Covariant Time-Frequency Distributions Based on Conjugate Operators

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Abstract— We propose classes of quadratic time-frequency distributions that retain the inner structure of Cohen’s class. Each of these classes is based on a pair of “conjugate” unitary operators satisfying time-frequency displacements. The classes satisfy covariance and marginal properties corresponding to these operators. For each class, we define a “central member” generalizing the Wigner distribution and the $Q$-distribution, and we specify a transformation by which the class can be derived from Cohen’s class.

I. INTRODUCTION

COHEN’S class with signal-independent kernels (Cohen’s class hereafter) consists of all quadratic time-frequency representations (QTFR)’s $T_g(t, f)$ that are covariant to time-frequency shifts $T_{g,x}(t, f) = T_g(t - x, f - v)$ [1]–[3]. Here, $x(t)$ is a signal with Fourier transform $X(f) = \int x(t) e^{-j2\pi ft} dt$, and $S_{x,v} = F_1 T_T$ with the time-shift operator $(T_T x)(t) = x(t-\tau)$. The properties of the operators $T_T$ and $F_1$ are characteristic of a commutative structure. In this letter, this structure will be worked out in a generalized framework. We construct QTFR classes that are based on pairs of “conjugate” operators and that satisfy generalized covariance and marginal properties [4], [5]. Due to space limitations, we summarize our results without providing proofs. The concept of conjugate operators has been developed independently in [6] and [7].

II. CONJUGATE OPERATORS

We consider two operators $A_\alpha$ and $B_\beta$ indexed by parameters $\alpha \in G$ and $\beta \in G$ with $G \subseteq \mathbb{R}$. They are assumed to be unitary on a linear signal space $X \subseteq L^2(\mathbb{R})$, and to satisfy identical composition laws $A_{\alpha_1} A_{\alpha_2} = A_{\alpha_1 \alpha_2}$ and $B_{\beta_1} B_{\beta_2} = B_{\beta_1 \beta_2}$ where $(G \bullet *)$ is a commutative group [4], [8], [9]. The eigenvalues $\lambda_{a,b}^A$ and eigenfunctions $\psi_{a,b}^\ast(t)$ of $A_\alpha$ are defined by $A_\alpha \psi_{a,b}^\ast(t) = \lambda_{a,b}^A \psi_{a,b}^\ast(t)$; they are indexed by a “dual parameter” $\alpha$. The A-Fourier transform (A-FT) [8] is defined as $X_A(\hat{\alpha}) = (F_A x)(\hat{\alpha}) = \frac{1}{\sqrt{2\pi}} \int x(t) \psi_{a,b}^\ast(t) dt$. Analogous definitions apply to $\lambda_{a,b}^B$, $\psi_{a,b}^\ast(t)$, and $X_B(\hat{\beta}) = (F_B x)(\hat{\beta})$. We now assume that applying one operator to an eigenfunction of the other operator merely shifts the eigenfunction parameter [4], [5].

Definition 1. Two operators $A_\alpha$ and $B_\beta$ as described above will be called conjugate if $\alpha \in G$, $\beta \in G$ and

\[
\{B_\beta \psi_{a,b}^\ast(t)\} = \psi_{a,b}^\ast(\xi(t)), \quad (A_\alpha \psi_{a,b}^\ast(t)) = \psi_{a,b}^\ast(\eta(t)).
\]

Two conjugate operators $A_\alpha$, $B_\beta$ can be shown to satisfy the following remarkable properties [4]:

1) Their eigenvalues can be written as $\lambda_{a,b}^A = e^{j2\pi \mu(a) \mu(\beta)}$ and $\lambda_{a,b}^B = e^{j2\pi \mu(a) \mu(\beta)} = \lambda_{a,b}^A^*$. Here, $\mu(g) \in \mathbb{R}$ maps $(G \bullet *)$ onto $(\mathbb{R}, +)$ in the sense that $\mu(g_1 \bullet g_2) = \mu(g_1) + \mu(g_2)$, $\mu(g_0) = 0$, and $\mu(g^{-1}) = -\mu(g)$, where $g_0$ is the identity element in $G$ and $g^{-1}$ denotes the group-inverse of $g$. In the following, we shall simply write $\lambda_{a,b}^A = \lambda_{a,b}$, $\lambda_{a,b}^B = \lambda_{a,b}^\ast$.

2) They commute up to a phase factor, $A_\alpha B_\beta = \lambda_{a,b} B_\beta A_\alpha$.

