Capacity Scaling Laws in Asynchronous Relay Networks

Rohit U. Nabar[1] and Helmut Bölcskei[2]

[1] Department of Electrical and Electronic Engineering
Imperial College of Science, Technology, and Medicine
Exhibition Road, London SW7 2BT, UK
Email: r.nabar@imperial.ac.uk

[2] Communication Technology Laboratory
Swiss Federal Institute of Technology (ETH) Zurich
ETF E122, Sternwartstrasse 7, 8092 Zürich, Switzerland
Email: boelcskei@nari.ee.ethz.ch

Abstract

This paper examines capacity scaling in asynchronous relay networks, where L single-antenna source-destination terminal pairs communicate concurrently through a common set of K single-antenna relay terminals using one-hop relaying. In the perfectly synchronized case, assuming perfect channel state information (CSI) at the relays, the network capacity is known [1] to scale (asymptotically in K for fixed L) as $C = \frac{L}{2}\log(K) + O(1)$. In the absence of CSI at the relay terminals, it was shown in [1] that a simple amplify-and-forward architecture, asymptotically in K for fixed L, turns the network into a point-to-point multiple-input multiple-output (MIMO) link with high-SNR capacity $C = \frac{L}{2}\log(\mathrm{SNR}) + O(1)$. In this paper, we demonstrate that lack of synchronization in the network, under quite general conditions on the synchronization error characteristics, leaves the capacity scaling laws for both scenarios fundamentally unchanged. However, synchronization errors do result in a reduction of the spatial multiplexing gain (pre-log in the capacity expression) and the effective signal-to-noise ratio at the destination terminals. Quantitative results for these losses, as a function of the synchronization error characteristics, are provided.

1 Introduction and Outline

We consider an asynchronous wireless (fading) relay network where L single-antenna source-destination terminal pairs communicate concurrently through one-hop relaying over a common set of K single-antenna relay terminals (see Fig. 1). For a perfectly synchronized network with fixed L, letting $K \to \infty$ the following results have been established in [1]:

- In the presence of perfect channel state information (CSI) at the relay terminals (coherent network), the network capacity scales as $C = \frac{L}{2}\log(K) + O(1)$ and can be achieved with matched-filtering at the relay terminals and independent (coherent) decoding at the destination terminals. This result implies that *distributed array gain* [2] and *spatial multiplexing gain* [3, 4, 5, 2] can be obtained in a completely distributed fashion, i.e., without cooperation between any of the terminals (not even the destination terminals).

- In the absence of CSI at the relay terminals (non-coherent network), a simple amplify-and-forward (AF) architecture turns the relay network into a point-to-point multiple-input multiple-output (MIMO) link. Provided that the destination terminals have perfect knowledge of the resulting effective MIMO channel, joint (across destination terminals) decoding achieves a high-SNR capacity of $C = \frac{L}{2}\log(\mathrm{SNR}) + O(1)$ and hence full spatial multiplexing gain.

