COHERENCE-BASED RECOVERY GUARANTEES FOR GENERALIZED BASIS-PURSUIT DE-QUANTIZING

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ABSTRACT

This paper deals with the recovery of signals that admit an approximately sparse representation in some known dictionary (possibly over-complete) and are corrupted by additive noise. In particular, we consider additive measurement noise with bounded ℓ_p-norm for $p \geqslant 2$, and we minimize the ℓ_q quasi-norm (with $q \in (0,1]$) of the signal vector. We develop coherence-based recovery guarantees for which stable recovery via generalized basis-pursuit de-quantizing $(\mathrm{BP\,DQ}_{p,q})$ is possible. We finally show that depending on the measurement-noise model and the choice of the ℓ_p-norm used in the constraint, $(\mathrm{BP\,DQ}_{p,q})$ significantly outperforms classical basis pursuit de-noising (BP DN).

Index Terms— Sparse signal recovery, sparse estimation, de-noising, de-quantizing, deterministic recovery guarantees.

1. INTRODUCTION

We consider the recovery of the sparse vector $\boldsymbol{x} \in \mathbb{C}^{n_a}$ from m linear and non-adaptive measurements

$$\boldsymbol{z} = \mathbf{A}\boldsymbol{x} + \boldsymbol{n}, \qquad (1)$$

where $\mathbf{A} \in \mathbb{C}^{m \times n_a}$, $n_a \geqslant m$ is a dictionary (i.e., a matrix having columns with unit ℓ_2-norm) and $\boldsymbol{n} \in \mathbb{C}^m$ represents additive measurement noise. We consider a general noise model where $\|\boldsymbol{n}\|_p \leqslant \eta$ for a given $p \geqslant 2$ and some known or estimated bound η. Algorithms making use of the ℓ_2-norm of the noise, i.e., with $p = 2$, have been extensively studied in [1–4] for both the strictly-sparse and approximately-sparse case (see Section 2). Virtually all existing recovery guarantees rely on finding a feasible solution with smallest ℓ_1-norm. However, as demonstrated in [5,6], other ℓ_q quasi-norms (with $q \in (0,1]$) can also be used—our results also give recovery guarantees for this case. The problem outlined above finds applications in, for example, image restoration [7] and de-quantization of audio or video signals [8,9].

For example, a linear quantizer $Q_\alpha(\cdot)$ rounds each entry of an m-dimensional signal vector $\boldsymbol{y}$ to the nearest integer

multiple of $\alpha > 0$. The quantized signal is $\boldsymbol{z} = Q_\alpha(\boldsymbol{y}) = \boldsymbol{y} + \boldsymbol{n}$, where $\boldsymbol{n}$ depends on $\boldsymbol{y}$ and has maximum value $\alpha/2$. We will show that if $\boldsymbol{y} = \mathbf{A}\boldsymbol{x}$ has a sparse representation $\boldsymbol{x}$ in dictionary $\mathbf{A}$, then we can stably recover $\boldsymbol{y}$ from $\boldsymbol{z}$ if certain conditions on $\mathbf{A}$ and the sparsity of $\boldsymbol{x}$ are satisfied.

Contributions: We develop coherence-based recovery guarantees providing conditions when we can stably recover $\boldsymbol{x}$ from $\boldsymbol{z}$ in (1) using generalized basis-pursuit de-quantizing $(\mathrm{BP\,DQ}_{p,q})$. In particular, we extend existing coherence-based recovery guarantees to the more general case $p \geqslant 2$ and $0 < q \leqslant 1$. Finally, we provide simulation results to demonstrate the importance of choosing the ℓ_p-norm in accordance with the considered measurement-noise model.

Notation: The number of nonzero components of the vector $\boldsymbol{x}$ (the sparsity) is denoted by $n_x = \|\boldsymbol{x}\|_0$ and $\mathrm{supp}_k(\boldsymbol{x})$ designates the indices of the k largest components (in magnitude) of $\boldsymbol{x}$; when $\mathcal{S}$ is a set of indices, $\boldsymbol{x}_{\mathcal{S}}$ is the vector that agrees with $\boldsymbol{x}$ on $\mathcal{S}$ and is 0 elsewhere. The coherence μ of $\mathbf{A}$ is defined as $\mu \triangleq \max_{i \neq j} |\langle \boldsymbol{a}_i, \boldsymbol{a}_j \rangle|$, where $\boldsymbol{a}_i$ is the ith column of $\mathbf{A}$.

