Diplomarbeit

Gabor Expansion and Frame Theory

ausgeführt am Institut für
Nachrichtentechnik und Hochfrequenztechnik
der Technischen Universität Wien

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durch
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Wien, im September 1994

To my parents

ABSTRACT

This thesis treats the Gabor expansion, one of the major linear time-frequency signal representations. The Gabor expansion, proposed by D. Gabor in 1946, is the decomposition of a signal into a set of time-shifted and modulated versions of an elementary window function. A significant part of this work is devoted to the theory of frames, which constitutes the mathematical background for the Gabor expansion.

We provide a general treatment of the Gabor expansion for arbitrary windows and arbitrary sampling densities in the time-frequency plane. First of all, a detailed presentation of the Zak transform and the theory of frames, mathematical tools necessary for a discussion of the Gabor expansion, is given. The Zak transform is a linear signal representation that is of fundamental theoretical and practical importance in the context of the Gabor expansion. The theory of frames in general, and Weyl-Heisenberg frames in particular, yields important information about the mathematical properties of the Gabor expansion (existence of the expansion, uniqeness of the expansion coefficients, numerical properties). Using the Zak transform and results from frame theory, we finally consider methods for the calculation of the Gabor coefficients.

KURZFASSUNG

Diese Diplomarbeit behandelt die Gabor-Entwicklung, eine der wichtigsten linearen Zeit-Frequenz-Signaldarstellungen. Die Gabor-Entwicklung wurde 1946 von D. Gabor eingeführt. Sie beschreibt die Zerlegung eines beliebigen Signals in Zeit-Frequenz-verschobene Versionen einer elementaren Fensterfunktion. Ein beträchtlicher Teil dieser Arbeit ist der Theorie der Frames gewidmet, welche den mathematischen Hintergrund für die Gabor-Entwicklung darstellt.

Wir geben eine allgemeine Abhandlung über die Gabor-Entwicklung, die beliebige Fensterfunktionen und beliebige Abtastraten in der Zeit-Frequenz-Ebene zuläßt. Zunächst werden die Zak-Transformation und die Frame-Theorie eingehend diskutiert. Diese mathematischen Werkzeuge sind notwendig für ein tieferes Verständnis der Gabor-Entwicklung. Die Zak-Transformation ist eine lineare Signaltransformation von grundlegender theoretischer und praktischer Bedeutung für die Gabor-Entwicklung. Die Frame-Theorie im allgemeinen und die Theorie der Weyl-Heisenberg-Frames im besonderen liefern wichtige Informationen über die mathematischen Eigenschaften der Gabor-Entwicklung (Existenz der Entwicklung, Eindeutigkeit der Koeffizienten, numerische Eigenschaften). Im letzten Abschnitt betrachten wir Methoden zur Berechnung der Gabor-Koeffizienten, wobei wir intensiv von der Zak-Transformation und der Frame-Theorie Gebrauch machen.

Acknowledgments

I would like to thank Dr. F. Hlawatsch for his continuous support and guidance during my research. He always had time for discussions, offered me valuable suggestions for my work, and helped to greatly improve this thesis. Furthermore, I am grateful to Prof. H. G. Feichtinger for many useful remarks and for the time he spent in countless discussions concerning my work. I am also indebted to Dipl.-Ing. W. Kozek for his suggestions and advice. Finally, I would like to express my thanks to Prof. W. Mecklenbräuker for his interest and encouragement.

Contents

1 Introduction

Time-frequency (TF) methods for signal processing have attracted substantial interest during the past few years [1]-[4]. They are especially useful for analyzing and processing nonstationary signals and for analyzing and implementing time-varying systems. Traditionally, a signal is represented either in the time domain or (using the Fourier transform) in the frequency domain. Unfortunately, the usual frequency domain description does not generally yield a readable characterization of the temporal evolution of a signal's spectral content. This motivates the use of *TF representations* which display the time-varying spectral features over a joint TF plane. TF methods have found widespread use in many signal processing applications.

There are many different TF representations, which are based on a variety of different notions of how to arrive at a joint TF description. However, a first distinction which has proved useful is between *linear* and *quadratic* TF representations. Examples of linear TF representations are the *short-time Fourier transform*, the *Gabor expansion*, and the *wavelet transform*. TF representations with quadratic structure are the *Wigner distribution*, the *Rihaczek distribution*, the *spectrogram*, the *ambiguity function*, and many others. Finally, there are TF representations which are neither linear nor quadratic, but are nonetheless related conceptually to linear or quadratic methods. This is especially true for various *signal-adaptive* TF representations.

This thesis treats the Gabor expansion (GE), an important linear TF representation. The GE has first been proposed by D. Gabor in 1946. In his paper Gabor suggests to decompose an arbitrary signal into a set of time-shifted and modulated Gaussian elementary signals. Gabor restricted himself to Gaussian windows and to the so-called case of 'critical sampling.' But his expansion also works for other windows and 'oversampling.'

A nontrivial problem with the GE is the calculation of the expansion coefficients (Gabor coefficients), since the set of time-shifted and modulated functions used for the decomposition is generally not an orthonormal set of functions (not even orthogonal). In 1980, M.J. Bastiaans suggested a method for the calculation of the Gabor coeffcients. His method is based on the so-called biorthogonality of two sets of functions.

Later it was recognized that the theory of *frames* is a powerful tool in the context of the GE. This theory, first introduced by R.J. Duffin and A.C. Schaeffer in 1952, allows the formulation of fundamental theorems on the existence of the GE and the uniqueness of the Gabor coefficients. Furthermore it provides useful measures for the numerical properties of the expansion. Bastiaans' approach to the calculation of the Gabor coefficients can be verified by the theory of frames. The type of frames which corresponds to the GE is called Weyl-Heisenberg frame. A Weyl-Heisenberg frame is a generally nonorthogonal set of functions obtained from a prototype function by time-shifts and modulations.

A linear signal transformation coming from quantum mechanics, called the Zak transform, is of fundamental importance in the context of Weyl-Heisenberg frames and hence the GE. This signal transformation has been introduced by J. Zak in solid state physics and has become popular in a signal theoretic context during the last few years. A.J.E.M. Janssen has been the first to systematically study the Zak transform from a signal processing point of view.

This thesis is organized as follows. First of all, a general survey of linear TF representations is given in Section 2. In Section 3, the Zak transform is studied. Section 4 is devoted to the general theory of frames. In Section 5 we discuss Weyl-Heisenberg frames. Section 6 treats the GE, making extensive use of the Zak transform and results from frame theory as discussed in Sections 3-5.

2 Survey of Linear Time-Frequency Representations

During the past few years, *linear TF representations* have become increasingly important for signal and image coding, feature extraction, and numerous other applications. They have also furnished a new interpretation of some traditional methods such as filter bank analysis/synthesis techniques and Laplacian pyramids. Since this thesis treats the Gabor expansion, an important linear TF representation, we shall give a survey of the major linear TF methods proposed so far.

2.1 The Short-Time Fourier Transform

The *short-time Fourier transform* (STFT) [5]-[10], which has been introduced by D. Gabor in [11], gives a picture of the local frequency content of the signal under analysis. The STFT can be interpreted in three different ways:

- At a specific time, the STFT is the Fourier transform of the signal multiplied by a local analysis window function.
- At a specific frequency, the STFT is essentially the output signal of a bandpass filter centered at the respective frequency.
- At a specific point in the TF plane, the STFT is the inner product of the signal with a version of an analysis window TF-shifted to the respective TF point.

The STFT result for a given signal depends on the choice of the analysis window. Good time resolution requires a narrow analysis window whereas good frequency resolution requires a narrowband (and thus long) analysis window. The uncertainty principle prohibits the existence of windows with arbitrarily small time duration and arbitrarily small bandwidth. Hence, the STFT suffers from a fundamental resolution tradeoff and resolution limitation. Once an analysis window is chosen, the time and frequency resolutions are fixed over the entire TF plane. This is an important difference from the wavelet transform to be discussed in Section 2.3.

A discretized version of the STFT can be implemented very efficiently using fast Fourier transform (FFT) techniques. It is also possible to recover the signal from its STFT ("STFT synthesis") [8]. Some signal processing schemes consist of STFT analysis, modification, and synthesis steps [6, 9].

The STFT has found widespread use in signal processing. Some of its applications are the analysis of time-varying signals [5, 6, 12], system identification and spectral estimation [13, 14], signal detection and parameter estimation [15]-[18], speech processing [6],[19]-[21] and nonlinear noise removal [22].

The STFT is very closely related to the *Gabor expansion* (GE) discussed in the next section. To a large extent, the GE can be considered a discretized STFT. However, the emphasis in the context of the GE is placed on the "signal expansion" aspect, which is largely equivalent to STFT synthesis.

2.2 The Gabor Expansion

In 1946, D. Gabor suggested to expand signals into TF shifted versions of an elementary Gaussian function [11]. The *Gabor expansion* (GE) [11],[23]-[26] provides a *partitioning (tiling) of the TF plane* into rectangles of equal size, as shown in *Fig. 1*. The expansion coefficients (Gabor coefficients) represent the signal's local content about the respective TF grid point. Gabor suggested a so-called *critical density* of the TF lattice, which means that the product of the time-shift parameter T and the frequency-shift parameter F is equal to 1. Subsequently, the GE was generalized to other elementary signals (besides the Gaussian) [24] and other lattice densities.

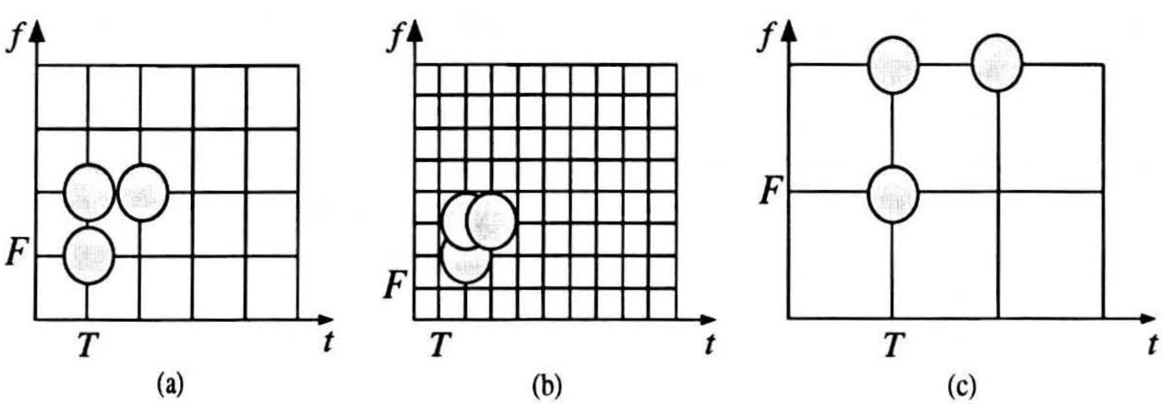


Fig. 1. Tiling of the time-frequency plane provided by the Gabor expansion. (a) Critical sampling, (b) oversampling, and (c) undersampling.

Since the TF shifted versions of the elementary signal do not generally constitute an orthonormal basis, the question of how to obtain the Gabor coefficients and related questions concerning the existence and uniqueness of the GE are rather involved. M.J. Bastiaans [25] suggested the use of *biorthogonal functions* for calculating the Gabor coefficients. This approach has been extended to the discrete-time case by J. Wexler and S. Raz in [26]. Later it was recognized that a systematic study of the properties of the GE requires the mathematical concept of *frames*. A frame is a complete but generally nonorthogonal set of functions with pleasing numerical properties. This concept, first introduced by R.J. Duffin and A.C. Schaeffer [27] and worked out by I. Daubechies [28] and several other authors [29, 30], yields satisfactory results on the existence and uniqueness of the GE as well as the calculation of the Gabor coefficients. In particular, Bastiaans' biorthogonality approach for computing the Gabor coefficients is also obtained from frame theory, and the Gabor coefficients are always calculated as samples of an STFT. Frames of the Gabor type are called *Weyl-Heisenberg frames*. A different frame type, the class of *affine frames*, is important in the context of the wavelet transform to be discussed in Section 2.3.