3) Their eigenfunctions are related as $\langle \psi_{a,b}^\ast(t) \rangle = \lambda_{a,b} \psi_{a,b}^\ast(t)$, $\int \psi_{a,b}^\ast(t) \psi_{a,b}^\ast(t') = \delta(\mu(a) \mu(b), \mu(a) \mu(b)) = \int \psi_{a,b}^\ast(t) \psi_{a,b}^\ast(t') dt'$, where $\delta(\cdot) = \delta(\mu(\alpha) \mu(\beta))$ denotes the Dirac delta function (cf. [10]).

4) The inner product of their kernels is $\int \int A_\alpha B_\beta \delta(\beta, \beta') = \delta(\mu(a) \mu(b), \mu(a) \mu(b)) = \int \psi_{a,b}^\ast(t) \psi_{a,b}^\ast(t') dt'$, and they are related as $X_B(\hat{\beta}) = \int X_A(\hat{\alpha}) \lambda_{a,b} \psi_{a,b}^\ast(\hat{\beta}) dt$ and $A_\alpha B_\beta = \int A_\alpha B_\beta \delta(\beta', \beta') dt$. [6].

We now compose two conjugate operators $A_\alpha$, $B_\beta$ and $D_\theta = B_\beta A_\alpha$ where $\theta = (\alpha, \beta) \in G^2$ with $G^2 = \tilde{G} \times \tilde{G}$. It is readily shown that $D_\theta$ is unitary on $X$ and satisfies the composition property [4], [11] $D_{\theta_1} D_{\theta_2} = \lambda_{\alpha_1, \alpha_2} D_{\beta_1, \beta_2}$ where $(G^2, \circ)$ is the commutative group with group operation $\theta_1 \circ \theta_2 = (\alpha_1, \beta_1) \circ (\alpha_2, \beta_2) = (\alpha_1 \circ \alpha_2, \beta_1 \circ \beta_2)$, identity element $\theta_0 = (g_0, g_0)$, and inverse elements $\theta^{-1} = (\alpha^{-1}, \beta^{-1})$.

Furthermore, $D_{\theta^{-1}} = \lambda_{a,b} D_{\theta^{-1}}$ and $D_{\theta} = I$ where $I$ is the identity operator on $X$.

Examples. The shift operators $T_T$, $F_1$ underlying Cohen’s class are conjugate with $(G \bullet *) = (\mathbb{R}, +)$, $\mu(g) = g$, eigenvalues $\lambda_{a,b} = e^{-j2\pi \mu(a) \mu(\beta)}$, $\lambda_{a,b}^* = e^{-j2\pi \mu(a) \mu(\beta)}$, eigenfunctions $\psi_{a,b}^\ast(t) = e^{j2\pi \mu(a) \mu(\beta)}$, and dual parameters $\tau = f = \nu$. The operators are conjugate since $\{F_1 \psi_{a,b}^\ast(t)\} = \psi_{a,b}^\ast(\tau(t))$ and $(T_T \psi_{a,b}^\ast(t)) = \psi_{a,b}^\ast(t)$. The operators underlying the hyperbolic QTFR’s class [12] are conjugate as well, but the operators underlying the affine class and the power classes [13]–[15] are not conjugate.

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III. COVARIANCE AND MARGINAL PROPERTIES

Let $u_3(t)$ and $\tau_3(f)$ denote the instantaneous frequency and group delay of the eigenfunctions $u_3(t)$ and $u_3(t)$, respectively. For any $\tilde{\beta} = (\tilde{\alpha}, \tilde{\beta}) \in \mathbb{G}^2$, the corresponding functions $u_3(t)$ and $\tau_3(f)$ are assumed to intersect in a unique time-frequency (TF) point $z = (t, f)$. Hence, $z = (\tilde{\theta})$ where $\tilde{\theta}$ will be called the localization function (LF) of the operator $D_0$. The LF is constructed by solving the system of equations $u_3(t) = f$, $\tau_3(f) = t$ for $(t, f) = z$. It is assumed to be invertible, i.e., $z = (\tilde{\theta}) = \tilde{\theta} = l^{-1}(z)$.

The LF describes the TF displacements caused by $D_0$. If a signal $x(t)$ is localized about a TF point $z = (t, f)$, then $(D_0 x)(t)$ will be localized about a new TF point $z' = (t', f')$. Since $z$ is the intersection of $u_3(t)$ and $u_3(t)$ with $(\tilde{\alpha}, \tilde{\beta}) = \tilde{\theta} = l^{-1}(z)$, $z'$ will be the intersection of $(D_0 u_3(t))$ and $(D_0 u_3(t))$. Due to the conjugate nature of $A_\alpha$ and $B_\beta$, $(D_0 u_3(t)) = \lambda_\alpha,\beta u_30,\beta(t)$ and $(D_0 u_3(t)) = \lambda_\beta,\beta,-\alpha0,\beta(t)$. Hence, $z' = (\tilde{\theta} \cdot \tilde{\alpha}, \tilde{\beta} \cdot \tilde{\alpha}) = l(\tilde{\theta} \cdot \tilde{\alpha}) = l^{-1}(z)$ with $\tilde{\theta} = (\beta, \alpha)$. This motivates the following definition.