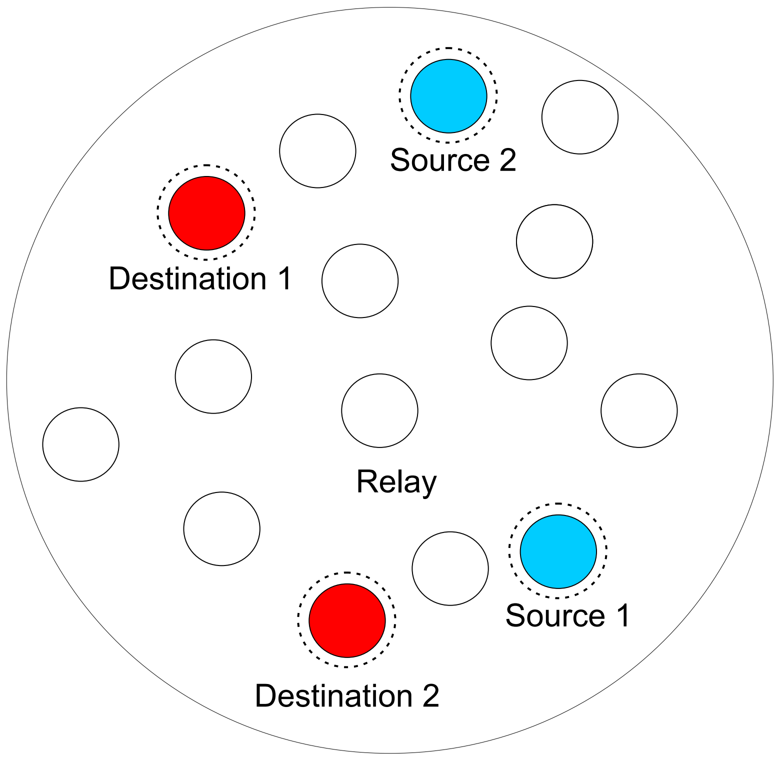

Figure 1: *Schematic of an asynchronous wireless relay network with multiple source-destination terminal pairs.*

Contributions. In practice perfect synchronization across geographically distributed network terminals is very difficult to realize. In this paper, we study the impact of lack of synchronization on the fixed L and large K capacity scaling of coherent and non-coherent (under AF) relay networks. We show that in both cases, under quite general conditions on the timing offset characteristics, the scaling laws remain fundamentally unchanged. However, the presence of timing errors does result in a reduction of spatial multiplexing gain (reflected in a reduced pre-log) and a reduction in the effective signal-to-noise ratio (SNR) at the destination terminals. These losses are quantified in terms of the timing offset characteristics.

Relation to previous work. Beginning with the pioneering work of Gupta and Kumar [6] and Gastpar and Vetterli [7], studying the ultimate performance limits of wireless networks through their asymptotic capacity scaling behavior has been attracting increasing interest [1, 8, 9, 10, 11, 12]. With the exception of [9] these papers focus on networks with perfect synchronization. In [9], accounting for synchronization errors through multiplicative phase terms in the complex-valued channel gains, it is concluded that modest synchronization errors leave the power efficiency scaling law in sensory and ad-hoc networks unchanged.

Notation. $\mathcal{E}$ denotes the expectation operator. A circularly symmetric complex Gaussian random variable is a random variable $Z = X + j\,Y \sim \mathcal{CN}(0, \sigma^2)$, where X and Y are i.i.d. $\mathcal{N}(0, \sigma^2/2)$. The notation $u(x) = O(v(x))$ denotes that $|u(x)/v(x)|$ remains bounded as $x \to \infty$. $|\mathcal{X}|$ stands for the cardinality of the set $\mathcal{X}$. Throughout the paper all logarithms are to the base 2.

Organization of the paper. The rest of this paper is organized as follows. Section 2 introduces the channel and signal models for the asynchronous relay network. In Section 3, we briefly summarize the capacity scaling results for the perfectly synchronized case reported in [1]. Section 4 contains the capacity scaling results for the coherent and non-coherent asynchronous cases. We conclude in Section 5.

2 Channel and Signal Model

General assumptions. We consider an asynchronous wireless network consisting of $K + 2L$ single-antenna terminals with L designated source-destination terminal pairs $\{\mathcal{S}_l, \mathcal{D}_l\}\,(l = 1, 2, ..., L)$ and K relay terminals $\mathcal{R}_k\,(k = 1, 2, ..., K)$. Source terminal $\mathcal{S}_l$ intends to communicate solely with destination terminal $\mathcal{D}_l$, a "dead-zone" of non-zero radius around each $\mathcal{S}_l$ and $\mathcal{D}_l$ is free of relay terminals, no direct links exist between the individual source-destination terminal pairs (caused for instance by large separation or heavy shadowing), and transmission takes place over two time slots (using one-hop relaying). None of the terminals can transmit and receive simultaneously. In the first time slot, the source terminals $\mathcal{S}_l$ broadcast their information to all relay terminals over one symbol period of duration T_s. After processing the received signals, in the second time slot the relay terminals broadcast the processed data to all destination terminals over one symbol period (of duration T_s) while the source terminals are silent. In order to accommodate propagation delay differences between the network terminals, we furthermore introduce guard periods so that the entire time duration spanned by the first (second) time slot is Δ_1 (Δ_2). For the sake of simplicity of exposition, we take $\Delta_1 > T_s$ and $\Delta_2 > T_s$ large enough to ensure that over the first hop each relay terminal receives each of the L data symbols of duration T_s entirely and likewise, over the second hop, each destination terminal receives each relay transmission entirely. Finally, we assume that the K relay terminals are located within a domain of fixed area (dense network) with random locations chosen independently across the $\mathcal{R}_k\,(k = 1, 2, ..., K)$.