Concerning the existence, uniqueness, and numerical properties of the GE, the theory of frames indicates that a distinction between the following three cases is appropriate (see Fig. 1):

- *critical sampling* where $TF = 1$,
- *oversampling* where $TF < 1$,
- *undersampling* where $TF > 1$.

Critical sampling results in poor numerical stability of the GE. In the *oversampled* case, better numerical stability is obtained at the cost of redundant and non-unique Gabor coefficients. In the case of *undersampling*, the GE will not exist for arbitrary signals.

The GE can be interpreted as a filterbank analysis. The impulse responses of the individual filters are simply modulated versions of the impulse response of an elementary lowpass filter. This type of filterbank is known as a *modulated filterbank* [31]. The resulting spectral decomposition is a "constant-bandwidth" decomposition since the filter bandwidths are all equal (*Fig. 2(a)*). This is consistent with Fig. 1 in which the frequency axis is separated into bands of equal width.

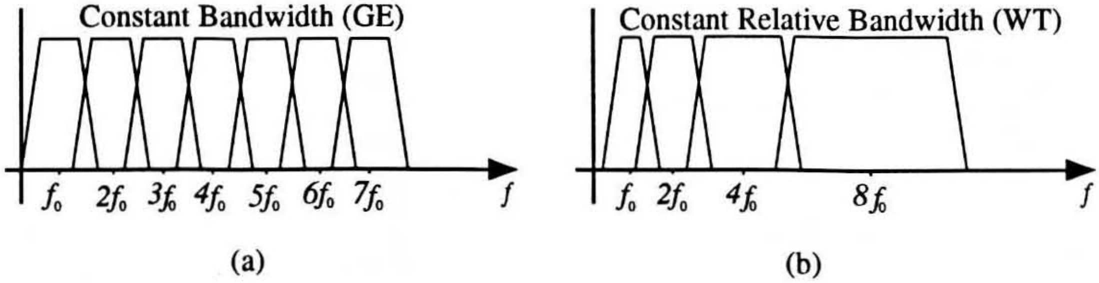


Fig. 2. Division of the frequency axis for (a) the GE and (b) the WT.

In the context of the GE, a linear signal transform known as the *Zak transform* (also called Weil-Brezin mapping or Gelfand mapping) is important for both theoretical and computational reasons. The Zak transform has been introduced by J. Zak [32] in the field of solid state physics, and studied by A.J.E.M. Janssen with a signal processing viewpoint [33]. In [34], R. Orr compares several algorithms for GE analysis and synthesis, including efficient methods based on the Zak transform. In the last few years, many other papers have discussed the implementation of the GE for 1-D and 2-D signals. Furthermore, numerous applications of the GE have been reported, such as image processing [35, 36], texture segmentation [37], the design of time-varying systems [38], and the detection of transients in noise [39].

2.3 The Wavelet Transform

During the last decade J. Morlet, A. Grossmann, Y. Meyer and others developed a time-scale representation of signals called the *wavelet transform* (WT) [28],[40]-[46]. I. Daubechies and S. Mallat were the first to establish connections to discrete-time signal processing, especially discrete-time filtering [41, 42]. Since then the number of publications on the WT has been increasing in an incredible manner.

Like the GE, the WT provides an expansion of the signal. The expansion coefficients represent the signal and can be used for signal analysis and processing. Conceptually, the WT analyzes signals at various scales and resolutions. Although originally defined as a time-scale representation, the WT provides a tiling of the TF plane analogous to the GE, and therefore can also be interpreted as a TF representation. The TF tiling corresponding to the WT is however quite different from that of the GE. Whereas the GE tiling is uniform over the entire TF plane, the WT tiling is such that higher frequencies are analyzed with better time resolution and poorer frequency resolution. Thus, the TF resolution is varied as a function of frequency. *Fig. 3* compares the TF tilings corresponding to the GE and the WT.

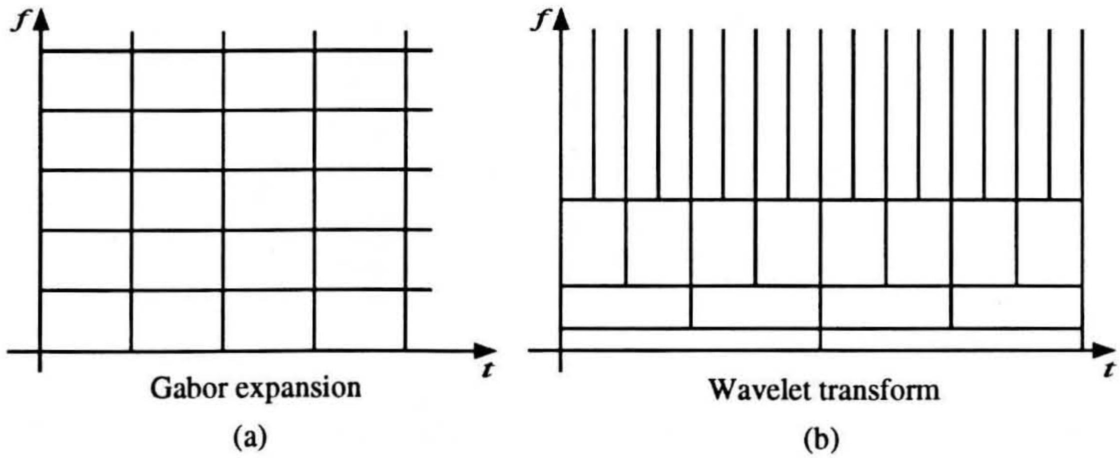


Fig. 3. Tilings of the time-frequency plane corresponding to (a) the GE and (b) the WT.

Like the GE, the WT can be interpreted as a filterbank analysis. The resulting spectral decomposition is a "constant-Q" decomposition where the filters' quality factor (Q) does not depend on the frequency (see *Fig. 2(b)*). This means that the filter bandwidths are proportional to the center frequencies, in contrast to the GE where the bandwidths are all equal.

Also for the analysis of the WT, the theory of frames is an appropriate mathematical tool. The functions into which the WT expands a given signal are all derived from an elementary function (called *mother wavelet*) by time shifts and scalings. The corresponding frame type is called an *affine frame*. The properties of affine frames are quite different from those of the Weyl-Heisenberg frames encountered in the context of the GE. Specifically, it is possible to construct affine frames where the frame signals have finite lengths, are orthogonal, and satisfy certain regularity properties corresponding to good frequency concentration. On the other hand, the calculation of the "dual frame" needed in the general (non-orthogonal) case is more expensive than for the GE.

Both the GE and the WT can also be interpreted as applying the signal to a *filter bank tree* (see *Fig. 4*). At each level, some (or all) of the frequency bands are split up into two subbands. Subband decompositions have long been used for speech compression applications [47]-[51].

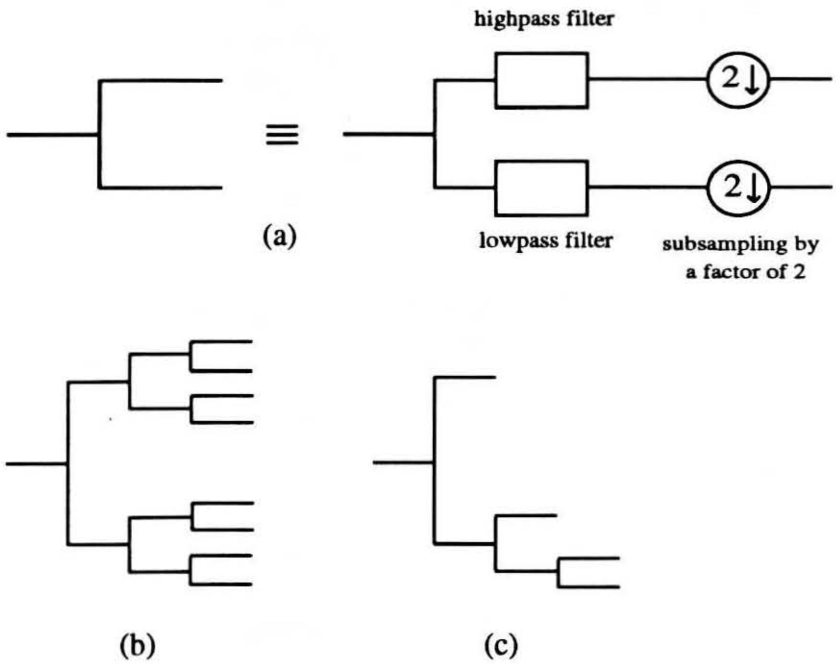


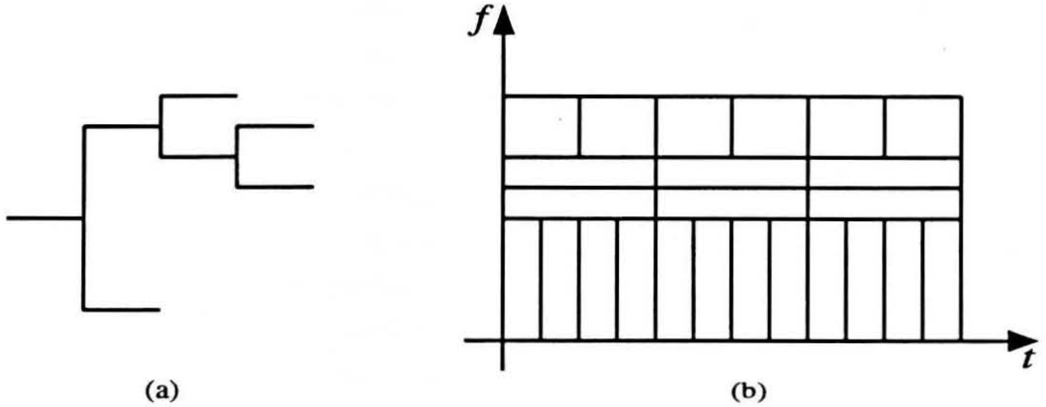
Fig. 4. Filter bank trees. (a) Elementary halfband filter structure, (b) filter bank tree corresponding to GE analysis, (c) filter bank tree corresponding to WT analysis.

Besides the analysis of nonstationary signals, a major application of the WT is the coding/compression of signals and images [52]-[58]. Other applications of the WT are speech analysis [59]-[61] and system analysis [62]. Statistical applications of the WT are the multiscale modeling of stochastic processes [63]-[65], the analysis of $1/f$ noise [66, 67], and the estimation of fractals [68]. Detailed discussions of the WT and its applications are provided in [62, 69, 70]. An excellent literature survey is [71].

2.4 The Wavepacket Transform

Both the GE and the WT correspond to specific tilings of the TF plane. The GE partitions the TF plane into rectangles of equal size while the WT provides a tiling where the tiles for higher frequencies become narrower in the time direction but wider in the frequency direction (see *Fig. 3(b)*). Recently, a generalized tiling of the TF plane was introduced by R.R. Coifman, Y. Meyer, S. Quaker, and M.V. Wickerhauser [72, 73], who called the resulting signal bases *wavepackets*. The wavepackets constitute an orthonormal basis for discrete-time functions [74]. The concept comprises all possible filter bank tree decompositions based on a dyadic partitioning. A specific wavepacket basis can be obtained by pruning the complete filter bank tree (see *Fig. 5(a)*). The individual wavepacket functions are characterized by the three parameters center time, center frequency, and scale.