**Definition 2.** A QTFR $T_a(z) = T_a(t, f)$ is called covariant to an operator $D_0$ if

$$T_{D_0}(z) = T_a\left(l^{-1}(z) \cdot \theta^{-1}\right)$$

with $\theta^{-1} = (\theta^{-1})^{-1} = (\beta^{-1}, \alpha^{-1})$.

The class of all QTFR's covariant to $D_0$ is characterized as follows (cf. [4], [11]):

**Theorem 1.** A QTFR $T_a(z) = T_a(t, f)$ is covariant to an operator $D_0$ if and only if

$$T_a(z) = \langle x, H^a x \rangle$$

$$= \int \int x(t_1, t_2) x^*(t_2) h_a^2(t_1, t_2, 1) dt_1 dt_2$$

with $H^a = D_0^{-1}(z)H D_0^{-1}(z)\tau_f$, i.e. $h_a^2(t_1, t_2) = \int_1 \int_1 D_0^{-1}(z)\tau_f(t_1, t_2) h(t_1, t_2) D_0^{-1}(z)\tau_f(t_1, t_2) dt_1 dt_2$. Here, $H$ is an arbitrary operator with kernel $h(t_1, t_2)$, assumed independent of $z(t)$, and $D_0(t_1, t_2)$ and $D_0^{-1}(t_1, t_2)$ are the kernels of $D_0$ and $D_0^{-1}$, respectively.

For given operator $D_0$, (2) defines a class of QTFR's parameterized by the 2-D kernel $h(t_1, t_2)$ of the operator $H$.

This class consists of all QTFR's satisfying the covariance (1). For $D_0 = S_{\mu, \nu} = F_{\mu} F_{\nu}$, (1) becomes the TF shift covariance $T_{S_{\mu, \nu}}(z) = T_{\nu}(t - \mu, f - \nu)$ and (2) becomes Cohen's class.

Besides the covariance property (1), the marginal properties

$$\int_0 T_a(l(\tilde{\theta})) \mu(\tilde{\alpha}) = \left| X_\alpha(\tilde{\alpha}) \right|^2;$$

$$\int_0 T_a(l(\tilde{\theta})) \mu(\tilde{\beta}) = \left| X_\beta(\tilde{\beta}) \right|^2$$

are of importance. A class of QTFR's satisfying (3) is

$$\tilde{T}(z) = \int_0 \Psi(\theta) A_{\theta}^2(\theta) \Lambda(l^{-1}(z), \theta) \mu(\theta)$$

where $\Lambda(\tilde{\theta}, \theta) = \lambda_{\alpha, \beta} \lambda_{\alpha, \beta}^*$, $A_{\theta}^2(\theta) = \langle D_{\theta} x, D_{\theta} x \rangle$ (the characteristic function), $\mu(\theta) = \mu(\alpha) \mu(\beta)$, and $\Psi(\theta) = \Psi(\alpha, \beta)$ is a kernel (assumed independent of $x(t)$) satisfying $\Psi(\alpha, 0) = \Psi(0, \beta) = 1$ [4], [8], [17]. In the case of the conjugate operators $T_\nu$ and $F_{\mu}$, the marginal properties (3) become $\int_0 T_a(l(\tilde{\theta})) dt = \left| X(f) \right|^2$ and $\int_0 T_a(l(\tilde{\theta})) df = \left| x(t) \right|^2$, $A_{\theta}^2(\theta) = A_\alpha^2(\nu, \nu)$ becomes the symmetric ambiguity function [3], and the QTFR class (4) becomes Cohen's class.

So far, we have formulated the QTFR class $T = \{T_a(z)\}$ in (2) comprising all QTFR's satisfying the covariance property (1), and the QTFR class $\tilde{T} = \{\tilde{T}(z)\}$ in (4) related to the marginal properties (3). These classes are equivalent in the conjugate case [4], [5]:

**Theorem 2.** For conjugate operators $A_\alpha, B_\beta$, there is $T = \tilde{T}$ or equivalently $T_a(z) = \tilde{T}_a(z)$ where the kernel $h(t_1, t_2)$ of $T_a(z)$ and the kernel $\Psi(\theta)$ of $\tilde{T}_a(z)$ are related as $h(t_1, t_2) = \int_0 \tilde{T}_a(l(\tilde{\theta})) dt_1 dt_2$.