Channel and signal model. In the following, we establish the channel and signal model for a single end-to-end use of the relay network over the time period $0 \le t \le \Delta_1 + \Delta_2$. For the sake of simplicity of exposition, we assume that all source transmissions are perfectly coordinated to begin at $t = 0$, while the relay transmissions are perfectly coordinated to begin at $t = \Delta_1$. In practice, these assumptions require a common clock for the $\mathcal{S}_l$ and a common clock for the $\mathcal{R}_k$. The more general case accounting for complete lack of coordination in the network can easily be dealt with [13], but requires tedious modifications of the derivations in Section 4.

We assume that starting at $t = 0$ the l-th source terminal transmits

$$s_l(t) = a_l u(t) e^{j\omega_c t}, \quad l = 1, 2, ..., L \tag{1}$$

where the a_l are i.i.d. $\mathcal{CN}(0, 1)$ data symbols, $u(t)$ compactly supported in $[0, T_s]$ is the real-valued baseband pulse-shaping function normalized according to $\int_0^{T_s} u^2(t)\,dt = 1$ and

ω_c denotes the carrier frequency (in rad/s) of the modulated waveform. Note that all source terminals employ the same pulse-shaping function $u(t)$ and operate at the same carrier frequency ω_c.

Throughout the paper, we assume that all channels in the network are independently frequency-flat block fading [14] with the same block length, and with independent realizations across blocks (the block duration is taken to be an integer multiple of $\Delta_1 + \Delta_2$). The signal received at the k-th relay terminal during the first time slot is given by

$$r_k(t) = \sum_{l=1}^{L} \sqrt{E_{k,l}}\, h_{k,l} s_l(t - \tau_{k,l}) + n_k(t), \quad k = 1, 2, ..., K \tag{2}$$

where $E_{k,l}$ is the average energy received at $\mathcal{R}_k$ through the $\mathcal{S}_l \rightarrow \mathcal{R}_k$ link[1] (having accounted for path loss and shadowing), $h_{k,l}$ and $\tau_{k,l}$ denote the corresponding $\mathcal{CN}(0, 1)$ backward channel gain and propagation delay, respectively, and $n_k(t)$ is additive white Gaussian noise (analytic representation) at the k-th relay terminal (independent across k) with power spectral density [15]

$$S(\omega) = \begin{cases} N_o, & \omega > 0 \\ \frac{N_o}{2}, & \omega = 0 \\ 0, & \omega < 0 \end{cases} . \tag{3}$$

Each relay terminal now demodulates the received signal through matched-filtering (MF) according to

$$r_k = \int_{\theta_k}^{\theta_k + T_s} r_k(t) u(t - \theta_k) e^{-j\omega_c(t - \theta_k)}\, dt, \quad k = 1, 2, ..., K \tag{4}$$

where $0 \le \theta_k \le \Delta_1 - T_s$ $(k = 1, 2, ..., K)$ is the demodulator start-time (can be chosen randomly or based on (partial) synchronization information) for the k-th relay terminal. Combining (1), (2), and (4), we have

$$r_k = \sum_{l=1}^{L} \left(\sqrt{E_{k,l}}\, \underbrace{h_{k,l}\, e^{j\omega_c \xi_{k,l}} \int_0^{T_s} u(t) u(t + \xi_{k,l})\, dt}_{\overline{h}_{k,l}}\, a_l \right) + n_k \tag{5}$$

where $\xi_{k,l} = \theta_k - \tau_{k,l}$ and $\overline{h}_{k,l}$ denote the timing offset (synchronization error) and effective channel (including the timing offset), respectively, for the $\mathcal{S}_l \rightarrow \mathcal{R}_k$ link, and n_k is $\mathcal{CN}(0, N_o)$ noise (independent across k). In the following, we assume that the timing offsets $\xi_{k,l}$ are i.i.d. random variables (independent across k and l).