2. BACKGROUND

As mentioned above, the special case $p = 2$ was investigated previously in [1–4]. The following well-known result guarantees the stable recovery of approximately sparse signals $\boldsymbol{x}$ from $\boldsymbol{z}$ in (1) using basis pursuit de-noising (BP DN).

Theorem 1 ([3, Thm. 2.1] and [4, Thm. 1])**.** *Let $\boldsymbol{z} = \mathbf{A}\boldsymbol{x} + \boldsymbol{n}$ with $\|\boldsymbol{n}\|_2 \leqslant \eta$ and $\mathcal{X} = \mathrm{supp}_{n_x}(\boldsymbol{x})$. If*

$$n_x = \|\boldsymbol{x}\|_0 < \left(1 + \mu^{-1}\right)/2, \qquad (2)$$

then the solution $\hat{\boldsymbol{x}}$ of

$$(\mathrm{BP\,DN}) \quad \text{minimize } \|\hat{\boldsymbol{x}}\|_1 \text{ s.t. } \|\boldsymbol{z} - \mathbf{A}\hat{\boldsymbol{x}}\|_2 \leqslant \varepsilon_2,$$

with $\eta \leqslant \varepsilon_2$ satisfies

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|_2 \leqslant \mathrm{C}_0(\varepsilon_2 + \eta) + \mathrm{C}_1 \|\boldsymbol{x} - \boldsymbol{x}_{\mathcal{X}}\|_1,$$

where C_0 and C_1 are constants that depend only on μ and n_x.

In the absence of measurement noise, we have $\eta = 0$, which, for $\varepsilon_2 = 0$, gives the classical (BP) recovery result [1].

Related recovery guarantees were also developed in [10] and are based on the restricted isometry constant (RIC) δ_k of the dictionary $\mathbf{A}$. For example, in [9], the (BP DN) problem was extended to basis-pursuit de-quantizing

$$(\text{BP DQ}_p) \quad \text{minimize } \|\hat{\boldsymbol{x}}\|_1 \text{ s.t. } \|\boldsymbol{z} - \mathbf{A}\hat{\boldsymbol{x}}\|_p \leqslant \varepsilon_p,$$

for $p \geqslant 2$. Corresponding recovery guarantees require the extended restricted isometry property (RIP) (p,q) of order k. A matrix $\mathbf{A}$ is said to satisfy the extended RIP if there exist constants $\delta_k > 0$ and $\mu_{p,q} > 0$ such that [9]

$$\mu_{p,q}^q \, (1 - \delta_k) \, \|\boldsymbol{x}\|_q^q \leqslant \|\mathbf{A}\boldsymbol{x}\|_p^q \leqslant \mu_{p,q}^q (1 + \delta_k) \, \|\boldsymbol{x}\|_q^q, \quad (3)$$

for every $\|\boldsymbol{x}\|_0 \leqslant k$. The following theorem guarantees stable recovery using (BP DQ_p).

Theorem 2 ([9, Thm. 2]). *Let $\boldsymbol{z} = \mathbf{A}\boldsymbol{x} + \boldsymbol{n}$ with $\|\boldsymbol{n}\|_p \leqslant \varepsilon_p$ and $\mathcal{X} = \text{supp}_{n_x}(\boldsymbol{x})$. If the extended RIP with $(p,2)$ of order $3n_x$ is satisfied, then the solution $\hat{\boldsymbol{x}}$ of (BP DQ_p) satisfies*

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|_2 \leqslant \text{C}_2\varepsilon_p + \text{C}_3 \|\boldsymbol{x} - \boldsymbol{x}_{\mathcal{X}}\|_1,$$

where C_2 and C_3 depend only on $\delta_{n_x}, \delta_{2n_x}$ and δ_{3n_x}.

