbb{R}_+^*$ is called (ϕ, φ)-continuous if there exist constants $c, C > 0$ such that

$$C^{-1} \leq \frac{w(x + x', \omega + \omega')}{w(x, \omega)} \leq C$$

for all $x, x', \omega, \omega' \in \mathbb{R}^d$ satisfying

$$\frac{\|x'\|_2}{\varphi(x, \omega)} + \frac{\|\omega'\|_2}{\phi(x, \omega)} \leq c. \tag{124}$$

The weight w is said to be (ϕ, φ)-temperate if it is (ϕ, φ)-continuous and there exist constants $C, \zeta > 0$ so that

$$w(x + x', \omega + \omega') \leq C\,w(x, \omega)\big[1 + \|x'\|_2\,\phi(x, \omega) + \|\omega'\|_2\,\varphi(x, \omega)\big]^{\zeta}, \tag{125}$$

for all $x, x', \omega, \omega' \in \mathbb{R}^d$.

Further, given a (ϕ, φ)-temperate weight w, Dauge–Robert define the symbol class $\Gamma(\mathbb{R}^{2d}; w, \phi, \varphi)$ as the set of complex-valued functions $\sigma \in C^{\infty}(\mathbb{R}^{2d})$ such that, for every pair of multi-indices $\alpha, \beta \in \mathbb{N}^d$, one has

$$\left|\partial_x^{\alpha}\,\partial_{\omega}^{\beta}\,\sigma(x, \omega)\right| \leq C_{\alpha,\beta}\frac{w(x, \omega)}{\varphi(x, \omega)^{|\alpha|}\,\phi(x, \omega)^{|\beta|}}, \qquad x, \omega \in \mathbb{R}^d, \tag{126}$$

with a constant $C_{\alpha,\beta} > 0$.

In order to control the spectral asymptotics of compact pseudodifferential operators with symbol decay, Dauge–Robert impose a set of structural conditions on the weight functions w, ϕ, φ governing localization and scaling in phase space. For ease of reference, we reproduce these assumptions below, using the same labels as in [19].

(H1) The functions ϕ^{-1} and φ^{-1} are (ϕ, φ)-continuous, $(1, 1)$-temperate, and bounded on $\mathbb{R}^{2d}$.

(H2) There exist constants $C, \zeta > 0$ such that

$$C\,(1 + \|x\|_2 + \|\omega\|_2)^{\zeta} \le \phi(x,\omega)\,\varphi(x,\omega), \qquad x,\omega \in \mathbb{R}^d. \tag{127}$$

(W) The weight w belongs to the symbol class $\Gamma(\mathbb{R}^{2d}; w, \phi, \varphi)$ and is (ϕ,φ)-temperate.

(N) There exist constants $K, K', \gamma, \gamma' > 0$ such that

$$K'\,w(x,\omega)^{\gamma'} \le \phi^{-1}(x,\omega)\,\varphi^{-1}(x,\omega) \le K\,w(x,\omega)^{\gamma}, \qquad x,\omega \in \mathbb{R}^d. \tag{128}$$

Under assumptions *(H1)–(N)*, Dauge–Robert establish the following Weyl-type asymptotic formula for the eigenvalue-counting function of compact pseudodifferential operators whose symbol σ belongs to $\Gamma(\mathbb{R}^{2d}; w, \phi, \varphi)$.

Theorem 8 ([19, Theorem 1.3])**.** *Let $w, \phi, \varphi \in C^\infty(\mathbb{R}^{2d})$ be weight functions taking values in $\mathbb{R}_+^*$ and satisfying (H1), (H2), (W), and (N). Let $\sigma \in \Gamma(\mathbb{R}^{2d}; w, \phi, \varphi)$ be real-valued non-negative and assume that the maps $\lambda \mapsto V_\sigma(\lambda)$ and $\lambda \mapsto V_w(\lambda)$ are regularly varying at zero of negative index with $V_w(\lambda) = O_{\lambda\to 0}(V_\sigma(\lambda))$. Then, $M_{|T_\sigma|}(\lambda) \sim M_{T_\sigma}(\lambda) \sim V_\sigma(\lambda)$, as $\lambda \to 0$, where T_σ is the Weyl quantization of σ.*