In the frequency domain, a wavepacket basis corresponds to a partitioning of the frequency axis into bands with different bandwidths. The larger the bandwidth, the better is the time resolution in the respective band. This corresponds to a specific tiling of the TF plane (see *Fig. 5(b)*). Note that this tiling does not change over time.



(a) (b)

Fig. 5. (a) Example of a wavepacket tree, (b) corresponding tiling of the TF plane.

The *wavepacket transform* (WPT) of a signal is obtained by calculating the inner products of the signal with all wavepacket functions. The GE and the WT are special cases of the WPT. Just as in the case of the GE and the WT, a major application of the WPT is the coding/compression of signals and images [72, 73]. The signal or image is represented by the "significant" WPT coefficients, meaning that small WPT coefficients are discarded for the sake of data reduction. This procedure corresponds to a projection of the signal onto a subspace of lower dimension. The choice of the wavepacket basis will affect both the efficiency of the representation and the reconstruction error. An appropriate choice of the wavepacket basis is influenced by the *a priori* knowledge about the signal, such as its bandwidth or the relative importance of various frequencies. In [75], K. Ramchandran and M. Vetterli consider the best basis in a *rate-distortion* sense. They formulate a fast algorithm for finding the subtree which minimizes the global distortion for a given coding bit budget.

Further applications of the WPT are adaptive filtering [76] and the classification of transient signals in background noise [77].

2.5 Time-Varying and Non-Rectangular Tilings of the TF Plane

The tiling of the TF plane corresponding to the WPT allows a largely arbitrary segmentation of the frequency axis. However, no variation over time is allowed. Recently, *time-varying* tilings of the TF plane have been proposed in order to better match the signal's properties.

A first approach to time-varying tilings is given by the so-called "Malvar's wavelets" which correspond to signal expansions known as *lapped orthogonal transforms* [78, 79]. Here, the signal under analysis is expanded into a set of modulated window functions with varying time support. Whereas the time resolution can be chosen by adjusting the width of the wavelets, the frequency resolution is determined by the time resolution. In particular, *no frequency variation is allowed*. In fact, the Malvar tiling can be viewed as a wavepacket tiling with the roles of time and frequency interchanged. In [79], a fast Malvar transform algorithm is proposed and applied to the analysis, synthesis and compression of speech signals.

Wavepacket tilings are frequency-varying but not time-varying whereas Malvar tilings are time-varying but not frequency-varying. TF tilings that are *both* time-varying and frequency-varying are proposed by C. Herley, J. Kovačević, K. Ramchandran and M. Vetterli in [80]. Time-varying orthogonal tree structures are realized by making use of so-called *boundary filters* and *transition filters*. An algorithm for deriving an optimum TF tiling for a given signal is also presented.

Another generalization of the WP tiling is a *non-rectangular* tiling. In [81], R. Baraniuk and D.L. Jones propose orthonormal bases and frames which comprise signals corresponding to non-rectangular (skew) geometries in the TF plane. The new bases are obtained by applying unitary transformations to existing Gabor, wavelet or Wilson bases. The signal transforms obtained are specifically matched to various types of frequency-modulated signals (e.g., linear FM or "chirp" signals). Some of these transforms can be viewed as discretizations of the *metaplectic transform* [82, 83]. The metaplectic transform maps a signal into a five-dimensional domain, where the five parameters correspond to time translation, frequency translation, time-frequency scaling, shear along the frequency axis, and shear along the time axis. The GE, WT, and WPT are special cases of discretized metaplectic transforms, where three or two of the five parameters are equal to zero.

2.6 Signal-Adaptive Linear TF Representations

Substantial performance gains can be obtained by adapting certain parameters of a linear signal transform to the specific signal under analysis. It should be noted that the resulting signal transform is then no longer linear. In the following, we give a brief survey of *signal-adaptive TF representations* derived from linear TF representations.

The wavepacket method (cf. Section 2.4) selects a WP basis that is optimally adapted to the global signal properties [72, 73]. The optimality criterion is based on the entropy of the expansion coefficients. The optimum basis selection corresponding to a general TF-varying tiling of the TF plane is discussed in [80]. This basis selection algorithm is optimum in a rate-distortion sense, i.e, the optimum tiling of the TF plane minimizes the total distortion subject to a maximum bit rate constraint.

In [84], signal-dependent TF representations are discussed in which the kernel or window adapts to the signal analyzed, in the sense that a short-time quality measure of the TF representation is maximized. For example, in the STFT case the length of the analysis window is adapted to the local properties of the signal. In [85], a window matching technique for the spectrogram (the squared magnitude of the STFT) is proposed. A set of "generalized instantaneous parameters" of the analyzing window is derived and matched to the signal to be analyzed. In [86], a time-varying discrete-time WT is proposed in which certain properties of the analysis wavelet such as its regularity and frequency characteristics are changed over time. Due to the strong relationship between two-band filter banks and the WT, the time-varying discrete-time WT can be studied in the context of time-varying filter banks [87]. Two design procedures based on the theory of time-varying filter banks are a tree structure of time-varying two-band filter banks and parallel non-uniform filter banks.

In [88], the signal to be expanded is modeled as a nonstationary random process, and the GE analysis window is optimally matched to the second-order statistics of this process. The optimality criterion is the minimum mutual correlation of the Gabor coefficients. A similar window optimization is proposed for the STFT in [89].

A signal-adaptive expansion method called *matching pursuits* has been proposed by S. Mallat and Z. Zhang in [90, 91]. The signal to be analyzed is decomposed into a set of waveforms that belong to a "dictionary of functions." The waveforms actually used to represent a specific signal are selected from the dictionary to match the signal in an optimum way. For this task, a recursive strategy is proposed. A similar method has been developed independently by S. Qian and D. Chen in [92]. This method performs a decomposition of the signal into a set of Gaussian functions whose spreads and TF locations are adapted to the signal.

The matching pursuits method is closely related to projection pursuit strategies developed for statistical parameter estimation [93]. It is also similar to the construction of code-books for vector quantization [94]. The matching pursuits method is different from the methods discussed so far in that it does not, in general, correspond to a regular tiling of the TF plane, and thus makes less structural assumptions about the signal to be processed. Such flexible decompositions are advantageous for representing signals with widely varying localizations in the TF plane. However, the computational complexity of the matching pursuits technique is relatively high.

3 The Zak Transform

The Zak transform (also called Weil-Brezin mapping or Gelfand mapping) has been introduced by J. Zak [32] in the field of solid state physics, and studied by A.J.E.M. Janssen with a signal processing viewpoint [33]. It is of fundamental relevance to the Gabor expansion for both theoretical and practical reasons. On the theoretical side, the Zak transform allows the formulation of theorems on the completeness and the existence of Gabor function sets (Weyl-Heisenberg frames). On the practical side, the Zak transform is an efficient tool for the numerical calculation of the Gabor coefficients. It will be shown later that, based on the Zak transform, Gabor analysis and synthesis can be performed efficiently using FFT methods.

This section is divided into four parts. First of all, the definition of the Zak transform is given. Then its properties are studied systematically in order to provide a sound basis for further discussions. After discussing the Zak transform of some important signals, a discrete-time version of the Zak transform is considered. Most of the relations given in Sections 3.1 and 3.2 can be found in [33].

3.1 Definition and Expressions

The Zak transform (ZT) of a signal $x(t)$ is defined as [33]

$$\mathcal{Z}_x^{(T)}(t,f) = \left(\mathbf{Z}^{(T)}x\right)(t,f) = \sum_{k=-\infty}^{\infty} x(t+kT)\, e^{-2\pi jkTf}. \tag{3.1}$$

The ZT operator $\mathbf{Z}^{(T)}$ is linear and contains the *sampling period* T as parameter. In the following we will often omit the superscript (T), writing $\mathcal{Z}_x(t,f)$ instead of $\mathcal{Z}_x^{(T)}(t,f)$ and $\mathbf{Z}$ instead of $\mathbf{Z}^{(T)}$.

Frequency-Domain Expression. The ZT can be written in terms of the signal's Fourier transform

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-2\pi jft}\, dt \tag{3.2}$$

as

$$\mathcal{Z}_x(t,f) = \frac{1}{T}\, e^{2\pi jft} \sum_{k=-\infty}^{\infty} X\left(f+\frac{k}{T}\right) e^{2\pi jk\frac{t}{T}}. \tag{3.3}$$

Proof: A time function $y(t)$ and the samples of its Fourier Transform $Y(f)$ are related by the Poisson summation formula

$$\sum_{k=-\infty}^{\infty} y(t+kT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} Y\left(\frac{k}{T}\right) e^{2\pi jk\frac{t}{T}}. \tag{3.4}$$

For $y(t) = x(t)\,e^{-2\pi j\tilde{f}t}$ we have $Y(f) = X(f+\tilde{f})$. We obtain

$$\sum_{k=-\infty}^{\infty} x(t+kT)\,e^{-2\pi j\tilde{f}(t+kT)} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\tilde{f}+\frac{k}{T}\right) e^{2\pi jk\frac{t}{T}}$$

and hence

$$\sum_{k=-\infty}^{\infty} x(t+kT)\,e^{-2\pi jkT\tilde{f}} = \frac{1}{T}e^{2\pi j\tilde{f}t}\sum_{k=-\infty}^{\infty} X\left(\tilde{f}+\frac{k}{T}\right) e^{2\pi jk\frac{t}{T}}.$$

Now set $\tilde{f} = f$ and (3.3) results. □

3.2 Properties and Relations

We shall now discuss the basic properties of $\mathcal{Z}_x(t,f)$.

Periodicity. $\mathcal{Z}_x(t,f)$ is *periodic* in the *frequency* variable and *quasiperiodic* in the *time* variable,

$$\mathcal{Z}_x(t+T,f) = e^{2\pi jTf}\,\mathcal{Z}_x(t,f) \qquad (3.5)$$

$$\mathcal{Z}_x\left(t,f+\frac{1}{T}\right) = \mathcal{Z}_x(t,f). \qquad (3.6)$$

As a consequence of relations (3.5) and (3.6), it is sufficient to consider $\mathcal{Z}_x(t,f)$ on the *fundamental rectangle* $[0,T)\times[0,\frac{1}{T})$.