The major drawback of RIP-based recovery guarantees is the fact that no deterministic construction is known that lead to matrices satisfying the RIP. One attempt to arrive at recovery guarantees that can be computed efficiently are obtained by bounding the extended RIP in terms of the coherence parameter μ. For $p \geqslant 2$, $q = 1$, and $k = 3n_x$, it follows from Lemmas 4 and 6 that $\delta_{3n_x} \leqslant \mu(3n_x - 1)$ in (3), implying that the matrix has the extended RIP $(p,2)$ of order $3n_x$ if

$$n_x < \left(1 + \mu^{-1}\right)/3. \quad (4)$$

We emphasize, however, that this recovery condition is stricter than that for (BP DN) in (2). We next show that a direct proof for (BP DQ_p) improves upon (4) and results in the same coherence-based recovery condition as for (BP DN).

3. COHERENCE-BASED RECOVERY GUARANTEES

We consider a generalized version of (BP DQ_p), referred to as generalized basis pursuit de-quantizing $(\text{BP DQ}_{p,q})$, where we minimize the ℓ_q quasi-norm (for $0 < q \leqslant 1$) of the vector $\hat{\boldsymbol{x}}$

$$(\text{BP DQ}_{p,q}) \quad \text{minimize } \|\hat{\boldsymbol{x}}\|_q \text{ s.t. } \|\boldsymbol{z} - \mathbf{A}\hat{\boldsymbol{x}}\|_p \leqslant \varepsilon_p.$$

The following theorem guarantees the stable recovery of $\boldsymbol{x}$ using $(\text{BP DQ}_{p,q})$ if (5) is satisfied.

Theorem 3. *Let $\boldsymbol{z} = \mathbf{A}\boldsymbol{x} + \boldsymbol{n}$ with $\|\boldsymbol{n}\|_p \leqslant \eta$ and $\mathcal{X} = \text{supp}_{n_x}(\boldsymbol{x})$. If*

$$n_x < \frac{1}{1 + \alpha_q\beta_q}\left(1 + \mu^{-1}\right), \quad (5)$$

where $\alpha_q = 2^{1/q-1}$ and $\beta_q = (n_x)^{1/q-1}$, then the solution, $\hat{\boldsymbol{x}}$, of $(\text{BP DQ}_{p,q})$ with $\eta \leqslant \varepsilon_p$ satisfies

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|_2 \leqslant \text{C}_4 \, m^{\frac{1}{2} - \frac{1}{p}}(\eta + \varepsilon_p) + \text{C}_5 \|\boldsymbol{x} - \boldsymbol{x}_{\mathcal{X}}\|_1, \quad (6)$$

where C_4 and C_5 depend only on μ, n_x and q, but not *on p.*

Proof. See Appendix A for the proof. $\qquad\square$

Note that in the case $q = 1$, we have $\alpha_q\beta_q = 1$, and (5) gives the same condition as (2). Moreover, for $q = 1$ and $p = 2$, we get the same constants as in Theorem 1. Hence, Theorem 3 results in a better (i.e., less restrictive) recovery guarantee than that obtained in (4).

An important question to ask is, how does this result compare to the existing results for (BP DN) given that they both have the same condition for stable recovery? Do we get an improvement by using a different norm constraint? To this end, we consider the quantization example given in Section 1, with $\alpha = 2\tau$. In this case, $\|\boldsymbol{n}\|_\infty \leqslant \tau$ and $\|\boldsymbol{n}\|_2 \leqslant \sqrt{m}\tau$, and by setting $\varepsilon_\infty = \tau$, we get the recovery condition

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}_\infty\|_2 \leqslant 2\text{C}_0\sqrt{m}\tau + \text{C}_1 \|\boldsymbol{x} - \boldsymbol{x}_{\mathcal{X}}\|_1. \quad (7)$$

But applying Theorem 1 with $\varepsilon_2 = \sqrt{m}\tau$ leads to same recovery bound as in (7), implying that using a more suitable norm does not necessarily result in better analytical recovery guarantees. However, as shown next, the use of (BP DQ_p) can give vastly superior recovery performance in practice.

