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Capacity scaling in large wireless networks

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Abstract

We consider networks consisting of nodes with radios, and without any wired infrastructure, thus necessitating all communication to take place only over shared wireless medium. We focus mainly on the flat fading channel model. We study capacity scaling laws in such systems when the number of nodes gets large. We show that in extended networks (network with constant density of users) capacity scales sub-linearly in number of nodes. Thus, when number of nodes gets large, performance of such system inevitably goes to zero. We show also that in the case of condensed networks (all the nodes are situated in some bounded area) capacity can scale super-linearly in the number of nodes. Finally, we present and analyze particular architecture which achieves capacity bound in the case of condensed network.

Contents

Notations

The superscripts ^{T}, † and * stand for transposition, conjugate transpose, and element-wise conjugation, respectively. $\mathcal{E}$ denotes the expectation operator. $\text{Tr}(\mathbf{A})$ and $|\mathbf{A}|$ stand for the trace and determinant of the matrix $\mathbf{A}$, respectively. $\mathbf{I}_m$ stands for the $m \times m$ identity matrix. $|\mathcal{X}|$ denotes the cardinality of the set $\mathcal{X}$. The notation $u(x) = O(v(x))$ denotes that $|u(x)/v(x)|$ remains bounded as $x \to \infty$. A circularly symmetric complex Gaussian random variable is a random variable $Z = X + j\,Y \sim \mathcal{CN}(0, \sigma^2)$, where X and Y are i.i.d. $\mathcal{N}(0, \sigma^2/2)$. $\text{VAR}(X)$ denotes the variance of the random variable X. Throughout the paper all logarithms unless specified otherwise are to the base 2. $\boldsymbol{A} \circ \boldsymbol{B}$ stands for the Hadamard product of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. $\sqrt{\boldsymbol{A}}$ denotes the matrix with components $\sqrt{A_{i,j}}$. The diagonal matrix with the components of the vector $\boldsymbol{x}$ is denoted by $diag(\boldsymbol{x})$.

Chapter 1

Introduction

1.1 Overview

Suppose Alice and Bob are communicating over some channel. In information theory, channel is usually characterized by conditional probability distribution of the symbols on the output of the channel, given a symbol on the input. Of course, real channels are not perfect. This means, for example, that two different symbols on input can result in a single symbol on the output of the channel because of the noise or any other causes. That is why if Alice wants to transmit her information reliably, she needs to use coding. It does not matter what type of coding is used, the idea remains the same. To send l bits of information to Bob, Alice should transmit greater number of bits n ($n > l$) over the channel. Additional bits can help Bob to recover the initial message. It depends on a particular coding scheme chosen, how Alice constructs codewords of length n given a message of length l. The ratio $R = l/n$ is usually called the rate of the code. This value is of a great interest, it characterizes the performance of the particular code. The probability that Bob has decoded a word that does not coincides with Alice's message is called a probability of error. Information theorist and engineers are trying to maximaze the rate, keeping the probability of error very low. Famous coding theorem (see [1]) gives a theoretical bound for rate. This bound C is called capacity. Theorem states that if Alice uses some code with $R > C$ then the probability of error is inevitably very hight. Reliable communication is impossible on this rate for any code. On the other hand, if Alice chooses the rate R which is less then C, then there exists a code which gives infinitely small probability of error. This means that reliable communication is principally possible. Unfortunatelly, coding theorem does not describe how to construct such a good code. Capacity is one of the basic properties of any communication system. It quantifies the quality of the channel between sender and receiver.

Consider now several senders and several receivers communicating over wireless environment with radios. If all senders and all receivers are transmitting at the same time in the same frequency band, then they are sharing the same common media, the same channel. Studying channels with many inputs and many outputs is the subject of network information theory [2]. Such channels are described by probability distribution of vectors of outputs, given a vector of inputs. Each user of the network can principally choose his own rate independently. Thus, network is fully characterized by the so-called capacity region, i.e. a set of simultaneously achievable rates. Finding capacity region of the network is very difficult task in most cases. That is why people often study sum capacity, i.e. the maximum of the sum of achievable rates. Rather then give complete information about properties of the channel, sum capacity provides some integral measure. In many papers the word capacity is used to denote sum capacity. We will also do so for brevity in the rest of this report.

To compute capacity of the system, one should first model its conditional probability function. An AWGN (additive channel with Gaussian noise) channel model is widely used. In this model channel is characterized by the channel matrix and the noise. The signal vector on input $\boldsymbol{x}$ is first multiplied by channel matrix $\boldsymbol{G}$ and then noise vector $\boldsymbol{n}$ is added (see Chapter 1 for details):

$$\boldsymbol{y} = \boldsymbol{G}\boldsymbol{x} + \boldsymbol{n}.$$

This model is linear and can be analyzed mathematically more effectively then other possible models. It is also very natural from the engineering point of view.

The state of the channel, captured by matrix $\boldsymbol{G}$, can change in time. The channels which change their state during communication period are called time-varying or fading. The theory of fading channels have been a subject of intensive research for many years (see [3] for review) and has many practical applications.

Throughout this work, we focus on fading wireless networks where M_S source terminals are transmitting data to M_D destinations terminals. We are particularly interested in the capacity scaling laws of such networks, i.e. how capacity of the system grows when M_S and M_D go to infinity. It is clear, that in order to allow senders to transmit their data simultaneously on the same rate, capacity should scale at least linearly with M_D. Otherwise achievable rate per sender will go to zero, when $M_T, M_D \to \infty$. Our main goal is to identify situations when desirable capacity grows can principally be achieved. Once such situations are identified, we will analyze particular architecture which reaches this bound.

1.2 Previous results

In his pioneering work [4], Telatar has derived expressions for capacity of MIMO system (wireless communication system with multiple transmit and receive antennas). He showed that with fixed transmitting power, capacity of the system scales linearly with number of antennas used. Transmitters and receivers are assumed to be spatially grouped together, i.e. system can be considered as a condensed network. To achieve the capacity joint coding and decoding is needed. The results are achieved for both fading and non-fading case.

For non-fading extended wireless networks first Gupta and Kumar [5] and then Leveque and Telatar [6] under less restrictive assumptions showed that capacity can not scale linearly in the number of users. This sets very strong limitation for possible architectures of large wireless extended networks.

Bölcskei, Nabar, Oyman and Paulraj [7] consider a dense network with fading and suggest a particular architecture which achieves the theoretical capacity bound. They assume that the number of transmit terminals is equal to the number of receive terminals ($M = M_S = M_D$). The idea of their approach is to transmit data through K additional relay terminals, situated between sources and destinations. Communication takes place over two time slots using a one-hop relaying ("listen-and-transmit") protocol. During the first time slot M sources simultaneously transmit their data to relays. Then the received data is processed by relays and retransmitted to destination during the second time slot. It is shown in [7] that for fixed M this architecture achieves capacity $C = (M/2)\log(K) + O(1)$ when K is large enough. It is very important that this result is achived without any cooperation in the system.

1.3 Contribution

The detailed contribution reported in this work can be summarized as follows:

- We explore extended wireless network with fading and prove (under certain assumptions) that capacity of such network scales sub-linearly with the number of users. This shows that results by Leveque and Telatar [6] can be applied for fading networks.

- For condensed networks, we investigate the connection between structure of channel matrix and an upper bound on the capacity of the system. We use results from the theory of large random matrices to see how fraction of active users and network connectivity result capacity bound. We calculate exact values of the capacity in two special cases.