Symmetry Properties. Some other basic properties of the ZT are

$$\mathcal{Z}_{x-}(t,f) = \mathcal{Z}_x(-t,-f) \qquad (3.7)$$

$$\mathcal{Z}_{x^*}(t,f) = \mathcal{Z}_x^*(t,-f), \qquad (3.8)$$

where $x-$ denotes $x(-t)$ and the asterisk means conjugation. When $x(t)$ is real we have

$$x(t)\in\mathbb{R} \;\Rightarrow\; \mathcal{Z}_x^*(t,-f) = \mathcal{Z}_x(t,f). \qquad (3.9)$$

When $x(t)$ is imaginary we obtain

$$\mathcal{Z}_x^*(t,-f) = -\mathcal{Z}_x(t,f).$$

For even functions $x(t)$, the ZT satisfies

$$x(-t) = x(t) \;\Rightarrow\; \mathcal{Z}_x(-t,-f) = \mathcal{Z}_x(t,f). \qquad (3.10)$$

For odd functions $x(t)$, we obtain

$$\mathcal{Z}_x(-t,-f) = -\mathcal{Z}_x(t,f).$$

For conjugate even functions we have

$$x^*(-t) = x(t) \;\Rightarrow\; \mathcal{Z}_x^*(-t,-f) = \mathcal{Z}_x(t,f). \tag{3.11}$$

For conjugate odd functions we have

$$x^*(-t) = x(t) \Rightarrow \mathcal{Z}_x^*(-t,-f) = -\mathcal{Z}_x(t,f).$$

Scaling. For $\tilde{x}(t) = x(at)$ we get

$$\mathcal{Z}_{\tilde{x}}^{(T)}(t,f) = \mathcal{Z}_x^{(aT)}\left(at, \frac{f}{a}\right). \tag{3.12}$$

Shift Properties. The *shift properties* of the ZT yield the following results:

$$\tilde{x}(t) = x(t-\tau) \;\Rightarrow\; \mathcal{Z}_{\tilde{x}}(t,f) = \mathcal{Z}_x(t-\tau,f) \tag{3.13}$$

$$\tilde{x}(t) = x(t)\,e^{2\pi j\nu t} \;\Rightarrow\; \mathcal{Z}_{\tilde{x}}(t,f) = e^{2\pi j\nu t}\,\mathcal{Z}_x(t,f-\nu). \tag{3.14}$$

We shall specialize (3.13) and (3.14) for $\tau = kT$ and $\nu = k\frac{1}{T}$. This yields

$$\tilde{x}(t) = x(t-kT) \;\Rightarrow\; \mathcal{Z}_{\tilde{x}}(t,f) = \mathcal{Z}_x(t,f) \tag{3.15}$$

$$\tilde{x}(t) = x(t)\,e^{2\pi jk\frac{t}{T}} \;\Rightarrow\; \mathcal{Z}_{\tilde{x}}(t,f) = e^{2\pi jk\frac{t}{T}}\,\mathcal{Z}_x(t,f). \tag{3.16}$$

Marginal Properties and Inversion. We can recover $x(t)$ and $X(f)$ from $\mathcal{Z}_x(t,f)$. The *first marginal property* reads:

$$T\int_0^{1/T} \mathcal{Z}_x(t,f)\,df = x(t). \tag{3.17}$$

Proof: With (3.1), the left-hand side of (3.17) is

$$T\int_0^{1/T}\sum_{k=-\infty}^{\infty} x(t+kT)e^{-2\pi jkfT}\,df = T\sum_{k=-\infty}^{\infty} x(t+kT)\underbrace{\left[\int_0^{1/T} e^{-2\pi jkfT}\,df\right]}_{\frac{1}{T}\delta[k]} = x(t). \quad \square$$

The *second marginal property* yields the Fourier transform of $x(t)$:

$$\int_0^T \mathcal{Z}_x(t,f)\,e^{-2\pi jft}dt = X(f). \tag{3.18}$$

Proof: Use (3.3) in (3.18), and proceed as in the proof of the first marginal property (3.17). □

Unitarity. The ZT is up to a constant factor a unitary mapping from $L_2(\mathbb{R})$ onto $L_2([0,T)\times[0,\frac{1}{T}))$.

Proof: Following the approach of I. Daubechies [95], we show that the ZT maps an orthonormal basis of $L_2(\mathbb{R})$ to an orthonormal basis of $L_2([0,T) \times [0,\frac{1}{T}))$, which implies the unitarity of the ZT. Consider the signals $g_{m,n}(t) = \frac{1}{T}\, e^{2\pi j n \frac{t}{T}}\, g(t-mT)$ with $g(t) = 1$ for $0 \le t < T$ and $g(t) = 0$ otherwise. This set is easily shown to constitute an orthonormal basis for $L_2(\mathbb{R})$. Due to (3.15) and (3.16) the ZT of $g_{m,n}(t)$ is given by $\mathcal{Z}_{g_{m,n}}(t,f) = \frac{1}{T}\, e^{2\pi j n \frac{t}{T}}\, e^{-2\pi j m T f}\, \mathcal{Z}_g(t,f)$. But $\mathcal{Z}_g(t,f) = 1$ on the fundamental rectangle, so that $\mathcal{Z}_{g_{m,n}}(t,f) = e^{2\pi j n \frac{t}{T}}\, e^{-2\pi j m T f}$. The set $T\, \mathcal{Z}_{g_{m,n}}(t,f)$ is easily seen to constitute an orthonormal basis for $L_2([0,T) \times [0,\frac{1}{T}))$. □

The unitarity of the ZT has two important consequences:

- To any $F(t,f) \in L_2([0,T) \times [0,\frac{1}{T}))$, we can find a signal $x(t) \in L_2(\mathbb{R})$ such that $\mathcal{Z}_x(t,f) = F(t,f)$. This $x(t)$ can be obtained according to the inversion formula (3.17) as

$$x(t) = T \int_0^{1/T} F_x(t,f)\, df.$$

- The *cross energy (inner product)*

$$\langle x, y\rangle = \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt$$

of two signals $x(t)$ and $y(t)$ is obtained by integrating $T\mathcal{Z}_x(t,f)\mathcal{Z}_y^*(t,f)$ over the fundamental rectangle,

$$T \int_0^T \int_0^{1/T} \mathcal{Z}_x(t,f)\, \mathcal{Z}_y^*(t,f)\, dt\, df = \langle x, y\rangle. \tag{3.19}$$

This equation can be compactly written as

$$T\langle \mathcal{Z}_x, \mathcal{Z}_y\rangle = \langle x, y\rangle. \tag{3.20}$$

In particular we have for $x(t) = y(t)$

$$T \int_0^T \int_0^{1/T} |\mathcal{Z}_x(t,f)|^2 \; dt\, df = \|x\|^2 \tag{3.21}$$

which can also be written as

$$T\|\mathcal{Z}_x\|^2 = \|x\|^2.$$

This shows that *the ZT is an energy preserving transformation* (up to a constant factor T).

Multiplication Property. Let us now study the effect of linear transformations on the ZT. The ZT of a *modulated signal* $\tilde{x}(t) = x(t)\,h(t)$ is obtained by convolving, with respect to frequency, the ZT of $x(t)$ with the ZT of $h(t)$:

$$\mathcal{Z}_{\tilde{x}}(t,f) = T \int_0^{1/T} \mathcal{Z}_x(t, f-\nu)\, \mathcal{Z}_h(t,\nu)\, d\nu. \tag{3.22}$$

Proof: Use the frequency domain expression (3.3) and note that

$$\tilde{x}(t) = x(t)\,h(t) \quad \Rightarrow \quad \tilde{X}(f) = X(f) * H(f) = \int_{-\infty}^{\infty} X(f-\nu) H(\nu)\, d\nu. \tag{3.23}$$

It follows that

$$\begin{aligned}
\mathcal{Z}_{\tilde{x}}(t,f) &= \frac{1}{T} e^{2\pi j f t} \sum_{k=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} X\left(f + \frac{k}{T} - \nu\right) H(\nu)\, d\nu \right] e^{2\pi j k \frac{t}{T}} \\
&= \frac{1}{T} e^{2\pi j f t} \sum_{k=-\infty}^{\infty} e^{2\pi j k \frac{t}{T}} \sum_{l=-\infty}^{\infty} \int_{l/T}^{(l+1)/T} X\left(f + \frac{k}{T} - \nu\right) H(\nu)\, d\nu \\
&= \frac{1}{T} e^{2\pi j f t} \sum_{k=-\infty}^{\infty} e^{2\pi j k \frac{t}{T}} \sum_{l=-\infty}^{\infty} \int_0^{1/T} X\left(f + \frac{k-l}{T} - \nu\right) H\left(\nu + \frac{l}{T}\right) d\nu \\
&= \frac{1}{T} e^{2\pi j f t} \sum_{k'=-\infty}^{\infty} e^{2\pi j k' \frac{t}{T}} \sum_{l=-\infty}^{\infty} e^{2\pi j l \frac{t}{T}} \int_0^{1/T} X\left(f + \frac{k'}{T} - \nu\right) H\left(\nu + \frac{l}{T}\right) d\nu \\
&= T \int_0^{1/T} \underbrace{\frac{1}{T} e^{2\pi j (f-\nu) t} \sum_{k=-\infty}^{\infty} X\left(f + \frac{k}{T} - \nu\right) e^{2\pi j k \frac{t}{T}}}_{\mathcal{Z}_x(t,f-\nu)} \\
&\qquad \underbrace{\frac{1}{T} e^{2\pi j \nu t} \sum_{l=-\infty}^{\infty} H\left(\nu + \frac{l}{T}\right) e^{2\pi j l \frac{t}{T}}\, d\nu}_{\mathcal{Z}_h(t,\nu)}. \quad \square
\end{aligned}$$

Convolution Property. Next, consider a linear time invariant-system with impulse response $h(t)$. The ZT of the output signal $y(t) = x(t) * h(t)$ is obtained by convolving, with respect to time, the ZT of the input signal $x(t)$ with the ZT of the impulse response $h(t)$,

$$\mathcal{Z}_y(t,f) = \int_0^T \mathcal{Z}_x(t-\tau, f)\, \mathcal{Z}_h(\tau, f)\, d\tau. \tag{3.24}$$

Proof: Insert

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

in (3.1) to obtain

$$\begin{aligned}
\mathcal{Z}_y(t,f) &= \sum_{k=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t+kT-\tau)\, h(\tau)\, d\tau \right] e^{-2\pi jkfT} \\
&= \sum_{k=-\infty}^{\infty} e^{-2\pi jkfT} \sum_{l=-\infty}^{\infty} \int_{lT}^{(l+1)T} x(t+kT-\tau)\, h(\tau)\, d\tau \\
&= \sum_{k=-\infty}^{\infty} e^{-2\pi jkfT} \sum_{l=-\infty}^{\infty} \int_{0}^{T} x(t+kT-lT-\tau)\, h(\tau+lT)\, d\tau \\
&= \int_0^T \underbrace{\sum_{k'=-\infty}^{\infty} x(t+k'T-\tau)\, e^{-2\pi jk'fT}}_{\mathcal{Z}_x(t-\tau,f)} \underbrace{\sum_{l=-\infty}^{\infty} h(\tau+lT)\, e^{-2\pi jlfT}}_{\mathcal{Z}_h(\tau,f)}\, d\tau. \quad \square
\end{aligned}$$

Hence we can see that the effects of convolution and multiplication in the time domain are reflected in the ZT by convolutions with respect to time and frequency, respectively.

Product of Two Zak Transforms. Another important formula, which relates the product of two Zak transforms to the sampled cross ambiguity function, is the 2D Fourier series expansion[1]

$$\mathcal{Z}_x(t,f)\, \mathcal{Z}_y^*(t,f) = \frac{1}{T} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle x, \tilde{y}^{(mT,n/T)} \rangle\, e^{2\pi jn\frac{t}{T}}\, e^{-2\pi jmTf}, \tag{3.25}$$

where

$$\tilde{y}^{(mT,n/T)} = y(t-mT)\, e^{2\pi jn\frac{t}{T}}.$$

With the *asymmetric cross-ambiguity function*

$$A_{x,y}^{(a)}(\tau,\nu) = \int_{-\infty}^{\infty} x(t)\, y^*(t-\tau)\, e^{-2\pi j\nu t}\, dt,$$

(3.25) can be rewritten as

$$\mathcal{Z}_x(t,f)\, \mathcal{Z}_y^*(t,f) = \frac{1}{T} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} A_{x,y}^{(a)}\left(mT, \frac{n}{T}\right) e^{2\pi jn\frac{t}{T}}\, e^{-2\pi jmTf}. \tag{3.26}$$

Conversely, the sampled cross ambiguity function of $x(t)$ and $y(t)$ is obtained from $\mathcal{Z}_x(t,f)\, \mathcal{Z}_y^*(t,f)$ as

$$A_{x,y}^{(a)}\left(mT, \frac{n}{T}\right) = \langle x, \tilde{y}^{(mT,n/T)} \rangle = T \int_0^T \int_0^{1/T} \mathcal{Z}_x(t,f)\, \mathcal{Z}_y^*(t,f)\, e^{-2\pi jn\frac{t}{T}}\, e^{2\pi jmTf}\, dt\, df.$$

[1] It will be shown presently that $\mathcal{Z}_x(t,f)\, \mathcal{Z}_y^*(t,f)$ is periodic with respect to t and f with period T and $\frac{1}{T}$, respectively.