The k-th $(k = 1, 2, ..., K)$ relay terminal processes r_k to produce a unit average[2] energy output b_k (i.e., $\mathcal{E}\{|b_k|^2\} = 1$), which is then modulated to form the signal $q_k(t)$ transmitted during the second time slot and given by

$$q_k(t) = b_k u(t - \Delta_1) e^{j\omega_c(t - \Delta_1)} \tag{6}$$

with $u(t)$ and ω_c as defined in (1). The signal received at the l-th destination terminal is given by

$$y_l(t) = \sum_{k=1}^{K} \sqrt{P_{l,k}}\, g_{l,k} q_k(t - \delta_{l,k}) + z_l(t), \quad l = 1, 2, ..., L \tag{7}$$

[1] $\mathcal{A} \rightarrow \mathcal{B}$ signifies communication from terminal $\mathcal{A}$ to terminal $\mathcal{B}$.

[2] Averaged over the backward channels $h_{k,l}$, data symbols a_l, and noise n_k.

where $P_{l,k}$ denotes the average energy received at the l-th destination terminal through the $\mathcal{R}_k \to \mathcal{D}_l$ link (having accounted for path loss and shadowing), $g_{l,k}$ and $\delta_{l,k}$ represent the corresponding $\mathcal{CN}(0,1)$ channel gain and propagation delay, respectively, and $z_l(t)$ is additive white Gaussian noise (analytic representation) at the l-th destination terminal (independent across l) with power spectral density specified in (3).

Finally, each destination terminal demodulates its received signal according to

$$y_l = \int_{\Delta_1+\phi_l}^{\Delta_1+\phi_l+T_s} y_l(t)u(t-\Delta_1-\phi_l)e^{-j\omega_c(t-\Delta_1-\phi_l)}\,dt, \quad l=1,2,...,L \tag{8}$$

where $\Delta_1+\phi_l$ is the demodulator start-time (ϕ_l can be chosen randomly or based on (partial) synchronization information) at the l-th destination terminal and $0 \le \phi_l \le \Delta_2 - T_s$. Combining (6), (7), and (8), we get

$$y_l = \sum_{k=1}^{K}\left(\sqrt{P_{l,k}}\,\underbrace{g_{l,k}\,e^{j\omega_c\eta_{l,k}}\int_0^{T_s} u(t)u(t+\eta_{l,k})\,dt}_{\overline{g}_{l,k}}\;b_k\right)+z_l \tag{9}$$

where $\eta_{l,k}=\phi_l-\delta_{l,k}$ and $\overline{g}_{l,k}$ represent the timing offset and effective channel, respectively, for the $\mathcal{R}_k \to \mathcal{D}_l$ link, and z_l is $\mathcal{CN}(0,N_o)$ noise (independent across l). Just as for the first time slot, we again assume that the timing offsets $\eta_{l,k}$ are i.i.d. random variables, independent of the $\xi_{k,l}$.

As already mentioned above, throughout the paper, path loss and shadowing statistics (i.e., large-scale fading) are captured by the $E_{k,l}\,(k=1,2,...,K,\ l=1,2,...,L)$ (for the first hop) and the $P_{l,k}\,(l=1,2,...,L,\ k=1,2,...,K)$ (for the second hop). We assume that these parameters are i.i.d. random variables, strictly positive and bounded. The exact statistics of the $E_{k,l}$ and $P_{l,k}$ will in general depend on the network topology. Furthermore, the $E_{k,l}$, $\xi_{k,l}$, $P_{l,k}$ and $\eta_{l,k}$ remain constant over the entire time period of interest (spanning multiple uses of the network). Finally, the $\xi_{k,l}$ and the $\eta_{l,k}$ are independent of $h_{k,l}, E_{k,l}, g_{l,k}, P_{l,k}\ \forall l,k$ and all noise terms.

3 Brief Review of the Synchronous Case

In this section, in order to set the stage for Section 4, we briefly review the capacity scaling results for the synchronous coherent and non-coherent (under AF) cases (i.e., $\xi_{k,l}=0$, $\eta_{l,k}=0\ \forall l,k$ and $\Delta_1=\Delta_2=T_s$) reported in [1].

3.1 The Coherent Case

We start with the *coherent case* where the k-th relay terminal has perfect knowledge of the individual channels (including path loss and shadowing factors) corresponding to the $\mathcal{S}_l \to \mathcal{R}_k$ and $\mathcal{R}_k \to \mathcal{D}_l\,(l=1,2,...,L)$ links.

Applying the *cut-set theorem* [16, Th. 14.10.1] by separating the source terminals $\mathcal{S}_l\,(l=1,2,...,L)$ from the rest of the network (broadcast cut), assuming that L is fixed and $K\to\infty$, the network (sum) capacity is upper bounded by

$$C_u = \frac{L}{2}\log(K)+O(1). \tag{10}$$

This upper bound is achieved when *all relay terminals can fully cooperate* and can furthermore convey the relays' transmit signals b_k in a lossless fashion to the *cooperating destination terminals*.