- In paper [7] it is assumed that the number of assisting relays K goes to infinity, whereas the number of users M remains constant. In this work we give the analysis of the same protocol for large M limit. We study relationships between the number of users M and the number of relays K which are needed to assist the transmission. We prove that if number of relays grows faster than M^5 then capacity $C = (M/2)\log(K) + O(1)$ is achieved. The question of what happens when the number of relays grows slower remains open as a topic for future research.

1.4 Organization of the report

The rest of this report is organized as follows. Chapter II describes the channel and signal model. In Chapter III, we investigate capacity upper bounds for condensed and extended networks and prove corresponding scaling laws. Chapter IV presents possible network architecture achieving the capacity bound and gives the analysis in case of large network size. We conclude in chapter V.

Chapter 2

Channel and signal model

We consider a wireless network consisting of M_S sending and M_D destination terminals with single antenna each. The sets of all sending and destinations terminals are denoted by $\mathcal{S}$ and $\mathcal{D}$ respectively. Throughout this report, frequency-flat fading over the bandwidth of interest is assumed, channel is memoryless. We will exclusively deal with a linear model in which the received vector $\boldsymbol{y} \in \mathbb{C}^{M_D}$ depends on the transmitted vector $\boldsymbol{x} \in \mathbb{C}^{M_S}$ via

$$\boldsymbol{y} = \boldsymbol{G}\boldsymbol{x} + \boldsymbol{n}$$

where $\boldsymbol{G} = (g_{ij})$ is the $M_D \times M_S$ corresponding channel matrix and $\boldsymbol{n}$ is $M_D \times 1$ circularly symmetric complex white Gaussian noise vector at the receiver with covariance matrix $\mathcal{E}\{\boldsymbol{n}\boldsymbol{n}^H\} = \sigma^2 \boldsymbol{I}_R$. We assume an ergodic block fading channel model with path-loss such that entries of channel matrix are given by $g_{ij} = h_{ij} * \sqrt{E_{ij}}$. $\boldsymbol{H} = (h_{ij})$ is $M_D \times M_S$ matrix consisting of i.i.d. $\mathcal{CN}(0, 1)$ entries. It describes fading in the channel, remains constant over the entire duration of a time slot and changes in an independent fashion across time slots. $M_D \times M_S$ matrix $\boldsymbol{E} = (E_{ij})$ captures path-loss and shadowing in the channel. We assume that parameters E_{ij} ($i = 1, 2, \ldots, M_S$, $j = 1, 2, \ldots, M_D$) are real, positive, and remain constant over the entire time period of interest. The exact values will in general depend on the topology of the network.

Finally, we assume that each transmit terminal is constrained in its total power P, and uses Gaussian codebooks, that is $\boldsymbol{x}$ is zero-mean $M_S \times 1$ circularly symmetric complex Gaussian vector satisfying $\mathcal{E}\{\boldsymbol{x}\boldsymbol{x}^\dagger\} = P\boldsymbol{I}_{M_S}$.

Chapter 3

Upper bounds

In this chapter, we first apply the well known cut-set bound to the described network and then study the behavior of the derived bound using several methods for several special cases. This will help us to achieve some intuition on the capacity scaling laws in the general case and give insight on the factors increasing the capacity.

3.1 Cut-set bound

The cut-set lemma is known to be very general theoretical upper bound on the network capacity. It has been proved in somewhat different ways for slightly different setups in the literature ([2], [4]). We will use the following form of this lemma.

Lemma 1. *The sum of achievable rates of the wireless network modeled with fading satisfies*

$$C = \sum_{i\in\mathcal{S},j\in\mathcal{D}} R_{ij} \leq C_{upper} = \mathop{\mathcal{E}}_{\boldsymbol{H}} \left\{ \log \left| \boldsymbol{I}_{M_D} + \rho \boldsymbol{G}\boldsymbol{G}^{\dagger} \right| \right\}, \tag{3.1}$$

where $\rho = P/\sigma^2$.

Proof. Making a cut between source and destination terminals, using that receiver knows channel state information, it follows that the capacity of the network is upper bounded by mutual information

$$\sum_{i\in\mathcal{S},j\in\mathcal{D}} R_{ij} \leq I(\boldsymbol{x}; (\boldsymbol{y}, \boldsymbol{H})).$$

First using that $\boldsymbol{x}$ is independent of $\boldsymbol{H}$ and then writing mutual information explicitly we conclude

$$I(\boldsymbol{x};(\boldsymbol{y},\boldsymbol{H})) = I(\boldsymbol{x};\boldsymbol{H}) + I(\boldsymbol{x};\boldsymbol{y}|\boldsymbol{H}) = I(\boldsymbol{x};\boldsymbol{y}|\boldsymbol{H}) =$$
$$= \underset{\boldsymbol{H}}{\mathcal{E}}\left\{\log\left|\boldsymbol{I}_{M_D} + \rho\left(\boldsymbol{H}\circ\sqrt{\boldsymbol{E}}\right)\left(\boldsymbol{H}\circ\sqrt{\boldsymbol{E}}\right)^{\dagger}\right|\right\}.$$

Substituting $\boldsymbol{G} = \boldsymbol{H}\circ\sqrt{\boldsymbol{E}}$ we conclude the proof. □

It is often very helpful to write down the cut-set bound 3.1 in terms of singular values of the channel matrix. This is done as follows. Let $\lambda_i(i = 1, 2, \ldots, M_D)$ be eigenvalues of matrix $\boldsymbol{G}\boldsymbol{G}^{\dagger}$. Let $\boldsymbol{\Lambda} = diag(\lambda_1 \ldots \lambda_{M_D})$. Now we can use eigenvalue decomposition of matrix $\boldsymbol{G}\boldsymbol{G}^{\dagger}$:

$$\log|I_{M_D} + \rho\boldsymbol{G}\boldsymbol{G}^{\dagger}| = \log|\boldsymbol{U}^{-1}\boldsymbol{U} + \rho\boldsymbol{U}^{-1}\boldsymbol{\Lambda}\boldsymbol{U}| = \log|\boldsymbol{U}^{-1}(\boldsymbol{I}_{M_D} + \rho\boldsymbol{\Lambda})\boldsymbol{U}| =$$
$$= \log|\boldsymbol{I}_{M_D} + \rho\boldsymbol{\Lambda}| = \log\prod_{i=1}^{M_D}(1+\rho\lambda_i) = \sum_{i=1}^{M_D}\log(1+\rho\lambda_i).$$

Substituting this expressions into the cut-set bound (3.1) we receive

$$C_{upper} = \underset{\boldsymbol{H}}{\mathcal{E}}\left\{\sum_{i=1}^{n}\log(1+\rho\lambda_i)\right\}. \tag{3.2}$$

3.2 Extended fading networks

In this section, we apply the general bound (3.1) to extended network, that is network with constant density of users. Generalizing results reported in [6] by Leveque and Telatar to fading environment we prove that rate per communication pair goes to zero when number of users gets large.

3.2.1 Uniformly distributed networks

First, we make additional assumptions on the geometry of the network under consideration. For brevity assume that the number of senders is equal to number of receivers. Set $M = M_S = M_D$. Assume that senders are uniformly distributed in d-dimensional region $\Omega_M^S = [-M^{1/d}, 0] \times [0, M^{1/d}]^{d-1}$ and receivers are in the symmetric region $\Omega_M^D = [0, M^{1/d}] \times [0, M^{1/d}]^{d-1}$. The locations of users and source-destination pairs are chosen once before communication session begins and remain constant forever. This meets the fact that energy path-loss matrix $\boldsymbol{E}$ is fixed, which was assumed in the previous chapter.