For the special case $x(t) = y(t)$ we have

$$A_x^{(a)}\left(mT, \frac{n}{T}\right) = T \int_0^T \int_0^{1/T} |\mathcal{Z}_x(t,f)|^2 \, e^{-2\pi jn\frac{t}{T}} \, e^{2\pi jmTf} \, dt \, df,$$

with the *auto-ambiguity function* $A_x^{(a)}(\tau,\nu) = A_{x,x}^{(a)}(\tau,\nu)$.

Proof: We shall first show that the function $\mathcal{Z}_x(t,f)\,\mathcal{Z}_y^*(t,f)$ is periodic in both its variables t and f with period T and $\frac{1}{T}$, respectively. With (3.5) and (3.6), we have

$$\mathcal{Z}_x(t+T,f)\,\mathcal{Z}_y^*(t+T,f) = e^{2\pi jTf}\mathcal{Z}_x(t,f)\,e^{-2\pi jTf}\,\mathcal{Z}_y^*(t,f) \;=\; \mathcal{Z}_x(t,f)\,\mathcal{Z}_y^*(t,f)$$

and

$$\mathcal{Z}_x\left(t, f+\frac{1}{T}\right)\,\mathcal{Z}_y^*\left(t, f+\frac{1}{T}\right) = \mathcal{Z}_x(t,f)\,\mathcal{Z}_y^*(t,f).$$

Since $\mathcal{Z}_x(t,f)\mathcal{Z}_y^*(t,f)$ is periodic, it can be represented by a Fourier series,

$$\mathcal{Z}_x(t,f)\,\mathcal{Z}_y^*(t,f) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_{m,n}\, e^{2\pi jn\frac{t}{T}}\, e^{-2\pi jmTf} \tag{3.27}$$

where the $a_{m,n}$ are the Fourier series coefficients. We now calculate these coefficients:

$$\begin{aligned}
a_{m,n} &= \int_0^T \int_0^{1/T} \mathcal{Z}_x(t,f)\,\mathcal{Z}_y^*(t,f)\,e^{-2\pi jn\frac{t}{T}}\,e^{2\pi jmTf}\,dt\,df \\
&= \int_0^T \int_0^{1/T} \sum_{k=-\infty}^{\infty} x(t+kT)\,e^{-2\pi jkTf} \sum_{l=-\infty}^{\infty} y^*(t+lT)\,e^{2\pi jlTf}\,e^{-2\pi jn\frac{t}{T}}\,e^{2\pi jmTf}\,dt\,df \\
&= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \int_0^T x(t+kT)\,y^*(t+lT)\,e^{-2\pi jn\frac{t}{T}}\,dt \int_0^{1/T} e^{2\pi j(l-k+m)Tf}\,df \\
&= \frac{1}{T}\sum_{l=-\infty}^{\infty} \int_0^T x(t+(l+m)T)\,y^*(t+lT)\,e^{-2\pi jn\frac{t}{T}}\,dt \\
&= \frac{1}{T}\sum_{l=-\infty}^{\infty} \int_{(l+m)T}^{(l+m+1)T} x(t)\,y^*(t-mT)\,e^{-2\pi jn\frac{t}{T}}\,dt \\
&= \frac{1}{T}\int_{-\infty}^{\infty} x(t)\,y^*(t-mT)\,e^{-2\pi jn\frac{t}{T}}\,dt = \frac{1}{T}\left\langle x, \tilde{y}^{(mT,n/T)}\right\rangle.
\end{aligned}$$

Use the last result for the $a_{m,n}$ in (3.27) to obtain (3.25). □

For the sake of completeness we shall also introduce the *symmetric cross-ambiguity function*

$$A_{x,y}^{(s)}(\tau,\nu) = \int_{-\infty}^{\infty} x\left(t+\frac{\tau}{2}\right)\,y^*\left(t-\frac{\tau}{2}\right)\,e^{-2\pi j\nu t}\,dt.$$

The asymmetric ambiguity function and the symmetric ambiguity function are equal up to a phase factor, i.e.

$$A^{(a)}_{x,y}(\tau,\nu) = e^{-\pi j\nu\tau}\, A^{(s)}_{x,y}(\tau,\nu).$$

Note that this equation evaluated on the grid $\tau = mT$ and $\nu = \frac{n}{T}$ reads

$$A^{(a)}_{x,y}(\tau,\nu) = (-1)^{mn}\, A^{(s)}_{x,y}(\tau,\nu).$$

Hence, we can rewrite (3.26) in terms of the symmetric ambiguity function as

$$\mathcal{Z}_x(t,f)\,\mathcal{Z}^*_y(t,f) = \frac{1}{T}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(-1)^{mn}\,A^{(s)}_{x,y}\left(mT,\frac{n}{T}\right)\,e^{2\pi jn\frac{t}{T}}\,e^{-2\pi jmfT}.$$

Relation with Wigner Distribution. The *cross-Wigner distribution* of two signals $x(t)$ and $y(t)$ is defined as

$$W_{x,y}(t,f) = \int_{-\infty}^{\infty} x\left(t+\frac{\tau}{2}\right)\,y^*\left(t-\frac{\tau}{2}\right)\,e^{-2\pi jf\tau}\,d\tau.$$

The Wigner distribution is related to the ZT according to

$$\mathcal{Z}_x(t,f)\,\mathcal{Z}^*_y(t,f) = \frac{1}{2T}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(-1)^{mn}\,W_{x,y}\left(t-m\frac{T}{2},f-n\frac{1/T}{2}\right). \tag{3.28}$$

Proof:

$$\begin{aligned}
\mathcal{Z}_x(t,f)\mathcal{Z}^*_y(t,f) &= \sum_{m=-\infty}^{\infty} x(t+mT)\,e^{-2\pi jmTf}\sum_{n=-\infty}^{\infty} y^*(t+nT)\,e^{2\pi jnTf}\\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x(t+mT)\,y^*(t+(n-m)T)\,e^{-4\pi jmTf}e^{2\pi jnTf}\\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} x(t+\tau)\,y^*(t+nT-\tau)\,e^{-4\pi jf\tau}\delta(\tau-mT)\,e^{2\pi jnTf}\,d\tau\\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} x(t+\tau)\,y^*(t+nT-\tau)\,e^{-4\pi jf\tau}e^{2\pi jnTf}\sum_{m=-\infty}^{\infty} e^{-2\pi jm\frac{\tau}{T}}\,d\tau\\
&= \frac{1}{2T}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} x\left(t+\frac{nT}{2}+\frac{\tau}{2}\right)\,y^*\left(t+\frac{nT}{2}-\frac{\tau}{2}\right)\,e^{-2\pi jf\tau}e^{-\pi jm\frac{\tau}{T}}(-1)^{mn}\,d\tau\\
&= \frac{1}{2T}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(-1)^{mn}\int_{-\infty}^{\infty} x\left(t+\frac{nT}{2}+\frac{\tau}{2}\right)\,y^*\left(t+\frac{nT}{2}-\frac{\tau}{2}\right)\,e^{-2\pi j(f+\frac{m}{2T})\tau}\,d\tau\\
&= \frac{1}{2T}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(-1)^{nm}\,W_{x,y}\left(t+\frac{nT}{2},f+\frac{m}{2T}\right)\\
&= \frac{1}{2T}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(-1)^{mn}\,W_{x,y}\left(t-m\frac{T}{2},f-n\frac{1/T}{2}\right). \quad \square
\end{aligned}$$

For the special case $x(t) = y(t)$ we have

$$|\mathcal{Z}_x(t,f)|^2 = \frac{1}{2T} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{mn} W_x\left(t - m\frac{T}{2}, f - n\frac{1/T}{2}\right)$$

with the *auto-Wigner distribution* $W_x(t,f) = W_{x,x}(t,f)$.

Zeros of the Zak Transform. It is of great importance to know whether or not $\mathcal{Z}_x(t,f)$ takes on the value 0. We show in Section 6 that the Gabor coefficients can be calculated via the ZT by integrating the quotient of the ZT of the function to be analyzed and the ZT of the synthesis function over the fundamental rectangle. It is evident that the zeros of the ZT of the synthesis function are in general the poles of the integrand.

Theorem 3.1: If $\mathcal{Z}_x(t,f)$ is continuous in t and f, then $\mathcal{Z}_x(t,f)$ has at least one zero in the fundamental rectangle $[0,T) \times [0, \frac{1}{T})$.

Proof [33]: Let $\mathcal{Z}_x(t,f)$ be continuous in t and f. We assume that $\mathcal{Z}_x(t,f) \neq 0$ on the fundamental rectangle $[0,T) \times [0, \frac{1}{T})$. Then there is a continuous real-valued function $\varphi(t,f)$ such that $\mathcal{Z}_x(t,f) = |\mathcal{Z}_x(t,f)|\, e^{j\varphi(t,f)}$ for $(t,f) \in [0,T) \times [0, \frac{1}{T})$. From (3.6) we have $\mathcal{Z}_x(t, \frac{1}{T}) = \mathcal{Z}_x(t,0)$ and from (3.5) it follows that $\mathcal{Z}_x(T,f) = e^{2\pi jTf}\mathcal{Z}_x(0,f)$. So we conclude that $e^{j\varphi(t,1/T)} = e^{j\varphi(t,0)}$ and $e^{j\varphi(T,f)} = e^{j[\varphi(0,f)+2\pi Tf]}$. Therefore for each t and f there are integers $l(t)$ and $k(f)$ such that $\varphi(t,\frac{1}{T}) = \varphi(t,0) + 2\pi l(t)$ and $\varphi(T,f) = \varphi(0,f) + 2\pi Tf + 2\pi k(f)$. Since $\varphi(t,f)$ is continuous, $\varphi(t,\frac{1}{T}) - \varphi(t,0) = 2\pi l(t)$ and $\varphi(T,f) - \varphi(0,f) - 2\pi Tf = 2\pi k(f)$ are continuous functions of t and f respectively. We conclude that $\varphi(t,\frac{1}{T}) - \varphi(t,0)$ is continuous if and only if the integer $l(t)$ is independent of t, i.e. $l(t) = l$. Otherwise there would be discontinuities at the points where $l(t)$ changes its value, because $l(t) \in \mathbb{Z}$. The same argument can be used to show that $k(f)$ is independent of f, i.e. $k(f) = k$. It follows that

$$\begin{aligned}
0 &= \phi(0,0) - \phi(T,0) + \phi(T,0) - \phi\left(T, \frac{1}{T}\right) \\
&+ \phi\left(T,\frac{1}{T}\right) - \phi\left(0,\frac{1}{T}\right) + \phi\left(0,\frac{1}{T}\right) - \phi(0,0) \\
&= -2\pi k - 2\pi l + 2\pi + 2\pi k + 2\pi l \\
&= 2\pi.
\end{aligned}$$

Since $0 = 2\pi$ is definitely incorrect, our initial assumption must be wrong. Hence $\mathcal{Z}_x(t,f)$ must have at least one zero in the fundamental rectangle. □

We now discuss the zeros of $\mathcal{Z}_x(t,f)$ in the case of signals with certain properties. When $x(t)$ is even we have $\mathcal{Z}_x(t,f) = \mathcal{Z}_x(-t,-f)$ (see (3.7)). Let us calculate $\mathcal{Z}_x(\frac{T}{2}, \frac{1}{2T})$

using two different ways. First it follows from (3.6) that $\mathcal{Z}_x(\frac{T}{2}, \frac{1}{2T}) = \mathcal{Z}_x(\frac{T}{2}, -\frac{1}{2T})$. From (3.5) we obtain $\mathcal{Z}_x(\frac{T}{2}, \frac{1}{2T}) = -\mathcal{Z}_x(-\frac{T}{2}, \frac{1}{2T}) = -\mathcal{Z}_x(\frac{T}{2}, -\frac{1}{2T})$. Thus, we have $\mathcal{Z}_x(\frac{T}{2}, \frac{1}{2T}) = -\mathcal{Z}_x(\frac{T}{2}, \frac{1}{2T})$. This can only be true if $\mathcal{Z}_x(\frac{T}{2}, \frac{1}{2T}) = 0$. Hence we have

$$x(-t) = x(t) \quad \Rightarrow \quad \mathcal{Z}_x\left(\frac{T}{2}, \frac{1/T}{2}\right) = 0. \tag{3.29}$$