Hence, in the conjugate case considered, the "covariance approach" and the "characteristic function approach" to the construction of QTFR classes are fully equivalent.

With $\Psi(\theta) \equiv 1$, the "central member" $W_a^2(z) = \int_0 W_a^2(\theta) \Lambda(l^{-1}(z), \theta) \mu(\theta)$ of the QTFR class $T = \tilde{T}$ is obtained [5], [18]. It can be expressed as

$$W_a^2(z) = \int \int X_\alpha^2(\beta) X_\alpha^2(\beta) \lambda_{\alpha, \beta}^2(\theta) \mu(\beta)$$

$$= \int_0 X_\alpha^2(\beta) \mu(\alpha) \mu(\beta)$$

with $\lambda_{\alpha, \beta}(\theta) = l^{-1}(z)$. Any QTFR $T_a(z)$ of $T = \tilde{T}$ can be derived from $W_a^2(z)$ as

$$T_a(z) = \int_0 W_a^2(l(\tilde{\theta})) \psi(l^{-1}(z) \cdot \theta^{-1}) \mu(\theta)$$

with $\psi(\theta) = \int_0 \Psi(\theta) \Lambda(\tilde{\theta}, \theta) \mu(\theta)$. In the special cases of Cohen's class and the hyperbolic class, the central member becomes the Wigner distribution and the Q-distribution, respectively [3], [12].

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1. In certain cases, this assumption holds if one uses the group delay of $u_3(t)$ and the instantaneous frequency of $u_3(t)$; here, an analog theory can be formulated.

2. $z$ is the intersection of $u_3(t)$ and $u_3(t)$ in the sense that $u_3(t)$ and $u_3(t)$ are concentrated, in the TF plane, along $u_3(t)$ and $\tau_3(f)$, respectively, and $z$ is the intersection of $u_3(t)$ and $\tau_3(f)$.

3. We note that $\theta^{1/2}$ is defined by $\theta^{1/2} = \theta^{1/2} \cdot \theta^{1/2} = \theta$, and that $\lambda_{\alpha, \beta}^{1/2} = (\lambda_{\alpha, \beta}^{1/2}) \mu(\alpha) \mu(\beta)$.
IV. TRANSFORMATION APPROACH, A FACT LINKING OUR THEORY TO THE "WARPING" THEORY IN THE OPERATORS COMMUTATIVE GROUP OTHER SPACE AND CLASS REFERENCE TIME CONSTANT.

A corresponding to function can be derived independently associated to be isometric isomorphisms on signal space and isometric isomorphism mapping the dual parameters (*). Thus, isometric isomorphisms V and one-to-one group transformations s(·) preserve the conjugateness property of two operators. The following theorem states that any QTFR class T = T corresponding to conjugate operators Aα, Bβ can be derived from Cohen's class using a transformation. Similar results have been derived independently in [6, 7].

Theorem 3: Let Aα, Bβ be conjugate with group (G, •) corresponding to function μ(·), so that λαβ = e^{2πfμ(α)(μ(β))}. If λαβ = e^{-2πfμ(α)(μ(β))} (− sign), then Aα = V T_{β,μ(α)}V^{-1} and Bβ = V F μ(β)/t V^{-1}, where t_0 > 0 is an arbitrary reference time constant, and (V_{β}^{-1})(t) = \frac{t_0}{t_0 - \tau} X_{β}(μ^{-1}(\frac{t}{t_0})) with μ^{-1}(·) denoting the function inverse to μ(·). Furthermore, any QTFR T_σ(z) = T_σ(t, f) of the QTFR class T = T associated to Aα, Bβ can be derived from a corresponding QTFR C_σ(t, f) of Cohen's class as

T_σ(z) = C_σ^{-1} \left( t_0, \mu(\tilde{\beta}), \frac{\mu(\tilde{\alpha})}{t_0} \right) |_{\tilde{t} = t^{-1}(z)}

where \tilde{t}^{-1}(·) is the inverse LF of D_θ = BβAα. If λαβ = e^{2πfμ(α)(μ(β))} (+ sign), then the above relations have to be replaced by Aα = V F μ(α)/t V^{-1} and Bβ = V T_{β,μ(α)}V^{-1}, (V_{β}^{-1})(t) = \frac{t_0}{t_0 - \tau} X_{β}(μ^{-1}(\frac{t}{t_0})) , and T_σ(z) = C_σ^{-1} \left( t_0, \mu(\tilde{\beta}), \frac{\mu(\tilde{\alpha})}{t_0} \right) |_{\tilde{t} = t^{-1}(z)}.

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