For each $i \in \mathcal{S}, | \in \mathcal{D}$ let r_{ij} denote distance between i-th sender and j-th receiver. To connect geometrical properties of the network with the model introduced earlier, we model channel attenuation as follows

$$E_{i,j} = \frac{e^{-\beta r_{i,j}}}{r_{i,j}^{\alpha}}, \tag{3.3}$$

for all $i \in \mathcal{S}$ and $j \in \mathcal{D}$. Parameter $\beta \geq 0$ is called an absorption constant (usually positive, unless over a vacuum), and $\alpha > 0$ is the path loss exponent. This model is physically motivated and widely used.

In the next subsection we use expression (3.1) to derive weaker but simpler bound. We then apply it to prove the scaling law for extended wireless network with fading.

3.2.2 Bound Based on Hadamard and Jensen Inequalities

Starting from formula (3.1) and applying first Hadamard and then Jensen's we receive

$$C_{upper} = \underset{\boldsymbol{H}}{\mathcal{E}} \left\{ \log \left| I_M + \rho \boldsymbol{G}\boldsymbol{G}^{\dagger} \right| \right\} \leq \underset{\boldsymbol{H}}{\mathcal{E}} \left\{ \log \prod_{i=1}^{M} \left(1 + \rho (\boldsymbol{G}\boldsymbol{G}^{\dagger})_{ii} \right) \right\} \leq$$
$$\leq \sum_{i=1}^{M} \log \left(1 + \rho \underset{\boldsymbol{H}}{\mathcal{E}} \left\{ (\boldsymbol{G}\boldsymbol{G}^{\dagger})_{ii} \right\} \right).$$

Note that for each $0 \leq i \leq M$

$$\underset{\boldsymbol{H}}{\mathcal{E}} \left\{ (\boldsymbol{G}\boldsymbol{G}^{\dagger})_{ii} \right\} = \underset{\boldsymbol{H}}{\mathcal{E}} \left\{ \sum_{k=1}^{M} (\boldsymbol{G})_{ik} (\boldsymbol{G}^{\dagger})_{ki} \right\} = \underset{\boldsymbol{H}}{\mathcal{E}} \left\{ \sum_{k=1}^{M} g_{ik} g_{ik}^{*} \right\} =$$
$$= \sum_{k=1}^{M} \underset{\boldsymbol{H}}{\mathcal{E}} \left\{ \left| h_{ki} \sqrt{E_{ki}} \right|^2 \right\} = \sum_{k=1}^{M} E_{ki}.$$

Finally

$$C_{upper} \leq \sum_{i=1}^{M} \log \left(1 + \rho \sum_{k=1}^{M} E_{ki} \right). \tag{3.4}$$

3.2.3 Capacity Scaling in Extended Networks

In this section, we apply bound (3.4) to prove that capacity of the extended wireless network of nodes uniformly distributed in the region as described earlier, almost surely (in respect to spacial distribution) scales sub-linearly in M. We will transform bound (3.4) in such a way that we can apply results proved by Leveque and Telatar [6] in the case of networks without fading.

Let us first formulate the results that we are going to use. The facts proved in [6] can be summarized in the following theorem.

Theorem 1. *Let x_k ($k = 1, 2, \ldots, n$) be uniformly distributed variables in the interval $[0, M^{1/d}]$. Let $g(r) = \frac{e^{-\beta r/2}}{r^{\alpha/2}}$. Let $K > 0$. Then if $\beta = 0$ and $\alpha > max(d, 2(d-2))$*

$$\frac{1}{n}\sum_{k=1}^{n} \log\left(1 + K\sqrt{n}g(x_k)\right) \to 0, n \to \infty; \tag{3.5}$$

if $\beta > 0$

$$\frac{1}{n}\sum_{k=1}^{n} \log\left(1 + Kn^2 g(x_k)^2\right) \to 0, n \to \infty. \tag{3.6}$$

Convergence in both formulas is almost surely with respect to choice of x_k ($k = 1, 2, \ldots, n$).

Using notation $\boldsymbol{x}_i = (x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(d)})$ for location of the sender number i ($i = (1, 2, \ldots, M)$), substituting (3.3) into (3.4) we deduce:

$$\begin{aligned} C_{upper} &\le \sum_{i=1}^{M} \log\left(1 + \rho\sum_{k=1}^{M} E_{ki}\right) = \sum_{i=1}^{M} \log\left(1 + \rho\sum_{k=1}^{M} g(r_{ki})^2\right) \le \\ &\le \sum_{i=1}^{M} \log\left(1 + \rho M g(x_i^{(1)})^2\right). \end{aligned} \tag{3.7}$$

Note that $x_i^{(1)}$ is uniformly distributed in the interval $[0, M^{1/d}]$. When $\beta = 0$ we write

$$\sum_{i=1}^{M} \log\left(1 + \rho M g(|x_i^{(1)}|)^2\right) \le \sum_{i=1}^{M} \log\left(1 + \rho M^2 g(|x_i^{(1)}|)^2\right)$$

and apply first part of the theorem; when $\beta > 0$ we write

$$\sum_{i=1}^{M} \log\left(1 + \rho M g(|x_i^{(1)}|)^2\right) \le 2\sum_{i=1}^{M} \log\left(1 + \sqrt{\rho M} g(|x_i^{(1)}|)\right)$$

and apply second part. Finally we conclude that in the case of fading capacity of extended network scales sub-linearly with the number of users.

3.2.4 Conclusion

Now we see that in the case of extended network sum capacity can not scale linearly with the number of nodes. This implyes that when the size of the network gets large, performance per each user goes to zero. Thus, some additional techniques should be used to build such networks. One of the possible solutions is to use wired backbone network as it is done in GSM. In the rest of this work we focus on condensed networks. In such networks at least some constant fraction of users stays in the bounded area when the number of users gets large.

3.3 Condensed networks

It is well known that capacity upper bound given by cut-set theorem can be achieved under certain constraints. For example, it is achieved in MIMO systems when source and destination terminals can fully cooperate (perform joint coding and decoding). Sometimes it can also be achieved without cooperation among senders and receivers. This topic is discussed in the last chapter of this report in more details. This section is devoted to studying the scaling behavior of the cut-set bound in condensed networks.

3.3.1 Cut-set bound in integral form

In this subsection, we will transform the upper bound (3.2) into the integral form in order to apply results from random matrix theory for further analysis. First we give two important definitions.