A protocol which achieves (10) (up to the $O(1)$ term) is described in detail in [1] and can briefly be summarized as follows. The relay terminals are partitioned into L subsets $\mathcal{M}_l\,(l = 1, 2, ..., L)$ each of which is assigned to one of the L source-destination terminal pairs. The relaying strategy for $\mathcal{R}_k \in \mathcal{M}_l$ is as follows. The received signal r_k is matched-filtered with respect to the (backward) channel $\mathcal{S}_l \to \mathcal{R}_k$ followed by matched-filtering with respect to the (forward) channel $\mathcal{R}_k \to \mathcal{D}_l$ subject to energy normalization so that $\mathcal{E}\{|b_k|^2\} = 1$. When the number of relay terminals $K \to \infty$ we need to ensure that $|\mathcal{M}_l| \to \infty$ $(l = 1, 2, ..., L)$ so that each source-destination terminal pair gets served by an infinite number of relay terminals. The l-th destination terminal is assumed to have perfect knowledge of the effective scalar channels between the $\mathcal{S}_i\,(i = 1, 2, ..., L)$ and $\mathcal{D}_l$ and the destination terminals perform independent decoding, i.e., there is no cooperation between the $\mathcal{D}_l\,(l = 1, 2, ..., L)$. In summary, we can conclude that the protocol described above turns the network into a point-to-point MIMO link with spatial multiplexing gain $\frac{L}{2}$ (pre-log in (10)) where each of the multiplexed data streams experiences a distributed array gain of K. Since the destination terminals do not need to cooperate in order to achieve C_u (up to the $O(1)$ term), we can conclude that *multi-stream interference* is eliminated in a fully decentralized fashion or equivalently the effective MIMO channel between the $\mathcal{S}_l$ and the $\mathcal{D}_l$ is diagonal.

3.2 The Non-Coherent Case

The MF-based protocol described in the previous subsection requires channel knowledge at the relay terminals. If this assumption is relaxed and each relay terminal simply amplifies-and-forwards its received signal subject to the power constraint $\mathcal{E}\{|b_k|^2\} = 1$, for fixed L and $K \to \infty$, the network is turned into a point-to-point MIMO link with high-SNR capacity[3] given by [1]

$$C = \frac{L}{2}\log(\mathrm{SNR}) + O(1). \tag{11}$$

Here, SNR is an effective signal-to-noise-ratio depending on N_o and the statistics of the $E_{k,l}$ and the $P_{l,k}$. In contrast to the coherent case described in Section 3.1, achieving (11) requires that the destination terminals cooperate and jointly decode the $L \times 1$ vector signal transmitted by the $\mathcal{S}_l$. Moreover, perfect knowledge of the composite $L \times L$ MIMO channel is required at the destination terminals.

4 The Asynchronous Case

In this section, we study the impact of lack of synchronization on the results summarized in Section 3. In order to simplify the exposition, we introduce the notation

$$\overline{h}_{k,l} = h_{k,l}\, f(\xi_{k,l}) \tag{12}$$

$$\overline{g}_{l,k} = g_{l,k}\, p(\eta_{l,k}) \tag{13}$$

where it follows from (5) and (9), respectively, that $f(\xi_{k,l}) = e^{j\omega_c \xi_{k,l}} \int_0^{T_s} u(t)u(t+\xi_{k,l})\,dt$ and $p(\eta_{l,k}) = e^{j\omega_c \eta_{l,k}} \int_0^{T_s} u(t)u(t+\eta_{l,k})\,dt$. Since the $\xi_{k,l}$ and the $\eta_{l,k}$ are random variables

[3]Note that here the $O(1)$ term is with respect to SNR.

the quantities $f(\xi_{k,l})$ and $p(\eta_{l,k})$ will be random variables as well with probability density functions (pdfs) depending on the pdfs of the $\xi_{k,l}$ and the $\eta_{l,k}$, respectively, as well as the pulse-shaping function $u(t)$.

Due to space restrictions, in the following, we provide only rough outlines of the proofs of our main results and refer the interested reader to [13] for detailed derivations.