We remark that both Theorems 7 and 8 are stated here in slightly simplified forms relative to their original counterparts in [53, Theorem 30.1] and [19, Theorem 1.3], respectively. More precisely, [53, Theorem 30.1] replaces the condition (123) by the weaker requirement

$$|z \cdot \nabla\sigma(z)| \ge C\,|\sigma(z)|^{1-\delta}, \qquad \text{for all } z \in \mathbb{R}^{2d} \text{ with } \|z\|_2 \ge R, \tag{129}$$

for some $\delta \in [0, \min\{\rho', 1\})$, where $\rho' > 0$ is a parameter depending on the symbol σ (with the universal lower bound $\rho' \ge \rho/m_+$). Under this assumption, Shubin establishes the refined asymptotic expansion

$$M_\sigma^{\uparrow}(\lambda) = V_\sigma^{\uparrow}(\lambda)\Big(1 + O_{\lambda\to\infty}\big(\lambda^{\delta-\rho'+\eta}\big)\Big), \qquad \text{for every } \eta > 0.$$

Similarly, [19, Theorem 1.3] provides a remainder estimate of the form

$$M_\sigma(\lambda) = V_\sigma(\lambda) + O_{\lambda\to 0}\Big(\lambda^{\zeta}\,V_w(\lambda)\Big), \qquad \text{for some } \zeta > 0,$$

under their original set of assumptions. In the present paper, only first-order asymptotics are required, and the finer remainder estimates available in the original results are not needed. For this reason, the simplified formulations given in Theorems 7 and 8 are sufficient for our purposes.

Additionally, we note that the regular-variation assumption imposed in Theorem 8 replaces the original condition (T) in [19]. This is made possible by the smooth variation theorem [9, Theorem 1.8.2], which guarantees that V_σ admits smooth regularly varying lower and upper equivalents $f_- \le V_\sigma \le f_+$ such that $f_\pm(\lambda) \sim V_\sigma(\lambda)$ as $\lambda \to 0$. In particular, $f_\pm$ satisfy (T) for sufficiently small arguments by [9, Chapter 1.8]. In the proof of [19, Theorem 1.3], one needs to replace V_σ by $f_\pm$ and apply the corresponding estimates to these smooth functions only at those points where (T) and its integrated form (T') are invoked; one keeps V_σ in the geometric identities. Since $f_\pm(\lambda) \sim V_\sigma(\lambda)$ as $\lambda \to 0$, the resulting replacement introduces only $o(V_\sigma(\lambda))$ errors, which can be safely ignored given that our statement considers only first-order asymptotics. A similar argument holds for V_w. This modification is introduced here in order to enable the subsequent connection between eigenvalue-counting functions and metric entropy, and therefore does not entail any loss of generality in the context of our results.

Let us also record here that the regular variation condition is independent of a hypoellipticity condition of the type $\sigma \in \mathrm{H}\Gamma^-(\mathbb{R}^{2d})$. Morally, regular variation of the phase-space volume controls the global decay of the symbol, while hypoellipticity—especially

the second part of (120)—controls its local regularity. Let us formalize this idea with two examples. First, for fixed $m > 0$, the fast oscillations of the symbol

$$\sigma(z) = \frac{1 + \|z\|_2^2 + \sin(\|z\|_2^2)}{(1 + \|z\|_2^2)^{\frac{m+2}{2}}}, \qquad z \in \mathbb{R}^{2d},$$

prevent it from satisfying the right part of (120), while $V_\sigma(\lambda)$ is asymptotically equivalent to $\omega_{2d}\lambda^{-2d/m}$ and thus regularly varying. Conversely, the hypoelliptic symbol

$$\sigma(z) = \frac{e^{a\sin(\log(1+\|z\|_2^2))}}{(1 + \|z\|_2^2)^{\frac{m}{2}}}, \qquad z \in \mathbb{R}^{2d} \quad \text{and} \quad a \in (0, m/2),$$

has slow oscillations preventing its volume function from being regularly varying.

Finally, the non-negativity assumption on the symbol σ can in principle be removed. In [19], this is achieved by separating the asymptotic analysis of the positive and negative parts of the spectrum of T_σ. When σ is strictly positive and of negative order, the negative spectrum does not contribute to the leading asymptotics, and $M_{|T_\sigma|}(\lambda) \sim M_{T_\sigma}(\lambda)$. Since our analysis focuses exclusively on the leading-order behavior, the formulation of Theorem 8 for non-negative symbols is compatible with the assumptions of Theorem 1.