Definition 1. *Let A be $n \times n$ Hermitian matrix. Then $F_A(x)$ denotes empirical eigenvalue distribution of matrix A:*

$$F_A(x) = \frac{|\{\lambda : \lambda \; is \; eigenvalue \; of \; A, \lambda < x\}|}{n}.$$

Definition 2. *Function $p_A(x) = dF_A(x)/dx$ is called density of empirical eigenvalue distribution of matrix A.*

Now using formula (3.2) we can write

$$C_{upper} = \underset{\boldsymbol{H}}{\mathcal{E}} \left\{ \sum_{i=1}^{M_D} \log(1 + \rho\lambda_i) \right\} = M_D \underset{\boldsymbol{H}}{\mathcal{E}} \left\{ \frac{1}{M_D} \sum_{i=1}^{M_D} \log(1 + \rho\lambda_i) \right\} =$$

$$M_D \underset{\boldsymbol{H}}{\mathcal{E}} \left\{ \int p_{\boldsymbol{GG}^\dagger}(\lambda) \log(1 + \rho\lambda)\, d\lambda \right\}.$$

It is convenient to normalize matrix $\boldsymbol{G}$ by factor $\sqrt{M_D}$. Let $\dot{\boldsymbol{G}} = \boldsymbol{G}/\sqrt{M_D}$. Let $\dot{\lambda}_i$ $(i = 1, 2, \ldots, M_D)$ denote the eigenvalues of $\dot{\boldsymbol{G}}\dot{\boldsymbol{G}}^\dagger$. Then

$$\begin{gathered}\dot{\lambda}_i = \lambda_i / M_D, \\ dF_{\boldsymbol{G}\boldsymbol{G}^\dagger}(\lambda) = dF_{\dot{\boldsymbol{G}}\dot{\boldsymbol{G}}^\dagger}(\lambda/M_D), \\ p_{\boldsymbol{G}\boldsymbol{G}^\dagger}(\lambda)d\lambda = p_{\dot{\boldsymbol{G}}\dot{\boldsymbol{G}}^\dagger}(\lambda/M_D)d(\lambda/M_D),\end{gathered}$$

and changing the variable of integration we conclude

$$\begin{aligned}\frac{C_{upper}}{M_D} &= \underset{\boldsymbol{H}}{\mathcal{E}}\left\{\int p_{\boldsymbol{G}\boldsymbol{G}^\dagger}(\lambda)\log(1+\rho\lambda)d\lambda\right\} = \\ &= \underset{\boldsymbol{H}}{\mathcal{E}}\left\{\int p_{\dot{\boldsymbol{G}}\dot{\boldsymbol{G}}^\dagger}(\lambda)\log(1+M_D\rho\lambda)d\lambda\right\}.\end{aligned} \tag{3.8}$$

3.3.2 Girko's law

The main advantage of studying large networks is that we can apply convergence results from the theory of large random matrices. These results are somewhat analogous to the law of large numbers in the classical probability theory. The basic idea is that in many cases when the size of the random matrix gets large, the density function of its eigenvalue distribution (which generally is random function, depending on random matrix) converges almost surely to some non-random limit. One of the relevant results of this type is due to Girko [8]. It is presented below.

Theorem 2 (Girko's law). *Let the $n \times n$ matrix $\boldsymbol{G}$ be composed of independent entries g_{ij} with zero-mean and variances w_{ij}/n such that all w_{ij} are uniformly bounded from above. Assume that the empirical joint distribution of variances $w : [0,1] \times [0,1] \to \mathbb{R}$ defined by $w_n(x,y) = w_{ij}$ for i, j satisfying*

$$\frac{i}{n} \leq x \leq \frac{i+1}{n}, \frac{j}{n} \leq y \leq \frac{j+1}{n}$$

converges to a bounded joint limit distribution $w(x,y)$ as $n \to \infty$. Then, almost surely, the empirical eigenvalue distribution of $\boldsymbol{G}\boldsymbol{G}^\dagger$ converges weakly to a limiting distribution whose Stieltjets transform is given by

$$G_{\boldsymbol{G}\boldsymbol{G}^\dagger}(s) = \int_0^1 u(x,s)dx \tag{3.9}$$

where $u(x,s)$ is a solution of the fixed point equation

$$u(x,s) = \left[-s + \int_0^\beta \frac{w(x,y)dy}{1 + \int_0^1 u(x',s)w(x',y)dx'}\right]^{-1}. \tag{3.10}$$

The solution to (3.10) exists and is unique in the class of functions $u(x, s)$*,* $Im(u(x, s)) \geq 0$*, analytic for* $Im(s) > 0$ *and continuous on* $x \in [0, 1]$*.*

Thus, the limiting probability density of eigenvalues of $\boldsymbol{G}\boldsymbol{G}^\dagger$ can be computed by taking the following limit (Stietljes inversion formula):

$$p_{\boldsymbol{G}\boldsymbol{G}^\dagger}(x) = \lim_{y \to 0+} \frac{1}{\pi} Im[G(x + iy)]. \tag{3.11}$$

3.3.3 Capacity Scaling Laws

For simplicity assume again that $M = M_S = M_D$. Note that in our notations normalized channel matrix $\dot{\boldsymbol{G}}$ has independent zero-mean entries $\dot{g}_{ij}$ with variances E_{ij}/M. From this point on, we assume that the empirical joint distribution of variances $E_M(x, y)$ (see definition in the Girko's theorem) converges to a bounded joint limit distribution $E(x, y)$ as $M \to \infty$.

Discussion Let us look briefly at the case of extended networks once again. Suppose that $d = 1$. This means that senders are uniformly distributed in the domain $[-M, 0]$ and receivers are in the domain $[0, M]$. Fix any positive distance $r > 0$. Let $\mathcal{S}_r$ and $\mathcal{D}_r$ denote the sets of those source (destination) terminals which are situated in the regions $[-r, 0]$ and $[0, r]$ respectively. Let $\mathcal{S}_{M-r} = \mathcal{S} \setminus \mathcal{S}_r$ and $\mathcal{D}_{M-r} = \mathcal{D} \setminus \mathcal{D}_r$. If $i \in \mathcal{S}_{M-r}$ or $j \in \mathcal{S}_{N-r}$, then $r_{ij} > r$ and by formula (3.3)

$$E_{i,j} < \frac{e^{-\beta r}}{r^\alpha}.$$

Thus we can see, that

$$\left| \left\{ x : E_M(x, y) < \frac{e^{-\beta r}}{r^\alpha} \; \forall y \in [0, 1] \right\} \right| \approx \frac{M - r}{M},$$

$$\left| \left\{ y : E_M(x, y) < \frac{e^{-\beta r}}{r^\alpha} \; \forall x \in [0, 1] \right\} \right| \approx \frac{M - r}{M}.$$

Setting $r = \sqrt{M}$ and taking the limit $M \to \infty$ we conclude (when $\alpha > 0$ or $\beta > 0$) that almost for all $x \in [0, 1]$ the joint limit distribution of variances $E(x, y) = 0$ for each $y \in [0, 1]$ and vise versa.

If the limiting function $E(x, y)$ exists then it captures all the information about the capacity of the system. The behavior of $E(x, y)$ described above in the case of extended networks results in the fact that $\frac{C_{upper}}{M} \to 0$.

To compute system capacity given $E(x, y)$ function, we need to solve equation (3.10), compute Stieltjes transform of the limiting density of eigenvalues by formula (3.9), apply formula Stilts inversion formula (3.11) to recover the density of eigenvalues itself, and, finally, substitute this density function into the formula (3.8) to achieve the upper bound on capacity. Solving equation (3.10) for arbitrary function $E(x,y)$ seems to be very difficult. That is why we restrict ourselves only to very specific functions $E(x,y)$ which nevertheless allow to get some intuition on what is happening in general case.