4. SIMULATION RESULTS

4.1. Comparison of different norms

We first demonstrate the impact of the ℓ_p-norm on the recovery performance. Take $\mathbf{A}$ to be a randomly sub-sampled Hadamard matrix of dimension 128×256, and assume $\boldsymbol{x}$ has $n_x = 8$ non-zero entries. The measurement noise is generated according to the following models: (i) i.i.d. Gaussian with zero mean, (ii) uniformly distributed random noise on an interval or discrete set, and (iii) noise as obtained via quantization. In each case, we normalize the signal to have unit energy and choose the noise variance so that the expected signal-to-noise-ratio SNR is $10\,\text{dB}$. Note that we only consider the case $q = 1$, since for $q < 1$, $(\text{BP DQ}_{p,q})$ is not a convex optimization problem. We use CVX [11] to solve the optimization problems and consider a genie-aided norm bound $\varepsilon_p = \|\boldsymbol{n}\|_p$, which is the smallest bound guaranteeing that the correct solution lies in the feasible set.

In Fig. 1, we plot the mean-square error MSE, obtained by averaging $\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|_2^2 / \|\boldsymbol{x}\|_2^2$ over $1\,000$ Monte-Carlo trials. For each noise model, we plot the MSE relative to the norm constraint that gives the lowest MSE for that noise model. Observe that as the noise becomes less concentrated around the mean, higher order norms seem to perform better e.g., when the noise is uniformly distributed on $[-1, 1]$, the ℓ_4 norm constraint performs best, but as the noise clusters around ± 1, e.g.,
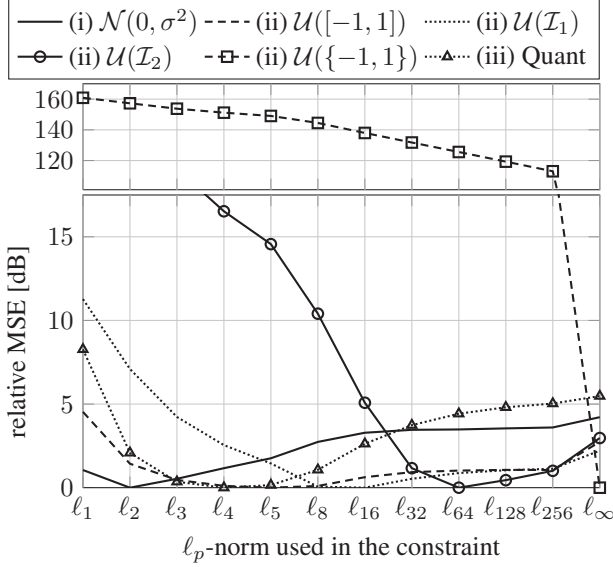
Fig. 1. Relative MSE for different noise models. "Quant" refers to quantization noise, $\mathcal{I}_1 = [-1, -0.5] \cup [0.5, 1]$ and $\mathcal{I}_2 = [-1, -0.9] \cup [0.9, 1]$. Here, $n_x = 8$ is considered; similar behavior is observed for other sparsity levels.

with $\mathcal{U}(\mathcal{I}_2)$, the ℓ_{64} norm gives better performance. This implies that the choice of the norm to be used in the constraint of $(\mathrm{BP\,DQ}_{p,q})$ is important and should be chosen in accordance with the considered measurement noise model.

4.2. De-quantization of band-limited audio signals

We now reconstruct a quantized audio signal. We take $m = 256$ sample segments of Mozart's Quintet K.452 at random, quantize it to 4 bit precision and reconstruct using $(\mathrm{BP\,DQ}_p)$. We plot the resulting SNR $= \|\boldsymbol{y}\|_2 / \|\mathbf{A}\hat{\boldsymbol{x}} - \boldsymbol{y}\|_2$ ($\boldsymbol{y}$ is the unquantized signal) in Fig. 2, relative to the performance of $(\mathrm{BP\,DN})$. We use two discrete cosine transform (DCT) matrices of dimension $m \times n$ ($n = 256$ or 512) as dictionaries. When $n > m$, the dictionary is overcomplete and may lead to a sparser vector $\boldsymbol{x}$. Note that here we deal with *approximately* sparse vectors. If we know $\|\boldsymbol{n}\|_p$ (which is often not the case), then using the ℓ_3 norm offers slightly improved reconstruction SNR (≈ 0.5dB). However, if we only have an upper-bound for the noise, e.g., $\varepsilon_p = 2^{-4} m^{1/p}$ (which is the ℓ_p-norm of the noise in the worse case scenario), then we see that higher-order norms offer a significant improvement over $(\mathrm{BP\,DN})$. Using an overcomplete DCT matrix, seems to make little difference, when compared to the standard DCT matrix.