For even functions $x(t)$ the ZT has a zero in the center of the fundamental rectangle. The next statements can be proved using similar arguments. We only give the results:

$$x(-t) = -x(t) \quad \Rightarrow \quad \mathcal{Z}_x(0,0) = 0, \;\; \mathcal{Z}_x\left(0, \frac{1/T}{2}\right) = 0, \;\; \mathcal{Z}_x\left(\frac{T}{2}, 0\right) = 0. \tag{3.30}$$

Time-limited Signals. We next consider the *ZT of time-limited signals.* Assume $x(t)$ is time-limited to $[-\frac{\tilde{T}}{2}, \frac{\tilde{T}}{2}]$, where $\tilde{T} \leq T$. We then have

$$\mathcal{Z}_x(t,f) = x(t) + x(t+T)\, e^{-2\pi jTf} + x(t-T)\, e^{2\pi jTf} + \ldots$$

It is obvious that

$$\mathcal{Z}_x(t,f) = x(t) \;\; \text{for } |t| \leq \frac{T}{2}. \tag{3.31}$$

Band-limited Signals. When $x(t)$ is *band-limited* to $[-\frac{\tilde{F}}{2}, \frac{\tilde{F}}{2}]$ where $\tilde{F} \leq \frac{1}{T}$, we have

$$\mathcal{Z}_x(t,f) = \frac{1}{T}\, e^{2\pi jft} X(f) \quad \text{for } |f| \leq \frac{F}{2}. \tag{3.32}$$

This immediately follows from (3.3) using the same arguments as above.

3.3 Some Signals

We shall now calculate the *ZT of some specific signals.*

Dirac Function. The ZT of the signal $x(t) = \delta(t - t_0)$ is given by

$$Z_x(t,f) = \delta(t - t_0).$$

Note that this expression is valid on the fundamental rectangle $[0,T) \times [0, 1/T)$.

Complex Exponential. The ZT of the signal $x(t) = e^{2\pi jf_0 t}$ is given as

$$\mathcal{Z}_x(t,f) = \frac{1}{T}\, e^{2\pi jf_0 t}\, \delta(f - f_0).$$

This expresison is valid on the fundamental rectangle.

Rectangular Pulse. For the rectangular pulse

$$x(t) = \begin{cases} 1, & 0 \leq t < T \\ 0, & \text{otherwise,} \end{cases}$$

we obtain

$$\mathcal{Z}_x(t,f) = 1 \quad \text{for} \quad 0 \le t < T.$$

So the ZT of $x(t)$ is equal to 1 in the fundamental rectangle.

Sinc Function. The counterpart of the rectangular pulse is the rectangular pulse in the frequency domain, that is

$$x(t) = \frac{1}{T}\, e^{j\pi\frac{t}{T}} \operatorname{sinc}\left(\pi \frac{t}{T}\right) \quad \Rightarrow \quad X(f) = \begin{cases} 1, & 0 \le f < \frac{1}{T} \\ 0, & \text{otherwise,} \end{cases}$$

where $\operatorname{sinc}(t) = \frac{\sin(t)}{t}$. We obtain

$$\mathcal{Z}_x(t,f) = \frac{1}{T}\, e^{2\pi j f t}$$

in the fundamental rectangle.

Periodic Function. For a periodic function with period T we obtain

$$\begin{aligned} \mathcal{Z}_x(t,f) &= \sum_{k=-\infty}^{\infty} x(t+kT)\, e^{-2\pi jkTf} = x(t) \sum_{k=-\infty}^{\infty} e^{-2\pi jkTf} \\ &= \frac{1}{T}\, x(t) \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right). \end{aligned}$$

One-Sided Exponential. The next example we consider is the *one-sided exponential* $x(t) = e^{-\alpha t}\, u(t)$, where $u(t)$ is the unit step function. We obtain for the ZT in the interval $0 \le t < T$

$$\begin{aligned} \mathcal{Z}_x(t,f) &= \sum_{k=-\infty}^{\infty} e^{-\alpha(t+kT)}\, u(t+kT)\, e^{-2\pi jkTf} \\ &= e^{-\alpha t} \sum_{k=0}^{\infty} e^{-\alpha kT} e^{-2\pi jkTf} \\ &= e^{-\alpha t} \sum_{k=0}^{\infty} \left[e^{-(\alpha T + 2\pi jTf)}\right]^k. \end{aligned}$$

The expression $u(t+kT)$ vanishes for $k < 0$ and $0 \le t < T$. When we evaluate the geometric series, we obtain the expression

$$\mathcal{Z}_x(t,f) = \frac{e^{-\alpha t}\, e^{2\pi jTf}}{e^{2\pi jTf} - e^{-\alpha T}}.$$

This result is valid on the fundamental rectangle.

Two-sided Exponential. The ZT of a two-sided exponential $x(t) = e^{-\alpha|t|}$ can be obtained from the last result by using the symmetry properties of the ZT. We can write the two-sided exponential as $x(t) = e^{-\alpha t}\, u(t) + e^{\alpha t}\, u(-t)$. With (3.7) we obtain

$$\mathcal{Z}_x(t,f) = \frac{e^{-\alpha t}\, e^{2\pi jTf}}{e^{2\pi jTf} - e^{-\alpha T}} + \frac{e^{\alpha t}\, e^{-2\pi jTf}}{e^{-2\pi jTf} - e^{-\alpha T}}.$$

3.4 The Discrete Zak Transform

For *numerical implementation* of the ZT there is a need for a discrete version. Such a discrete ZT has been introduced by M.J. Bastiaans in [96] and by L. Auslander et al. in [97]. However, it seems that a systematic study of the properties of the discrete ZT has not been performed so far.

Definition. The discrete ZT (DZT) $\mathcal{Z}_x[m,n]$ is obtained by sampling the ZT of a continuous-time signal with respect to both time and frequency. The samples of $\mathcal{Z}_x(t,f)$ are obtained by setting $t = m\frac{T}{M}$ and $f = n\frac{1/T}{N}$ in (3.1), where $M \geq 1$ and $N \geq 1$:

$$\mathcal{Z}_x[m,n] \stackrel{\text{def}}{=} \mathcal{Z}_x\left(m\frac{T}{M}, n\frac{1/T}{N}\right) = \sum_{k=-\infty}^{\infty} x\left(\frac{m}{M}T + kT\right) e^{-2\pi jk\frac{n}{N}} \tag{3.33}$$

with

$$0 \leq m \leq M-1, \quad 0 \leq n \leq N-1.$$

Note that the number of ZT samples on the fundamental rectangle $[0,T) \times [0,1/T)$ is

$$N_1 \stackrel{\text{def}}{=} MN.$$

We can write the DZT as

$$\mathcal{Z}_x[m,n] = \sum_{k=-\infty}^{\infty} x[m+kM]\, e^{-2\pi jk\frac{n}{N}} \tag{3.34}$$

with the discrete-time signal

$$x[m] \stackrel{\text{def}}{=} x\left(m\frac{T}{M}\right) \tag{3.35}$$

obtained by sampling $x(t)$ with sampling period $\frac{T}{M}$.

Fourier Transform Interpretation. We shall now give an interpretation of the DZT. The Fourier transform of a discrete-time signal $x[k]$ is defined as

$$X(e^{2\pi j\theta}) = \sum_{k=-\infty}^{\infty} x[k]\, e^{-2\pi j\theta k},$$

where $\theta = \frac{f}{f_s}$ is the normalized frequency and f_s is the sampling frequency. For $\theta = \frac{n}{N}$,

$$X(e^{2\pi j\frac{n}{N}}) = \sum_{k=-\infty}^{\infty} x[k]\, e^{-2\pi jk\frac{n}{N}}.$$

Comparing with (3.34), we see that the DZT is a frequency-sampled version of the Fourier transform of the discrete-time signal

$$x_m[k] = x[m+kM].$$

For each m with $0 \leq m \leq M-1$, we have to compute the Fourier transform of the signal $x_m[k]$ and then sample the result at the N points $\theta = \frac{n}{N}$, with $0 \leq n \leq N-1$.

Time-limited Signals and DFT Formulation. Let us next consider the case where the signal $x(t)$ is time-limited in the interval $[0, NT)$. This means that the discrete-time signal $x[m]$ in (3.35) consists of $\frac{NT}{T/M} = MN = N_1$ samples, i.e. is time-limited in the interval $[0, N_1]$. It follows that $x_m[k] = x[m+kM]$ consists of N samples for any m. We thus have

$$\mathcal{Z}_x[m,n] = \sum_{k=-\infty}^{\infty} x_m[k]\, e^{-2\pi jk\frac{n}{N}} = \sum_{k=0}^{N-1} x_m[k]\, e^{-2\pi jk\frac{n}{N}}, \quad 0 \leq m \leq M-1. \tag{3.36}$$

But this is the N-point DFT of the signal $x_m[k]$ with respect to k. So for a properly time-limited signal $x(t)$ we have to do M N-point DFTs to obtain the DZT of $x(t)$ on the fundamental rectangle. It is convenient to introduce the matrix

$$\mathbf{A} = \begin{pmatrix} x[0] & x[1] & \dots & x[M-1] \\ x[M] & x[M+1] & \dots & x[2M-1] \\ \vdots & \vdots & \vdots & \\ x[N_1-M] & x[N_1-M+1] & \dots & x[N_1-1] \end{pmatrix}.$$

The DZT

$$\mathcal{Z}_x[m,n] = \sum_{k=0}^{N-1} x[m+kM]\, e^{-2\pi jk\frac{n}{N}}$$

is obtained by taking the DFT of the columns of the matrix $\mathbf{A}$.

If the signal is not of finite duration, the computation of the DZT can still be performed as in (3.36) if we replace $x_m[k]$ by a periodized version of $x_m[k]$ defined as

$$\tilde{x}[m] = \sum_{l=-\infty}^{\infty} x[m+lN_1], \quad 0 \leq m \leq N_1 - 1. \tag{3.37}$$

This function is periodic with period N_1. A straightforward computation shows that

$$\mathcal{Z}_x[m,n] = \sum_{k=0}^{N-1} \tilde{x}[m+kM]\, e^{-2\pi jk\frac{n}{N}} = \sum_{k=0}^{N-1} \tilde{x}_m[k]\, e^{-2\pi jk\frac{n}{N}} \tag{3.38}$$

which has the same structure as (3.36).

We conclude that, for any signal $x[m]$, the computation of the DZT can be reduced to the computation of a number of DFTs, for which efficient FFT algorithms are available.