Model 1: nodes are active partly

Choose $0 \le x_1 < x_2 \le 1$ and $0 \le y_1 < y_2 \le 1$. Suppose

$$E(x,y) = \begin{cases} 1, x_1 < x < x_2 \; and \; y_1 < y < y_2 \\ 0, \; otherwise \end{cases} \tag{3.12}$$

Denote $d_x = x_2 - x_1$, $d_y = y_2 - y_1$. Then fixed point equation (3.10) has the form

$$\frac{1}{u(x,s)} = -s + \frac{d_y}{1 + \int_{x_1}^{x_2} u(x',s)dx'},$$

when $x \in [x_1, x_2]$, and the form

$$\frac{1}{u(x,s)} = -s, \tag{3.13}$$

when $x \notin [x_1, x_2]$.

Assume $x \in [x_1, x_2]$. Supposing that u does not depend on x in this interval: $u(x,s) = v(s)$ we achieve the following equation on v

$$\frac{1}{v} = -s + \frac{d_y}{1 + d_x v}, \tag{3.14}$$

or

$$v^2 s d_x + v(d_x - d_y + s) + 1 = 0.$$

Solving this quadric equation, combining the result with solution of (3.13), using (3.9) and performing the integration we receive

$$G_{\dot{\boldsymbol{G}}\dot{\boldsymbol{G}}^\dagger}(s) = d_x \left(-\frac{d_x - d_y + s}{2sd_x} + \frac{i}{2d_x}\sqrt{\frac{4d_x}{s} - \left(\frac{d_x - d_y}{s} + 1\right)^2} \right) - \frac{1 - d_x}{s}.$$

So

$$p_{\dot{\boldsymbol{G}}\dot{\boldsymbol{G}}^\dagger}(x) \to \lim_{y \to 0+} \frac{1}{\pi} Im[G(x+iy)] =$$

$$= \begin{cases} \frac{1}{2\pi}\sqrt{\frac{4d_x}{x} - (\frac{d_x - d_y}{x} + 1)^2}, x \in [d_y + d_x - 2\sqrt{d_y d_x}, d_y + d_x + 2\sqrt{d_y d_x}] \\ 0, \; otherwise \end{cases}$$

almost surely when $M \to \infty$. Finally we only note that limiting empirical eigenvalue distribution $p_{\dot{G}\dot{G}^\dagger}(x)$ does not depend on the realization of $\boldsymbol{G}$. Thus, taking the expectation in formula (3.8) is trivial and we conclude that

$$C_{upper} \to$$
$$\to \frac{M}{2\pi}\int_{d_y+d_x-2\sqrt{d_y d_x}}^{d_y+d_x+2\sqrt{d_y d_x}} \sqrt{\frac{2}{\lambda}(d_x+d_y)-\left(\frac{d_x-d_y}{\lambda}\right)^2-1}\,\log(1+\rho M\lambda)d\lambda \tag{3.15}$$

when $M \to \infty$. We can interpret this result as follows. Total number of senders (receivers) in the network is M. Only $d_x M$ senders and $d_y M$ receivers among them are active. Other terminals either do not work at all or lie in the unbounded area (distances between them go to infinity when $M \to \infty$).

Formula (3.15) says that while $d_x > 0$ and $d_y > 0$ (i.e. some fixed part of users remains active), $\frac{C_{upper}}{M}$ grows with M. Thus capacity scales super-linearly in M in contrast to extended network case. This is shown in Fig. 3.1 and Fig. 3.2.

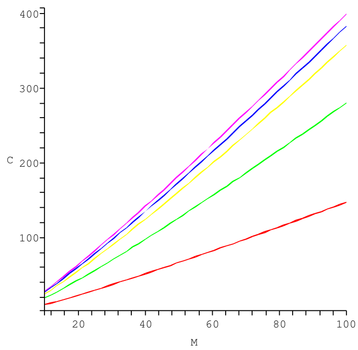

Figure 3.1: Capacity (in nats) vs. M, $0.2 \le d_x \le 1$ with increments 0.2, $d_y = 0.5$, $P = 40$dB, $\sigma^2 = 1$dB.

Fig. 3.3 shows how capacity depends on the ratio of active terminals (senders and receivers). We can note that plot is symmetric vs. d_x and d_y.

Model 2: Cyclic Connectivity

All the results reported above meet the conclusions made by Telatar [4] in his seminal paper on MIMO channels. The only difference is that in our case each node of the network has independent power constraint. Thus, the total power of the system scales linearly with M, whereas in case of multiple antenna system in ([4]) it remains constant. This results in the fact that capacity in our situation scales super-linearly in M (Fig. 3.1, Fig. 3.2). In case of MIMO it scales exactly linearly.

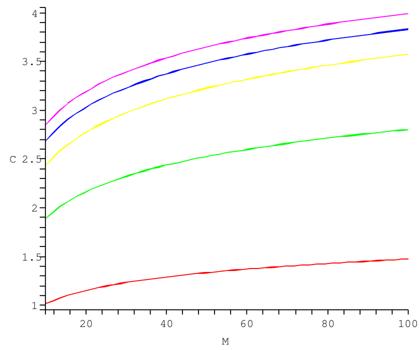

Figure 3.2: Capacity (in nats) divided by M vs. M, $0.2 \leq d_x \leq 1$ with increments 0.2, $d_y = 0.5$, $P = 40$dB, $\sigma^2 = 1$dB.

We will now slightly change assumption about behavior of function E and explore how it results the capacity. Suppose that E is doubly stochastic function, i.e. there exists constant S such that for each $x \in [0, 1]$ and $y \in [0, 1]$

$$\int_0^1 E(x', y)dx' = \int_0^1 E(x, y')dy' = S. \tag{3.16}$$

Function E with this property may appear, for example, if senders and receivers are uniformly distributed on the concentric circles of the fixed radius. The other motivation is as follows. In the situation described in the previous section, each active sender had equally good connection with all of the receivers. This only can be the case if all senders (and all receivers) are spatially grouped together like in the multiple antenna systems. If all nodes in the network are active, but each sender can communicate only to the certain fraction of receivers (other links can not be established for some reason, for example because corresponding terminals are too far away from each other) S can be interpreted as a connectivity of the network (i.e relative fraction of receivers with which each sender can communicate). Cyclic property (3.16) crucially reduces the complexity of Girko equation (3.10) and allows us to solve it successfully. Assuming again that function $u(x, s)$ does not depend on x when $x \in [0, 1]$: $u(x, s) = v(s)$, we can write equation on $v(s)$

$$\frac{1}{v(s)} = -s + \frac{S}{1 + Sv(s)}.$$

Solving this equation and following the computations performed earlier (formulas

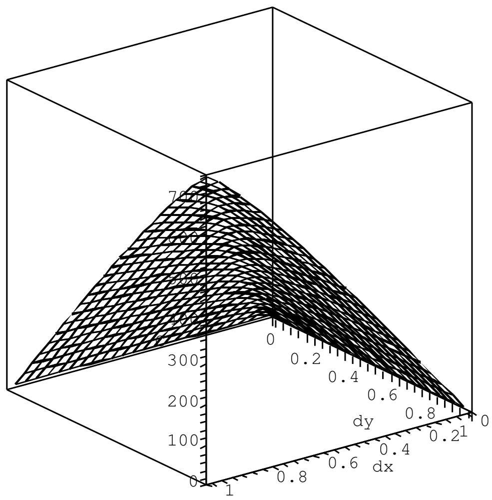

Figure 3.3: Capacity vs. d_x and d_y, $M = 100$, $P = 40$dB, $\sigma^2 = 1$dB.