5. CONCLUSION

In this paper, we have generalized the coherence-based results of [3, 4] to include the cases where a sparse signal is corrupted by noise with bounded ℓ_p norm ($p \geqslant 2$). We have shown that by using the appropriate optimization program, we
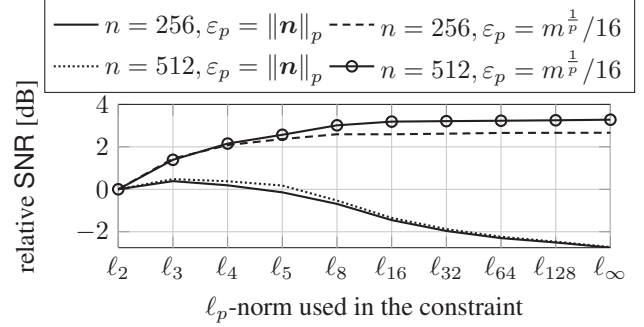

Fig. 2. Reconstruction SNR (relative to the ℓ_2 case) of an audio signal recovered after 4 bit quantization.

can achieve improved recovery performance when compared to solving $(\mathrm{BP\,DN})$. Furthermore, our analytical guarantees also enable the stable recovery if we minimize the ℓ_q quasi-norm ($0 < q \leqslant 1$) of $\boldsymbol{x}$. We presented simulation results to demonstrate that the choice of the norm employed in the constraint is an important design consideration and should be chosen in accordance with the measurement noise model for optimum performance.

A. PROOF OF THEOREM 3

We now present the proofs of our results for $(\mathrm{BP\,DQ}_{p,q})$. In this section, the *dual* p' of a number $p \in [1, \infty]$ satisfies $1/p + 1/p' = 1$. We first restate a number of well-known bounds.

Lemma 4 (Section 2.8 of [12]). *For any $\boldsymbol{x} \in \mathbb{C}^m$, $1 \leqslant p \leqslant s \leqslant \infty$ with duals p' and s'*

$$\|\boldsymbol{x}\|_p \, m^{1/p'} \leqslant \|\boldsymbol{x}\|_s \, m^{1/s'}. \qquad (8)$$

Lemma 5. *For any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^m$, $0 < q \leqslant 1, p \geqslant 1$ and $\alpha_q \triangleq 2^{1/q-1}$, we have*

$$\|\boldsymbol{a}\|_1 \leqslant \|\boldsymbol{a}\|_q \leqslant m^{\frac{1}{q} - \frac{1}{p}} \|\boldsymbol{a}\|_p \qquad (9a)$$

$$(|a|^q + |b|^q)^{\frac{1}{q}} \leqslant \alpha_q |a| + \alpha_q |b| \qquad (9b)$$

$$\|\boldsymbol{a}\|_q^q - \|\boldsymbol{b}\|_q^q \leqslant \|\boldsymbol{a} - \boldsymbol{b}\|_q^q \qquad (9c)$$

$$\|\boldsymbol{a} + \boldsymbol{b}\|_q \leqslant \alpha_q \big(\|\boldsymbol{a}\|_q + \|\boldsymbol{b}\|_q \big). \qquad (9d)$$

Proof. Equation (9a) is in [12, §2.8] and (9b) follows from the case with $m = 2$ and $p = 1$. Equation (9c) follows by applying Jensen's inequality to the mapping $t \mapsto t^q$. Finally, (9d) is a consequence of (9b) and (9c). $\square$