4.1 The Coherent Case

As in the synchronous case described in Section 3.1, we partition the relay terminals into L subsets each of which is assigned to one of the L source-destination terminal pairs. Two different scenarios are considered.

Scenario 1: Relay terminal $\mathcal{R}_k \in \mathcal{M}_l$ performs matched-filtering of r_k with respect to the effective (including timing offset) backward channel $\overline{h}_{k,l}$ and the effective forward channel $\overline{g}_{l,k}$ with appropriate normalization to meet the average transmit power constraint. This setup corresponds to a scenario where the backward and forward channels, including the timing offsets, are learned (i.e., estimated) and the network operates in an asynchronous mode. Subsequently, this mode of operation will be called the "synchronization error compensated matched-filtering mode".

Theorem 1. *For an asynchronous relay network operating in the synchronization error compensated matched-filtering mode, assuming a fixed number of source-destination terminal pairs L, the $K \to \infty$ network capacity scales as*

$$C = \frac{LT_s}{\Delta_1 + \Delta_2}\log(K) + O(1) \tag{14}$$

if $\mathcal{E}\{|f(\xi_{k,l})|\} > 0$ and $\mathcal{E}\{|p(\eta_{l,k})|\} > 0$ for $l = 1, 2, ..., L$, $k = 1, 2, ..., K$. Moreover, network capacity is achieved with independent decoding at the destination terminals assuming that each $\mathcal{D}_l$ has perfect knowledge of the effective scalar channels between the $\mathcal{S}_i$ $(i = 1, 2, ..., L)$ and $\mathcal{D}_l$.

Outline of proof: The proof is along the lines of the proofs of Theorems 1 and 2 in [1] with $h_{k,l}$ and $g_{l,k}$ in [1] replaced by the effective channels $\bar{h}_{k,l}$ and $\bar{g}_{l,k}$, respectively. Intuitively, the conditions $\mathcal{E}\{|f(\xi_{k,l})|\} > 0$ and $\mathcal{E}\{|p(\eta_{l,k})|\} > 0$ $\forall l, k$ ensure that despite asynchronicity, on average, an infinite number of relay terminals participates in the relaying process. □

Scenario 2: Relay terminal $\mathcal{R}_k \in \mathcal{M}_l$ performs matched-filtering of r_k with respect to the synchronization error uncompensated backward and forward channels $h_{k,l}$ and $g_{l,k}$, respectively. Appropriate normalization ensures that the average transmit power constraint at the relay terminals is satisfied. This setup corresponds to a scenario where the backward and forward channels are learned (i.e., estimated) with the network operating in a perfectly synchronous mode, and synchronicity is subsequently lost. We call this mode of operation the "synchronization error uncompensated matched-filtering mode".

Theorem 2. *For an asynchronous relay network operating in the synchronization error uncompensated matched-filtering mode, assuming a fixed number of source-destination terminal pairs L, the $K \to \infty$ network capacity scales as*

$$C = \frac{LT_s}{\Delta_1 + \Delta_2}\log(K) + O(1) \tag{15}$$

if

$$\mathcal{E}\{f(\xi_{k,l})\} \neq 0,\ \mathcal{E}\{p(\eta_{l,k})\} \neq 0 \quad \text{for } l = 1, 2, ..., L,\ k = 1, 2, ..., K. \tag{16}$$

Outline of proof: In contrast to Theorem 1 above, we perform matched-filtering with respect to $h_{k,l}$ and $g_{l,k}$ which leaves us with terms of the form $|h_{k,l}|^2 f(\xi_{k,l})$ and $|g_{l,k}|^2 p(\eta_{l,k})$. Condition (16) ensures that the effective scalar channels $\mathcal{S}_l \rightarrow \mathcal{D}_l$ have strictly positive SNR despite the presence of synchronization errors. Moreover, (16) implies that $\mathcal{E}\{|f(\xi_{k,l})|\} > 0$, $\mathcal{E}\{|p(\eta_{l,k})|\} > 0\ \forall l, k$ which ensures that, on average, an infinite number of relay terminals participates in the relaying process. The remainder of the proof follows along the same lines as the proofs of Theorems 1 and 2 in [1]. □

The results in Theorems 1 and 2 above can be interpreted as follows:

- Irrespectively of whether the synchronization error is compensated or not, network capacity continues to exhibit the same scaling behavior as in the perfectly synchronous case (linear in L and logarithmic in K).
- The use of guard periods causes a reduction of the spatial multiplexing gain from $\frac{L}{2}$ to $\frac{LT_s}{\Delta_1+\Delta_2} < \frac{L}{2}$. For large timing offsets, i.e., $(\Delta_1 + \Delta_2) \gg 2T_s$ this reduction can be significant.
- The presence of uncompensated synchronization errors results in a reduction of the effective per-stream SNR at the $\mathcal{D}_l$ reflected in the $O(1)$ term in (15) (detailed expressions provided in [13]). This effect is quantified in the simulation example below.
- For the synchronization error uncompensated case, condition (16) requires that the timing offsets are small compared to the carrier frequency. For $\int_0^{T_s} u(t)u(t+\xi_{k,l})\, dt \approx 1$ and $\int_0^{T_s} u(t)u(t+\eta_{l,k})\, dt \approx 1\ \forall l, k$ and $\omega_c \xi_{k,l}$ and $\omega_c \eta_{l,k}$ uniformly distributed in $[0, 2\pi)$, we have $\mathcal{E}\{e^{j\omega_c \xi_{k,l}}\} = 0$, $\mathcal{E}\{e^{j\omega_c \eta_{l,k}}\} = 0\ \forall l, k$ and hence condition (16) will be violated. We can therefore conclude that for high carrier frequencies the synchronization requirements will be very strict.

Simulation example. We conclude this section with a simulation example quantifying the impact of uncompensated synchronization errors on network capacity. The system parameters are chosen as $L = 2$, $T_s = 10^{-3}$s, $\Delta_1 = \Delta_2 = T_s + 0.5 \times 10^{-6}$s and the $\xi_{k,l}$ and the $\eta_{l,k}$ are assumed to be uniformly distributed in the interval $[0, 0.25 \times 10^{-6}]s\ \forall l, k$. Note that over this range of timing offsets any reasonably behaved pulse-shaping function will result in $\int_0^{T_s} u(t)u(t+\xi_{k,l})\, dt \approx 1$ and $\int_0^{T_s} u(t)u(t+\eta_{l,k})\, dt \approx 1\ \forall l, k$. Consequently, the presence of timing offsets causes only phase differences between the physical channels and their corresponding effective channels. Fig. 2 shows the network capacity (obtained through Monte Carlo simulation) as a function of K for the synchronization error uncompensated mode with $\omega_c = 2\pi \times 10^6$rad/s and $\omega_c = 2\pi \times 3 \times 10^6$rad/s, respectively. For reference, we show the corresponding network capacity for the perfectly synchronous case and the synchronization error compensated mode. As expected, in all four cases, we can see that C grows logarithmically in K on account of distributed array gain. The performance in the synchronization error compensated mode is independent of the carrier frequency. In the uncompensated mode, however, we observe a significant reduction in C with increasing carrier frequency which is due to the fact that the effective per-stream SNR at the $\mathcal{D}_l$ decreases for increasing support of $\omega_c \xi_{k,l}$ and $\omega_c \eta_{l,k}$. Note that $\omega_c = 2\pi \times 10^6$rad/s results in $\omega_c \xi_{k,l}$ and $\omega_c \eta_{l,k}$ being uniformly distributed in $[0, \pi/2]$ whereas for $\omega_c = 2\pi \times 3 \times 10^6$rad/s the distribution is uniform in $[0, 3\pi/2]$.