(3.14)-(3.15)) we receive

$$\begin{aligned} C_{upper} &= \frac{M}{2\pi}\int_0^{4S}\frac{1}{S}\sqrt{\frac{4S}{\lambda}-1}\log(1+\rho M\lambda)d\lambda = \\ &= \frac{M}{2\pi}\int_0^{4}\sqrt{\frac{4}{\lambda}-1}\log(1+S\rho M\lambda)d\lambda. \end{aligned} \tag{3.17}$$

It is very interesting to compare expressions (3.15) and (3.17). First note that when $d_x = d_y = 1$ and $S = 1$ these two formulas coincide. This is the situation of $MIMO$ system ([4]). Note now, that in formula (3.17) parameter S stands under the logarithm, whereas in formula (3.15) parameters d_x and d_y determine the limits of integration and the pre-log. We will see now, that influence of S is much less then influence of d_x and d_y in the large M limit. To show this compare two networks. In the first network 90% of senders and 100% of recievers are active. Each active node has full connectivity, i.e. it can communicate with all other active terminal equally well. Mathematically this description corresponds to network of the first described type with parameters $d_x = 0.9, d_y = 1$. Consider now the second network and suppose that all nodes there are active. At the same time suppose that each active sender can communicate only with a half of receivers. This is the network of second type with parameter $S = 1/2$. Suppose that powers of terminals and levels of noise are equal in both cases. Denote the capacity of the first system C^1_{upper} and the capacity of the second one C^2_{upper}. Fig. 3.4 shows the scaling of the difference between these two when M increases. We conclude that

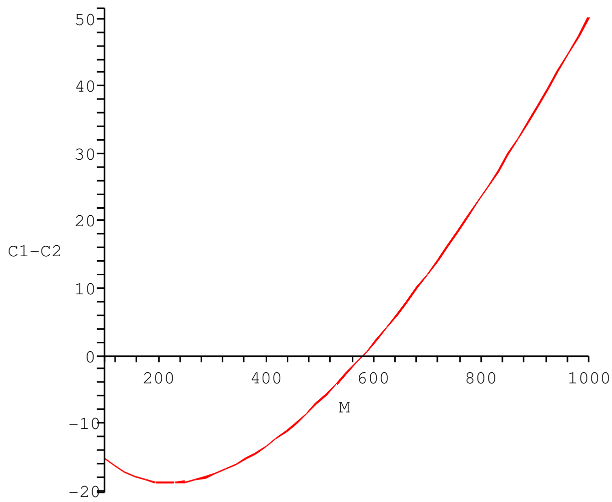

Figure 3.4: $C^2_{upper} - C^1_{upper}$ in nats vs. M, $P = 40$dB, $\sigma^2 = 1$dB.

even thought when values of M are small the influence of small connectivity is high, when M gets larger the most important factor is the number of active users. The network with slightly bigger number of active users and with much lower connectivity has greater capacity (see Fig. 3.4).

Chapter 4

Relay Wireless Networks

In this section we analyse particulary simple ralaying architecture proposed in [7]. The main advantage of this sheme is that it allows to achieve capacity bound without any cooperation between the terminals in the network (senders, recievers and relayes). The difference between our analysis and analysis made by Bölcskei, Nabar, Oyman and Paulraj is that we study the case when number of senders $M \to \infty$ where as in [7] M is fixed.

4.1 Architecture

Suppose again $M = M_S = M_D$. Suppose that before the transmission begins sources and destinations form communication pairs. Each sourse terminal chooses exactly one destination terminal which he is going to communicate to. Assume that between the set of sources $\mathcal{S} = \{S_1, S_2, \ldots, S_M\}$ and the set of destinations $\mathcal{D} = (D_1, D_2, \ldots, D_M)$ there is a set of K relay terminals $\mathcal{R} = (R_1, R_2, \ldots, R_K)$. We assume that K is greater then M and that M devides K. We partition all the relayes into M disjoint groups:

$$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \ldots \cup \mathcal{R}_{\mathcal{M}}$$

and define a surjective mapping $c : K \to M$ using the rule

$$c(k) = i \iff R_k \in \mathcal{R}_i.$$

For each $i = (1, 2, \ldots, M)$ the relayes of the group $\mathcal{R}_i$ are called assosiated relayes of the i-th source-destination pair. As we will see later, assosiated relayes of the i-th pair establish in fact direct link between R_i and D_i cancelling the interferiance of other active terminals.

Suppose that $M \times K$ matrix $\boldsymbol{E} = (E_{i,j})$ is the energy path-loss matrix of the channel between sources and relyes; $M \times K$ matrix $\boldsymbol{H} = (h_{i,j})$ is a fading matrix of the channel between sources and relyes; $K \times M$ matrix $\boldsymbol{P} = (P_{i,j})$ is energy path-loss in the channel between relayes and destinations, and, finally, $K \times M$ matrix $\boldsymbol{F} = (f_{i,j})$ is a fading matrix of the channel between relayes and destinations. $K \times 1$ vector $\boldsymbol{n}$ captures noise on the relayes and $M \times 1$ vector $\boldsymbol{z}$ describes noise on destinations. Similarly to Chapter 1 we assume that all $E_{i,j}$ and $P_{i,j}$ are strictly positive and bounded from above; $h_{i,j}$ and $f_{i,j}$ are i.i.d. Gaussian; $\boldsymbol{n}$ and $\boldsymbol{z}$ are Gaussian with variance σ^2. To avoid using additional simbols we assume that senders and relayes are constrained to power $P = 1$. Relayes and distanation nodes have full channel state information.

Transmitting protocol consists of two hops. On the first hop, senders simultaneously and independently transmit their data to relayes. The signal recived on k-th relay after the first hop is

$$r_k = \sum_{l=1}^{M} \sqrt{E_{k,l}} h_{k,l} s_l + n_k, \; k = 1, 2 \ldots, K.$$

After the first hop relays process the recieved information and prepare it for the second hop transmition to destinations. Relay number k (assosiated with the source-destination pair number $c(k)$) transforms the recieved signal by matched-filtering it with respect to the channel $S_{c(k)} \to R_k$, multiplying by energy normalization factor τ_k and co-phazing with respect to the forward channel $R_k \to D_{c(k)}$ as follows:

$$t_k = \frac{f^*_{c(k),k}}{|f_{c(k),k}|} \tau_k h^*_{k,c(k)} r_k,$$

where $\tau_k = (2E_{c(k),k} + \sum_{j \neq c(k)} E_{j,k} + N_0)^{1/2}$, and t_k is the signal after transformation. Note, that this process does not require big computational effort (relayes do not need to decode the signal). Nevertheless, relayes need to know both input and output channels.

Finally, the m-th destination node recieves

$$\begin{aligned} y_m =& \sum_{k=1}^{K} \sqrt{P_{m,k}} f_{m,k} t_k + z_l = \\ =& \sum_{k=1}^{K} \sqrt{P_{m,k}} f_{m,k} \frac{f^*_{c(k),k}}{|f_{c(k),k}|} u_k + z_m = \\ =& \sum_{k=1}^{K} \sqrt{P_{m,k}} f_{m,k} \frac{f^*_{c(k),k}}{|f_{c(k),k}|} \tau_k h^*_{k,c(k)} r_k + z_m = \\ =& \sum_{k=1}^{K} \sqrt{P_{m,k}} f_{m,k} \frac{f^*_{c(k),k}}{|f_{c(k),k}|} \tau_k h^*_{k,c(k)} \left(\sum_{l=1}^{L} \sqrt{E_{k,l}} h_{k,l} s_l + n_k \right) + z_m, \end{aligned} \tag{4.1}$$

$m = (1, 2, \dots, M)$.