Lemma 6. *Let $\mathbf{A} \in \mathbb{C}^{m \times n_a}$ have columns of unit ℓ_2 norm. Let μ be its coherence and $\boldsymbol{x} \in \mathbb{C}^{n_a} \backslash \{\mathbf{0}\}$ be n_x-sparse. Then*

$$\|\boldsymbol{x}\|_2^2 \, (1 - \mu(n_x - 1)) \leqslant \|\mathbf{A}\boldsymbol{x}\|_2^2 \leqslant \|\boldsymbol{x}\|_2^2 \, (1 + \mu(n_x - 1)).$$

Proof. The proof follows by bounding the off-diagonal entries of $\mathbf{A}^*\mathbf{A}$ with the coherence parameter μ and applying Geršgorin's disk theorem [13, Thm. 6.1.1]. $\square$

Proof of Theorem 3. Set $\boldsymbol{h} \triangleq \hat{\boldsymbol{x}} - \boldsymbol{x}$, and let $\mathcal{X}$ be the support of the best n_x-sparse approximation to the signal $\boldsymbol{x}$. The goal is to bound $\|\boldsymbol{h}\|_2$. Write $\mathbf{A}^*\mathbf{A} = \mathbf{1} + \mathbf{M}$, so that $\mathbf{M}$ has a zero diagonal, then

$$\|\mathbf{A}\boldsymbol{h}\|_2^2 = \|\boldsymbol{h}\|_2^2 + \boldsymbol{h}^*\mathbf{M}\boldsymbol{h} \geqslant \|\boldsymbol{h}\|_2^2 - \mu(\|\boldsymbol{h}\|_1^2 - \|\boldsymbol{h}\|_2^2).$$

Rearranging this result leads to

$$\|\boldsymbol{h}\|_2^2 \leqslant \frac{\|\mathbf{A}\boldsymbol{h}\|_2^2 + \mu\|\boldsymbol{h}\|_1^2}{1+\mu} \overset{(8)}{\leqslant} \frac{\gamma_p^2\|\mathbf{A}\boldsymbol{h}\|_p^2 + \mu\|\boldsymbol{h}\|_1^2}{1+\mu}, \quad (10)$$

with $\gamma_p = m^{\frac{1}{2}-\frac{1}{p}}$. First we have the so-called *tube* constraint

$$\|\mathbf{A}\boldsymbol{h}\|_p \leqslant \|\mathbf{A}\boldsymbol{x} - \boldsymbol{z}\|_p + \|\mathbf{A}\hat{\boldsymbol{x}} - \boldsymbol{z}\|_p \leqslant \eta + \varepsilon_p. \quad (11)$$

Since $\hat{\boldsymbol{x}}$ is a minimizer, we have

$$\begin{aligned}
\|\boldsymbol{x}\|_q^q = \|\boldsymbol{x}_{\mathcal{X}^c}\|_q^q + \|\boldsymbol{x}_{\mathcal{X}}\|_q^q &\geqslant \|\hat{\boldsymbol{x}}\|_q^q \\
&= \|\boldsymbol{h}_{\mathcal{X}^c} + \boldsymbol{x}_{\mathcal{X}^c}\|_q^q + \|\boldsymbol{h}_{\mathcal{X}} + \boldsymbol{x}_{\mathcal{X}}\|_q^q \\
&\overset{(9c)}{\geqslant} \|\boldsymbol{h}_{\mathcal{X}^c}\|_q^q - \|\boldsymbol{x}_{\mathcal{X}^c}\|_q^q - \|\boldsymbol{h}_{\mathcal{X}}\|_q^q + \|\boldsymbol{x}_{\mathcal{X}}\|_q^q,
\end{aligned}$$

which corresponds to the *cone* constraint and

$$2\|\boldsymbol{x}_{\mathcal{X}^c}\|_q^q + 2\|\boldsymbol{h}_{\mathcal{X}}\|_q^q \geqslant \|\boldsymbol{h}_{\mathcal{X}^c}\|_q^q + \|\boldsymbol{h}_{\mathcal{X}}\|_q^q = \|\boldsymbol{h}\|_q^q. \quad (12)$$