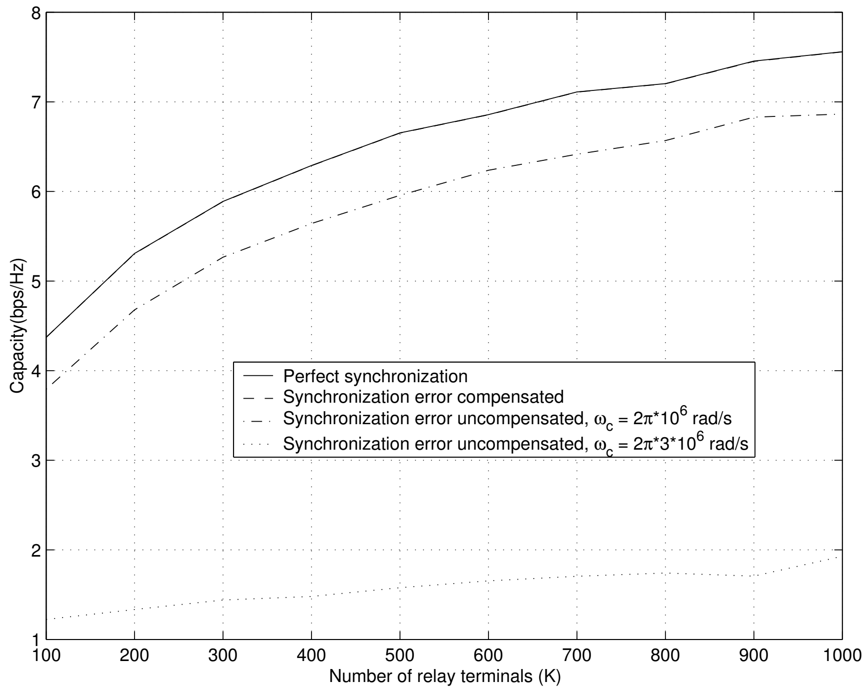

Figure 2: *Impact of synchronization errors on network capacity as a function of K.*

4.2 The Non-Coherent Case

In the non-coherent case, described for perfect synchronization in Section 3.2, the k-th relay terminal simply amplifies the signal received during the first time slot subject to a transmit power constraint and transmits the result during the second time slot. Our main result for the asynchronous non-coherent case is summarized in

Theorem 3. *For an asynchronous non-coherent relay network with a fixed number of source-destination terminal pairs L, under joint decoding at the destination terminals $\mathcal{D}_l$ and perfect knowledge of the effective MIMO channel between the $\mathcal{S}_l$ and the $\mathcal{D}_l$, the high-SNR capacity of the amplify-and-forward architecture, in the large K limit, satisfies*[4]

$$C_{AF}^{\infty} = \frac{LT_s}{\Delta_1 + \Delta_2} \log(\text{SNR}) + O(1) \tag{17}$$

provided that $\mathcal{E}\{|f(\xi_{k,l})|\} > 0$ and $\mathcal{E}\{|p(\eta_{l,k})|\} > 0$ for $l = 1, 2, ..., L$, $k = 1, 2, ..., K$. The effective signal-to-noise ratio is given by

$$\text{SNR} = \frac{\mathcal{E}\left\{\frac{L\, E_{k,l}|f(\xi_{k,l})|^2}{\sum_{m=1}^{L} E_{k,m}|f(\xi_{k,m})|^2 + N_o}\right\}}{N_o\, \mathcal{E}\left\{\frac{1}{\sum_{m=1}^{L} E_{k,m}|f(\xi_{k,m})|^2 + N_o}\right\}}. \tag{18}$$

Outline of proof: The proof follows from the proof of Theorem 3 in [1] with the channels $h_{k,l}$ and $g_{l,k}$ in [1] replaced by the effective channels $\overline{h}_{k,l}$ and $\overline{g}_{l,k}$. The condition corresponding to condition (17) in [1] is satisfied since $|\int_0^{T_s} u(t)u(t+x)\, dt| \leq 1 \;\; \forall x \in \mathbb{R}$ which implies $|f(\xi_{k,l})| \leq 1$ and $|p(\eta_{l,k})| \leq 1 \;\; \forall l, k$. $\square$

Like in the perfectly synchronous case, the simple AF architecture turns the network into an $L \times L$ point-to-point MIMO link with effective SNR given by (18) and the same

[4]Note that here the $O(1)$ term is with respect to SNR.

spatial multiplexing gain as in the coherent asynchronous case. The absence of CSI at the relay terminals results in a lack of distributed array gain, reflected in (18) not depending on K. The effective $L \times L$ MIMO channel between the $\mathcal{S}_l$ and the $\mathcal{D}_l$ is not diagonal (as in the coherent case) so that joint decoding at the destination terminals is crucial to achieve capacity. From (18) we can furthermore conclude that loss of synchronicity and hence $|f(\xi_{k,l})| < 1$ results in a reduction of the effective SNR.

5 Conclusions

We showed that, under quite general conditions on the timing offset characteristics, the capacity scaling laws in wireless relay networks with a finite number of source-destination terminal pairs are not affected by the lack of synchronicity. However, synchronization errors do result in capacity reduction through a reduced spatial multiplexing gain (pre-log) and reduced effective per-stream signal-to-noise ratio.

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