4.2 Analysis of the System

We assume that each destinations terminal decodes its massage separatelly, without any cooperation with others. He needs to separate the message adressed to him of messages adressed to other user of the network (interference). We rewrite expression (4.1) separating signal, interference, and noise contributions:

$$\begin{aligned} y_m =& \sum_{k=1}^{K} \sum_{l=1}^{M} \tau_k \sqrt{P_{m,k} E_{k,l}} f_{m,k} \frac{f^*_{c(k),k}}{|f_{c(k),k}|} h^*_{k,c(k)} h_{k,l} s_l + \\ &+ \sum_{k=1}^{K} \sqrt{P_{m,k}} f_{m,k} \frac{f^*_{c(k),k}}{|f_{c(k),k}|} \tau_k h^*_{k,c(k)} n_k + z_m = \\ =& s_m (\sum_{k \in \mathcal{R}_m} \tau_k \sqrt{P_{m,k} E_{k,m}} |f_{m,k}| |h_{k,m}|^2 + \\ &+ \sum_{k \notin \mathcal{R}_m} \tau_k \sqrt{P_{m,k} E_{k,l}} f_{m,k} \frac{f^*_{c(k),k}}{|f_{c(k),k}|} h^*_{k,c(k)} h_{k,m}) + \\ &+ \sum_{l \neq m} s_l \sum_{k=1}^{K} \tau_k \sqrt{P_{m,k} E_{k,l}} |f_{m,k}| h^*_{k,m} h_{k,l} + \\ &+ \sum_{k=1}^{K} \sqrt{P_{m,k}} f_{m,k} \frac{f^*_{c(k),k}}{|f_{c(k),k}|} \tau_k h^*_{k,c(k)} n_k + z_m. \end{aligned} \tag{4.2}$$

To simplify this expresion we introduce the notations

$$d_k^{m,l} = \tau_k \sqrt{P_{m,k} E_{k,l}} g_{m,k} \frac{g^*_{c(k),k}}{|g_{c(k),k}|} h_{k,l} h^*_{k,c(k)},$$

and

$$f_k^m = \tau_k \sqrt{P_{m,k}} g_{m,k} \frac{g^*_{c(k),k}}{|g_{c(k),k}|} h^*_{k,c(k)}.$$

Now we can rewrite the signal recieved by m-th destination terminal in the following form:

$$y_m = s_m \left(\sum_{k \in \mathcal{R}_m} d_k^{m,m} + \sum_{k \notin \mathcal{R}_m} d_k^{m,m} \right) + \sum_{l \neq m} s_l \sum_{k=1}^{K} d_k^{m,l} + \sum_{k=1}^{K} f_k^m n_k + z_m.$$

Recalling that we assumed independent encoding and decoding and full channel knowledge we can now write expresion for the capacity of this system [7]:

$$C_{lower} = \sum_{i=1}^{M} \underset{\boldsymbol{E},\boldsymbol{F}}{\mathcal{E}} \{I_i\} \tag{4.3}$$

where

$$I_i = \frac{1}{2} \log\left(1 + SINR_i\right)$$

is mutual information between i-th sender and i-th reciever and

$$SINR_i = \frac{E_i^S}{E_i^{N+I}} = \frac{|\sum_{k\in\mathcal{R}_m} d_k^{m,m} + \sum_{k\notin\mathcal{R}_m} d_k^{m,m}|^2}{\sum_{l\neq m} |\sum_{k=1}^{K} d_k^{m,l}|^2 + \sigma^2 \sum_{k=1}^{K} |f_k^m|^2 + \sigma^2}.$$

is the corresponding signal to noise plus interference ratio.

We are now going to find such conditions on M and K, that it is possible to bound $SINR_i$ from below. Building the uniform lower bound for all $SINR_i$ ($i = 1, 2, \ldots, M$) we build lower bound on the capacity which scales at least lineary in M.

We are going to use Chebyshev's inequality to estimate $SINR_i$.

Theorem 3 (Chebyshev's inequality). *Let $X_1, X_2, \ldots, X_n$ be independent random variables with means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$ respectively. Let $S = X_1 + \ldots + X_n$, $\mu = \mu_1 + \ldots + \mu_n$, and $\sigma^2 = \sigma_1^2 + \ldots + \sigma_n^2$. Then for any t*

$$Pr\left\{|S - \mu| > t\sigma\right\} \leq \frac{1}{t^2}.$$

To use this theorem we first compute means and variances of $d_k^{m,l}$ and f_k^m with respect to $\boldsymbol{H}$ and $\boldsymbol{F}$.

First we compute expectations:

$$\mathcal{E}\left\{d_k^{m,l}\right\} = \begin{cases} 0, k \notin L_m \; or \; m \neq l \\ \sqrt{\frac{P_{m,k}E_{k,m}}{2E_{c(k),k}+\sum_{j\neq c(k)} E_{j,k}+N_0}}\, \mathcal{E}\left\{|g_{m,k}|\right\}, k \in L_m, m = l; \end{cases} \tag{4.4}$$

$$\mathcal{E}\left\{|f_k^m|^2\right\} = \frac{P_{m,k}}{2E_{c(k),k} + \sum_{j\neq c(k)} E_{j,k} + N_0}. \tag{4.5}$$

Now compute variances:

$$VAR\left(d_k^{m,l}\right) = \frac{P_{m,k}E_{k,l}}{2E_{c(k),k} + \sum_{j\neq c(k)} E_{j,k} + N_0}\, \mathcal{E}\left\{|h_{k,l}h_{k,c(k)}^*|^2\right\} - \left(\mathcal{E}\left\{d_k^{m,l}\right\}\right)^2; \tag{4.6}$$

$$VAR\left(|f_k^m|^2\right) =$$
$$= \left(\frac{P_{m,k}}{2E_{c(k),k} + \sum_{j \neq c(k)} + N_0}\right)^2 \mathcal{E}\left\{\frac{|g_{m,k} g^*_{c(k),k}|^4}{|g_{c(k),k}|^4} |h^*_{k,c(k)}|^4\right\} - \left(\mathcal{E}\left\{|f_k^m|^2\right\}\right)^2. \tag{4.7}$$

Then, applying Chebyshev's inequality for each of the sums separately we conclude that probability of each of the following events is t^{-2} or less:

$$\left|\sum_{k=1}^{K} d_k^{m,m} - \sum_{k \in \mathcal{R}_m} \mathcal{E}\left\{d_k^{m,m}\right\}\right| \geq t\sqrt{\sum_{k=1}^{K} VAR\left(d_k^{m,m}\right)};$$

when $l \neq m$

$$\left|\sum_{k=1}^{K} d_k^{m,l}\right| \geq t\sqrt{\sum_{k=1}^{K} VAR\left(d_k^{m,l}\right)};$$

and

$$\left|\sum_{k=1}^{K} |f_k^m|^2 - \sum_{k=1}^{K} \mathcal{E}\,|f_k^m|^2\right| \geq t\sqrt{\sum_{k=1}^{K} VAR\left(|f_k^m|^2\right)}.$$

Note that we have $M^2 + M$ inequalities here. For each of these inequalities, the probability of that it is satisfied is less then t^{-2}. Resume that for any events $A_1, A_2, \ldots, A_n$

$$Pr\left(A_1 \cup A_2 \cup \ldots \cup A_n\right) \leq \sum_{i=1}^{n} Pr(A_i).$$

Using this for $M^2 + M$ inequalities we conclude that probability of that at least one of them is satisfied is less then $\frac{M^2+M}{t^2}$. Finally, we conclude that probability of that no one of these inequalities is satisfied (i.e. all opposite inequalities are satisfied all together) is greater then $1 - \frac{M^2+M}{t^2}$. From this point on we assume that this is fullfield.