Frequency-Domain Expression. The DZT can be expressed in terms of the Fourier transform of $x[n]$, $X(e^{2\pi j\theta}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-2\pi j\theta n}$, as

$$\mathcal{Z}_x[m,n] = \frac{1}{M}\, e^{2\pi j\frac{mn}{N_1}} \sum_{k=0}^{M-1} X\left(e^{2\pi j\frac{n+kN}{N_1}}\right) e^{2\pi j\frac{mk}{M}}. \tag{3.39}$$

Proof:

$$\begin{aligned}
\mathcal{Z}_x[m,n] &= \sum_{k=-\infty}^{\infty} x[m+kM]\, e^{-2\pi jk\frac{n}{N}} = \sum_{k=-\infty}^{\infty} \left[\int_0^1 X(e^{2\pi j\theta})\, e^{2\pi j\theta(m+kM)}\, d\theta \right] e^{-2\pi jk\frac{n}{N}} \\
&= \int_0^1 X(e^{2\pi j\theta})\, e^{2\pi j\theta m} \underbrace{\left[\sum_{k=-\infty}^{\infty} e^{2\pi jk(\theta M-\frac{n}{N})} \right]}_{\frac{1}{M}\sum_{k=-\infty}^{\infty} \delta\left(\theta-\frac{n}{MN}-\frac{k}{M}\right)} d\theta \\
&= \frac{1}{M} \sum_{k=-\infty}^{\infty} \int_0^1 X(e^{2\pi j\theta})\, e^{2\pi j\theta m}\, \delta\left(\theta - \frac{n}{MN} - \frac{k}{M}\right) \\
&= \frac{1}{M} e^{2\pi j\frac{mn}{N_1}} \sum_{k=0}^{M-1} X\left(e^{2\pi j\frac{n+kN}{N_1}}\right) e^{2\pi j\frac{mk}{M}}. \quad \square
\end{aligned}$$

We shall next discuss some basic properties of the DZT $\mathcal{Z}_x[m,n]$.

Periodicity. The DZT is quasi-periodic in the time variable and periodic in the frequency variable,

$$\mathcal{Z}_x[m+M,n] = e^{2\pi j\frac{n}{N}} \mathcal{Z}_x[m,n] \tag{3.40}$$

$$\mathcal{Z}_x[m,n+N] = \mathcal{Z}_x[m,n]. \tag{3.41}$$

As in the continuous time case, it is sufficient to consider $\mathcal{Z}_x[m,n]$ on the fundamental rectangle $[0,M) \times [0,N)$.

Symmetry. Other basic properties of the DZT are

$$\mathcal{Z}_{x^-}[m,n] = \mathcal{Z}_x[-m,-n] \tag{3.42}$$

$$\mathcal{Z}_{x^*}[m,n] = \mathcal{Z}_x^*[m,-n]. \tag{3.43}$$

When $x[m]$ is real, imaginary, even, odd, conjugate even, or conjugate odd we have respectively

$$x[m] \in \mathbb{R} \;\Rightarrow\; \mathcal{Z}_x^*[m,-n] = \mathcal{Z}_x[m,n] \tag{3.44}$$

$$x[m] \text{ imaginary} \;\Rightarrow\; \mathcal{Z}_x^*[m,-n] = -\mathcal{Z}_x[m,n] \tag{3.45}$$

$$x[-m] = x[m] \;\Rightarrow\; \mathcal{Z}_x[-m,-n] = \mathcal{Z}_x[m,n] \tag{3.46}$$

$$x[-m] = -x[m] \;\Rightarrow\; \mathcal{Z}_x[-m,-n] = -\mathcal{Z}_x[m,n] \tag{3.47}$$

$$x^*[-m] = x[m] \;\Rightarrow\; \mathcal{Z}_x^*[-m,-n] = \mathcal{Z}_x[m,n] \tag{3.48}$$

$$x^*[-m] = -x[m] \;\Rightarrow\; \mathcal{Z}_x^*[-m,-n] = -\mathcal{Z}_x[m,n]. \tag{3.49}$$

Inversion. The inversion of the DZT can be accomplished using the inversion formula of the DFT. Applying the IDFT to (3.38), we obtain

$$\tilde{x}_m[k] = \frac{1}{N}\sum_{n=0}^{N-1} \mathcal{Z}_x[m,n]\, e^{2\pi jk\frac{n}{N}}, \qquad 0 \le m \le M-1. \tag{3.50}$$

Note that the IDFT only gives the periodized version $\tilde{x}[m] = \sum_{l=-\infty}^{\infty} x[m+lN_1]$ of the signal $x[m]$; it is not possible to obtain the original signal $x[m]$ unless the signal has finite length $L \le N_1 = MN$.

Inner product. We define the inner product of two periodic (or finite-length) sequences $\tilde{x}[m]$, $\tilde{y}[m]$ with period (or length) N_1 by

$$\langle \tilde{x}, \tilde{y}\rangle = \sum_{m=0}^{N_1-1} \tilde{x}[m]\,\tilde{y}^*[m]. \tag{3.51}$$

The formula

$$\frac{1}{N}\langle \mathcal{Z}_x, \mathcal{Z}_y\rangle = \frac{1}{N}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \mathcal{Z}_x[m,n]\mathcal{Z}_y^*[m,n] = \langle \tilde{x}, \tilde{y}\rangle \tag{3.52}$$

is analogous to the inner product property in the continuous-time case. Note however that the right-hand side of (3.52) is not the inner product of $x[m]$ and $y[m]$ but rather of the periodized signal versions $\tilde{x}[m]$ and $\tilde{y}[m]$.

Proof:

$$\begin{aligned}
&\frac{1}{N}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\sum_{k=-\infty}^{\infty} x[m+kM]\, e^{-2\pi jk\frac{n}{N}} \sum_{l=-\infty}^{\infty} y^*[m+lM]\, e^{2\pi jl\frac{n}{N}} \\
&= \frac{1}{N}\sum_{m=0}^{M-1}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} x[m+kM]\, y^*[m+lM] \underbrace{\sum_{n=0}^{N-1} e^{2\pi j(l-k)\frac{n}{N}}}_{N\sum_{i=-\infty}^{\infty}\delta[l-k-iN]} \\
&= \sum_{m=0}^{M-1}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} x[m+kM]\, y^*[m+lM]\,\delta[l-k-iN] \\
&= \sum_{m=0}^{M-1}\sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} x[m+kM]\, y^*[m+kM+iN_1].
\end{aligned}$$

Setting $k = jN + l$ with $0 \le l \le N-1$ and $-\infty < j < \infty$, we obtain further

$$\begin{aligned}
&= \sum_{m=0}^{M-1}\sum_{l=0}^{N-1}\sum_{j=-\infty}^{\infty} x[m+lM+jN_1] \sum_{i=-\infty}^{\infty} y^*[m+lM+iN_1+jN_1] \\
&= \sum_{m=0}^{M-1}\sum_{l=0}^{N-1} \tilde{x}[m+lM]\,\tilde{y}^*[m+lM] = \sum_{m=0}^{N_1-1} \tilde{x}[m]\,\tilde{y}^*[m]. \qquad \square
\end{aligned}$$

Product of Two Zak Transforms. As in the continuous-time case, the product $\mathcal{Z}_x[m,n]\,\mathcal{Z}_y^*[m,n]$ of two DZTs is periodic in both its variables m and n, with period M

and N, respectively. We are thus able to compute the two-dimensional Fourier series expansion of this product. One obtains

$$\mathcal{Z}_x[m,n]\,\mathcal{Z}_y^*[m,n] = \frac{1}{M}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}\langle\tilde{x},\tilde{y}_{l,k}\rangle\, e^{2\pi jk\frac{m}{M}}\, e^{-2\pi jl\frac{n}{N}}, \tag{3.53}$$

where

$$\tilde{y}_{l,k}[m] = \tilde{y}[m-lM]\, e^{2\pi jk\frac{m}{M}}$$

is a time-frequency shifted version of the signal $\tilde{y}[m]$.

Proof: The two-dimensional Fourier series representation for $\mathcal{Z}_x[m,n]\,\mathcal{Z}_y^*[m,n]$ is

$$\mathcal{Z}_x[m,n]\,\mathcal{Z}_y^*[m,n] = \frac{1}{N_1}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} a_{k,l}\, e^{2\pi jk\frac{m}{M}}\, e^{-2\pi jl\frac{n}{N}},$$

where the Fourier series coefficients $a_{k,l}$ are given by

$$\begin{aligned}
a_{k,l} &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\mathcal{Z}_x[m,n]\,\mathcal{Z}_y^*[m,n]\, e^{-2\pi jk\frac{m}{M}}\, e^{2\pi jl\frac{n}{N}} \\
&= \sum_{u=-\infty}^{\infty}\sum_{v=-\infty}^{\infty}\sum_{m=0}^{M-1} x[m+uM]\, y^*[m+vM]\, e^{-2\pi jk\frac{m}{M}} \underbrace{\sum_{n=0}^{N-1} e^{2\pi j(v+l-u)\frac{n}{N}}}_{N\sum_{r=-\infty}^{\infty}\delta[v+l-u-rN]} \\
&= N\sum_{u=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\sum_{m=0}^{M-1} x[m+uM]\, y^*[m+(u+rN-l)M]\, e^{-2\pi jk\frac{m}{M}} \\
&= N\sum_{m=0}^{M-1}\sum_{u=-\infty}^{\infty} x[m+uM+lM]\, e^{-2\pi jk\frac{m}{M}} \sum_{r=-\infty}^{\infty} y^*[m+uM+rN_1] \\
&= N\sum_{m=0}^{M-1}\sum_{u=0}^{N-1}\sum_{j=-\infty}^{\infty} x[m+uM+jN_1+lM]\, e^{-2\pi jk\frac{m}{M}} \sum_{r=-\infty}^{\infty} y^*[m+uM+rN_1+jN_1] \\
&= N\sum_{m=0}^{M-1}\sum_{u=0}^{N-1}\tilde{x}[m+uM+lM]\, e^{-2\pi jk\frac{m}{M}}\, \tilde{y}^*[m+uM] \\
&= N\sum_{m=0}^{N_1-1}\tilde{x}[m+lM]\, e^{-2\pi jk\frac{m}{M}}\, \tilde{y}^*[m] \\
&= N\sum_{m=0}^{N_1-1}\tilde{x}[m]\, \tilde{y}^*[m-lM]\, e^{-2\pi jk\frac{m}{M}} \\
&= N\langle\tilde{x},\tilde{y}_{l,k}\rangle. \quad \square
\end{aligned}$$

4 The Theory of Frames

4.1 Motivation and Definition of Frames

Linear signal spaces and the associated concepts of orthonormal bases are of fundamental importance for signal theory. Let us first consider an orthonormal basis for a given Hilbert Space $\mathcal{H}$, with the basis signals $g_j(t)$. Every function $x(t) \in \mathcal{H}$ can be expressed as

$$x(t) = \sum_j \langle x, g_j \rangle \, g_j(t). \tag{4.1}$$

However, a disadvantage of orthonormal bases is that, in general, in the case of local changes of the function $x(t)$ the whole table of expansion coefficients $\langle x, g_j \rangle$ is affected. One way of solving this problem is to cut $\mathbb{R}$ (the real axis) into disjoint intervals of equal length and to construct the $g_j(t)$ starting from an orthonormal basis for one interval. Consider $h_n(t)$, an orthonormal basis of $L^2[0,T)$,[2] and define

$$g_{nm}(t) = \begin{cases} h_n(t - mT) & \text{for} \quad mT \le t < (m+1)T \\ 0 & \text{otherwise} \end{cases}$$

for $n, m \in \mathbb{Z}$. We can see that if the function $x(t)$ undergoes a local change in a certain interval $[kT, lT]$ only the values $\langle x, g_{nm} \rangle$ related to that interval are affected. The major drawback of this choice for the $g_j(t)$ is that some of the $g_j(t)$ are likely to be discontinuous at the edges of the intervals. Thus, we introduced *discontinuities in the analysis of* $x(t)$ which have not been present in $x(t)$.

These undesirable features can be avoided by the use of frames. The theory of frames has been introduced by R.J. Duffin and A.C. Schaeffer [27]. It has been worked out by I. Daubechies [28] and several other authors [29, 30]. In this section, we shall review the general theory of frames. The special type of Weyl-Heisenberg frames, which is the type of frames relevant to the Gabor expansion, will be discussed in Section 5. Throughout this section, we shall make extensive use of basic linear operator theory, which is discussed in [98].