With (9b), we arrive at

$$2^{\frac{2}{q}-1}\left(\|\boldsymbol{x}_{\mathcal{X}^c}\|_q + \|\boldsymbol{h}_{\mathcal{X}}\|_q\right) \geqslant \|\boldsymbol{h}\|_q. \quad (13)$$

To bound $\|\boldsymbol{h}_{\mathcal{X}}\|_q$ set $\kappa_x = \sqrt{1+\mu(n_x-1)}$ and $\theta_x = n_x^{1/q-1/2}/(1-\mu(n_x-1))$, which enables us to write

$$\begin{aligned}
\|\boldsymbol{h}_{\mathcal{X}}\|_2\|\boldsymbol{h}_{\mathcal{X}}\|_q &\overset{(9a)}{\leqslant} n_x^{\frac{1}{q}-\frac{1}{2}}\|\boldsymbol{h}_{\mathcal{X}}\|_2^2 \leqslant \theta_x\|\mathbf{A}\boldsymbol{h}_{\mathcal{X}}\|_2^2 \\
&= \theta_x(\mathbf{A}\boldsymbol{h})^*(\mathbf{A}\boldsymbol{h}_{\mathcal{X}}) - \theta_x(\mathbf{A}\boldsymbol{h}_{\mathcal{X}^c})^*(\mathbf{A}\boldsymbol{h}_{\mathcal{X}}) \\
&\leqslant \theta_x\|\mathbf{A}\boldsymbol{h}\|_2\|\mathbf{A}\boldsymbol{h}_{\mathcal{X}}\|_2 + \theta_x\mu\|\boldsymbol{h}_{\mathcal{X}^c}\|_1\|\boldsymbol{h}_{\mathcal{X}}\|_1 \\
&\overset{(9a)}{\leqslant} \theta_x\|\boldsymbol{h}_{\mathcal{X}}\|_2\left(\kappa_x\gamma_p(\eta+\varepsilon_p) + \mu\sqrt{n_x}\|\boldsymbol{h}_{\mathcal{X}^c}\|_q\right). \quad (14)
\end{aligned}$$

Now, we can use (12) and (9b) to bound $\|\boldsymbol{h}_{\mathcal{X}^c}\|_q$

$$\|\boldsymbol{h}_{\mathcal{X}^c}\|_q \leqslant 2^{\frac{1}{q}-1}\left(2^{\frac{1}{q}}\|\boldsymbol{x}_{\mathcal{X}^c}\|_q + \|\boldsymbol{h}_{\mathcal{X}}\|_q\right).$$

Combining with (14) and (9a) we arrive at

$$(1-\theta_x\mu\alpha_q n_x^{\frac{1}{2}})\|\boldsymbol{h}_{\mathcal{X}}\|_q \leqslant \theta_x(\kappa_x\gamma_p(\eta+\varepsilon_p) + \mu\alpha_q n_x^{\frac{1}{2}}\|\boldsymbol{x}_{\mathcal{X}^c}\|_q).$$

If we assume that $1 - \theta_x\mu 2^{\frac{1}{q}-1} > 0$, that is

$$\frac{1}{\mu} + 1 > (n_x)^{\frac{1}{q}}2^{\frac{1}{q}-1} + n_x = n_x\left((2n_x)^{\frac{1}{q}-1} + 1\right)$$

which implies (5). We obtain the desired result by combining

the last inequality with (10), (11), and (13), and with constants

$$C_0 = \frac{1}{\sqrt{1+\mu}} + \frac{2^{\frac{1}{q}}\alpha_q\sqrt{\mu n_x}\sqrt{1+\mu(n_x-1)}}{\sqrt{1+\mu}(1-\mu((1+\alpha_q\beta_q)n_x-1))}$$

$$C_1 = \frac{\alpha_q\sqrt{\mu}\left[\mu n_x\left(\alpha_q 2^{\frac{1}{q}}-1-\alpha_q\beta_q\right)+1+\mu\right]}{\sqrt{1+\mu}\left(1-\mu((1+\beta_q\alpha_q)n_x-1)\right)},$$

where $\alpha_q = 2^{1/q-1}$ and $\beta_q = n_x^{1/q-1}$. $\qquad\square$

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