Then, the following bounds on signal and noise-interference energy can be written for each m, n and i simultaneously:

$$E_i^S \geq \left(max\left[0, \sum_{k \in \mathcal{R}_m} \mathcal{E}\left\{d_k^{m,m}\right\} - t\sqrt{\sum_{k=1}^{K} VAR\left(d_k^{m,m}\right)}\right]\right)^2, \tag{4.8}$$

$$E_i^{N+I} \leq$$

$$\leq \sum_{l\neq m} t^2 \sum_{k=1}^{K} VAR\left(d_k^{m,l}\right) + \sigma^2 \left(\sum_{k=1}^{K} \mathcal{E}\left\{|f_k^m|^2\right\} + t\sqrt{\sum_{k=1}^{K} VAR\left(|f_k^m|^2\right) + 1} \right). \tag{4.9}$$

Now we will simplify these expretions by uniformly bounding variances and mean values in them. We recall that the network is bounded and there is some minimal separation distance between nodes. This means that there are constants $0 < P_* \leq P^*$ and $0 < E_* \leq E^*$ such that $P_* \leq P_{i,j} \leq P^*$ and $E_* \leq E_{i,j} \leq E^*$. Using the following notations for brevity: $A_1 = \mathcal{E}\left\{|g_{m,k}|\right\}$, $A_2 = \mathcal{E}\left\{|h_{k,l}h^*_{k,c(k)}|^2\right\}$ and $A_3 = \mathcal{E}\left\{\frac{|g_{m,k}g^*_{c(k),k}|^4}{|g_{c(k),k}|^4}|h^*_{k,c(k)}|^4\right\}$, from formulas (4.4), (4.5), (4.6), and (4.7) we deduce

$$\mathcal{E}\left\{d_k^{m,m}\right\} \geq \sqrt{\frac{P_*E_*}{E^*(M+1)+N_0}}A_1,$$

$$\mathcal{E}\left\{|f_k^m|^2\right\} \leq \frac{P^*}{E_*(M+1)+N_0},$$

$$VAR\left(d_k^{m,l}\right) \leq \frac{P^*E^*}{E_*(M+1)+N_0}A_2,$$

$$VAR(|f_k^m|^2) \leq \left(\frac{P^*}{E_*(M+1)+N_0}\right)^2 A_3.$$

Substituting these expressions into (4.8) and (4.9) we conclude that

$$E_i^S \geq$$

$$\geq \left(max\left[0, |\mathcal{R}_m|\sqrt{\frac{P_*E_*}{E^*(M+1)+N_0}}A_1 - t\sqrt{K\frac{P^*E^*}{E_*(M+1)+N_0}A_2}\right]\right)^2,$$

$$E_i^{N+I} \leq$$

$$\leq \frac{(M-1)t^2KP^*E^*A_2}{E_*(M+1)+N_0} + \sigma^2\left(\frac{KP^*}{E_*(M+1)+N_0} + \frac{t\sqrt{KP^*A_3}}{E_*(M+1)+N_0} + 1\right),$$

and

$$SNIR_i \geq \frac{\left(max\left[0, |\mathcal{R}_m|\sqrt{P_*E_*}A_1 - t\sqrt{KP^*E^*A_2}\right]\right)^2}{(M-1)t^2KP^*E^*A_2 + \sigma^2(KP^* + t\sqrt{KP^*A_3} + E_*(M+1)+N_0)}. \tag{4.10}$$

Finally, suppose that $K = M^\gamma$. We want to choose parameter γ in such a way that right hand side of (4.10) does not go to zero when $M \to \infty$ almost surely with respect to $\boldsymbol{G}$ and $\boldsymbol{F}$. We know that the relayes are equally distributed among the source destination pairs, i.e. $|\mathcal{R}_m| = M^{\gamma-1}$ for each $m = (1, 2, \ldots, M)$. Now set $t^2 = (L^2 + L)\delta$. Substitute the values of t and $\mathcal{R}$ into formula (4.10) and ommit terms of lower order in M. We conclude the following. Simalteniously for all $i = (1, 2, \ldots, M)$ with probability at least $1 - \frac{1}{\delta}$ assimptotically in M

$$SNIR_i \geq \frac{M^{2(\gamma-1)} P_* E_*}{M^{\gamma+3} \delta P^* E^* A_2} = \frac{M^{\gamma-5} P_* E_*}{\delta P^* E^* A_2}.$$

This proves the following lemma.

Lemma 2. *There exists constant Q such that for any $\delta > 0$*

$$Pr\left\{ SNIR_i \geq Q \frac{M^{\gamma-5}}{\delta} \; \forall i = (1, 2, \ldots, M) \right\} \geq 1 - \frac{1}{\delta}.$$

From this lemma and formula (4.3) we achieve the main theorem.

Theorem 4. *Suppose number of relayes $K = M^\gamma$. Then there exists a constant Q such that for any $\delta > 0$*

$$Pr\left\{ C_{lower} \geq \frac{M}{2} \log \left(1 + Q \frac{M^{\gamma-5}}{\delta} \right) \right\} \geq 1 - \frac{1}{\delta}.$$

When $\gamma > 5$ this theorem sais that capacity of the system scales superlinearly in M.

Chapter 5

Conclusion

In this work we have compared capacity scaling laws for different wireless systems over Gaussian channels with fading. We have shown that in the case of extended networks, capacity scales sublinearly in the number of terminals. This may serve as a hint on thumb for engeneers. It is not likely that the extended network can be built without backbone wired infrastructure. We have studyed several types of condensed networks and have shown how relayes can be used to assist communication in such cases. We should note that the bound $\gamma = 5$ given in the lust section does not seem to be tight. We are currently working on improving this bound. If we manage to do so, this will mean that less relayes are needed for reliable transmission in the described network.

Bibliography

[1] R. G. Gallager, *Information Theory and Reliable Communication*, Wiley, 1968.

[2] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, Wiley, New York, 1991.

[3] E. Biglieri, J. Proakis, and S. Shamai, "Fading channels: Information-theoretic and communications aspects," *IEEE Transactions on Information Theory*, vol. 44, no. 6, pp. 2619–2692, Oct. 1998.

[4] I. E. Telatar, "Capacity of multi-antenna Gaussian channels," *European Transactions on Telecommunications*, vol. 10, pp. 585–595, Nov./Dec. 1999.

[5] P. Gupta and P. R. Kumar, "The capacity of wireless networks," *IEEE Transactions on Information Theory*, vol. 46, no. 2, pp. 388–404, Mar. 2000.

[6] O. Leveque and E. Telatar, "Information theoretic upper bounds on the capacity of large extended ad-hoc wireless networks," *submitted to IEEE Transactions on Information Theory, Feb 2004*.

[7] H. Bölcskei, R. Nabar, O. Oyman, and A. Paulraj, "Capacity scaling laws in mimo relay networks," *presented in part at the Allerton Conference on Communications, Control, and Computing, Monticello, IL, Oct. 2003*.

[8] Vjaceslav L. Girko, *Theory of Random Determinants*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.