In the following, let $\{g_j(t)\}$ ($j \in J$, J a denumerable set) be a nonorthonormal (and nonorthogonal) set of functions taken from the Hilbert space $\mathcal{H}$. It is convenient to define a *linear operator* $\mathbf{T}$ which assigns to each signal $x(t)$ the sequence of inner products $(\mathbf{T}x)_j = \langle x, g_j \rangle$.

Definition 4.1: The linear operator $\mathbf{T}$ maps the Hilbert space $\mathcal{H}$ into the

[2] $L^2[0,T)$ denotes the space of square-integrable functions on the interval $[0,T)$, i.e., $\int_0^T |x(t)|^2 dt < \infty$ for $x(t) \in L^2[0,T)$.

space $l^2(J)$ of square-summable complex sequences,[3]

$$\mathbf{T}: \mathcal{H} \to l^2(J);$$

it assigns to each signal $x(t)$ the sequence of inner products $\langle x, g_j \rangle$,

$$\mathbf{T}: x \to \langle x, g_j \rangle_{j \in J} \quad \text{or equivalently} \quad (\mathbf{T}x)_j = \langle x, g_j \rangle.$$

Note that $\|\mathbf{T}x\|^2 = \sum_j |(\mathbf{T}x)_j|^2 = \sum_j |\langle x, g_j \rangle|^2$, i.e., the energy $\|\mathbf{T}x\|^2$ of $\mathbf{T}x$ can be expressed as

$$\|\mathbf{T}x\|^2 = \sum_j |\langle x, g_j \rangle|^2. \tag{4.2}$$

We shall next formulate the properties which the $g_j(t)$ and the operator $\mathbf{T}$ should satisfy:

1. *The signal $x(t)$ can be perfectly reconstructed from the expansion coefficients $\langle x, g_j \rangle$.* This means that we want $\langle x, g_j \rangle = \langle y, g_j \rangle$ for all $j \in J$ to imply that $x(t) = y(t)$. We conclude that, in order to satisfy this requirement, the operator $\mathbf{T}$ has to be left invertible,[4] which means that $\mathbf{T}$ *is invertible on its range* $\mathrm{Ran}(\mathbf{T})$. Note that in general $\mathrm{Ran}(\mathbf{T})$ is only a subspace of $l^2(J)$. The left inverse of $\mathbf{T}$ will be denoted $\mathbf{T}^{-1}$.

2. *The linear operator $\mathbf{T}$ is continuous and hence bounded.*[5] This requirement guarantees that the expansion coefficients $\langle x, g_j \rangle$ have finite energy if the signal $x(t)$ has finite energy, i.e., the expansion coefficients are square-summable.

3. *The reconstruction of the signal $x(t)$ from the sequence $\langle x, g_j \rangle$ is numerically stable.* If two sequences $c_j = \langle x, g_j \rangle$ and $d_j = \langle y, g_j \rangle$ are "close" in $l^2(J)$, then $x(t)$ and $y(t)$ are "close" in $L^2(\mathbb{R})$. Thus, for example, small errors in transmitted analysis coefficients or round-off errors in the reconstruction process would not necessarily be disastrous. Thus we require that *the left inverse $\mathbf{T}^{-1}$ be continuous and hence bounded.* Note that the mere continuity of $\mathbf{T}$ does not say anything about the continuity of $\mathbf{T}^{-1}$.

4. *Local changes in $x(t)$ are reflected by local changes in the $\langle x, g_j \rangle$.*

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[3] $\sum_{j \in J} |c_j|^2 < \infty$ for all sequences $c_j \in l^2(J)$.

[4] A mapping $\mathbf{L} : \mathcal{X} \to \mathcal{Y}$ is said to be left invertible if a mapping $\mathbf{M} : \mathcal{Y} \to \mathcal{X}$ exists such that $\mathbf{ML} = \mathbf{I}$ (the identity mapping) on the space $\mathcal{X}$.

[5] A linear operator $\mathbf{L} : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X}$ and $\mathcal{Y}$ are normed linear spaces is said to be bounded if there is a real number $M \geq 0$ such that $\|\mathbf{L}x\| \leq M\|x\|$ for all $x(t)$ in $\mathcal{X}$. If the operator satisfies $m\|x\| \leq \|\mathbf{L}x\|$ for all $x(t)$ in $\mathcal{X}$ it is said to be bounded below. Note that in our case $\mathcal{X}$ is a Hilbert space with L^2-norm and $\mathcal{Y}$ is the space of square-summable complex sequences with l^2-norm.

Requirement 2 implies that

$$\|\mathbf{T}x\|^2 \le B\|x\|^2, \tag{4.3}$$

with a positive constant B. With (4.2), this can be rewritten as

$$\sum_j |\langle x, g_j\rangle|^2 \le B\|x\|^2. \tag{4.4}$$

In requirement 3 we demand that the left inverse $\mathbf{T}^{-1}$ is continuous. This is the case if and only if the operator $\mathbf{T}$ is bounded below, i.e.,

$$A\|x\|^2 \le \|\mathbf{T}x\|^2,$$

with $A > 0$. With (4.2) this condition can be rewritten as

$$A\|x\|^2 \le \sum_j |\langle x, g_j\rangle|^2.$$

Combining the last two requirements, we are now able to give the following definition.

Definition 4.2: A set of signals $g_j(t) \in \mathcal{H}$ where $j \in J$ is called a *frame* for the Hilbert space $\mathcal{H}$ if

$$A\|x\|^2 \le \sum_j |\langle x, g_j\rangle|^2 \le B\|x\|^2 \qquad \forall\, x(t) \in \mathcal{H} \tag{4.5}$$

with $A, B \in \mathbb{R}$ and $0 < A \le B < \infty$. The constants A and B are called the *frame bounds.*

Using (4.2) in (4.5) it follows that

$$A\|x\|^2 \le \|\mathbf{T}x\|^2 \le B\|x\|^2. \tag{4.6}$$

This means that the energy of $\mathbf{T}x$ is bounded by the product of the frame bounds and the signal's energy.

We next define the operator $\mathbf{T}^\times$ which maps the space of square-summable complex sequences to the Hilbert space $\mathcal{H}$.

Definition 4.3: The linear operator $\mathbf{T}^\times$ is defined as

$$\mathbf{T}^\times : l^2(J) \to \mathcal{H} \tag{4.7}$$

$$(\mathbf{T}^\times c_j)(t) = \sum_j c_j\, g_j(t), \text{ where } c_j \in l^2(J). \tag{4.8}$$

We shall next show that the operator $\mathbf{T}^\times$ is the adjoint $\mathbf{T}^*$ of $\mathbf{T}$.[6] Consider an arbitrary sequence $c_j \in l^2(J)$. We have to prove that

$$\langle \mathbf{T}x, c_j\rangle = \langle x, \mathbf{T}^\times c_j\rangle.$$

[6]Let $\mathbf{L} : \mathcal{X} \to \mathcal{Y}$ be a bounded linear operator where $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces. Furthermore $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. The operator $\mathbf{L}^*$ which satisfies $\langle \mathbf{L}x, y\rangle = \langle x, \mathbf{L}^* y\rangle$ for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ is the adjoint of $\mathbf{L}$.

This can be seen by noting that

$$\begin{aligned}\langle \mathbf{T}x, c_j\rangle &= \sum_j \langle x, g_j\rangle c_j^* \\ \langle x, \mathbf{T}^\times c_j\rangle &= \Big\langle x, \sum_j c_j g_j \Big\rangle \\ &= \sum_j \langle x, g_j\rangle c_j^*.\end{aligned}$$

So we showed that the adjoint operator of $\mathbf{T}$ is $\mathbf{T}^\times$,

$$\mathbf{T}^\times = \mathbf{T}^*. \tag{4.9}$$

In what follows, we shall always write $\mathbf{T}^*$ instead of $\mathbf{T}^\times$.

4.2 The Frame Operator

Definition 4.4: The operator

$$\begin{aligned}&\mathbf{S} = \mathbf{T}^*\mathbf{T}, \\ &(\mathbf{S}x)(t) = \sum_{j\in J} \langle x, g_j\rangle\, g_j(t)\end{aligned}$$

is said to be the *frame operator*.

We can write

$$(\mathbf{S}x)(t) = \int_{t'} S(t,t')\, x(t')\, dt',$$

where $S(t,t')$, the kernel of the frame operator $\mathbf{S}$, is given by

$$S(t,t') = \sum_{j\in J} g_j(t)\, g_j^*(t'). \tag{4.10}$$

Note that $\mathbf{S}$ maps $\mathcal{H}$ into $\mathcal{H}$, and that

$$\sum_j |\langle x, g_j\rangle|^2 = \|\mathbf{T}x\|^2 = \langle \mathbf{S}x, x\rangle. \tag{4.11}$$

We are now able to formulate the frame condition with the frame operator $\mathbf{S}$. Definition 4.2 can also be written as

$$A\|x\|^2 \le \langle \mathbf{S}x, x\rangle \le B\|x\|^2. \tag{4.12}$$

An equivalent formulation of (4.12) is [7]

$$A\mathbf{I} \le \mathbf{S} \le B\mathbf{I} \tag{4.13}$$

[7] This relation is to be understood in the sense of quadratic forms, i.e., $A\langle x, x\rangle \le \langle \mathbf{S}x, x\rangle \le B\langle x, x\rangle$.

where $\mathbf{I}$ is the identity operator on $\mathcal{H}$. From (4.12), it also follows that

$$\|\mathbf{S}\| \leq B. \tag{4.14}$$

This is seen as follows:

$$\begin{aligned}\|\mathbf{S}\| &= \sup \frac{\|\mathbf{S}x\|}{\|x\|} = \sup \sqrt{\frac{\langle \mathbf{S}x, \mathbf{S}x \rangle}{\|x\|^2}} = \sqrt{\sup \frac{\langle \mathbf{S}^2 x, x \rangle}{\|x\|^2}} \\ &= \sqrt{\lambda_{sup}^2} = \lambda_{sup} \leq B.\end{aligned}$$

We shall now formulate an important theorem on the eigenvalues of the frame operator.

Theorem 4.1: Let λ_{inf} (λ_{sup}) denote the infimum (supremum) of the eigenvalues λ of the frame operator $\mathbf{S}$. Then $A \leq \lambda_{inf}$ and $B \geq \lambda_{sup}$.

Proof: With [98]

$$\lambda_{inf} \leq \frac{\langle \mathbf{S}x, x \rangle}{\|x\|^2} \leq \lambda_{sup}$$

and (4.12) the proof follows immediately. □

While the frame bounds A, B are not uniquely defined, this theorem shows that the *tightest possible* frame bounds are given by λ_{inf} and λ_{sup}, respectively.

We shall now discuss the properties of $\mathbf{S}$. The frame operator $\mathbf{S}$ is

1. linear,
2. positive definite, i.e. $\langle \mathbf{S}x, x \rangle > 0$ for all $x(t) \in \mathcal{H}$,
3. self-adjoint, $\mathbf{S}^* = \mathbf{S}$,
4. invertible on $\mathcal{H}$, i.e., the inverse operator $\mathbf{S}^{-1}$ exists and is bounded as

$$\frac{1}{B}\mathbf{I} \leq \mathbf{S}^{-1} \leq \frac{1}{A}\mathbf{I}.$$

The linearity of $\mathbf{S}$ follows from the fact that $\mathbf{S}$ is obtained by cascading a linear operator and its adjoint. To see that $\mathbf{S}$ is positive definite note that, with (4.6),

$$\langle \mathbf{S}x, x \rangle = \|\mathbf{T}x\|^2 \geq A\|x\|^2 > 0$$

for all $x(t) \neq 0$. Hence, $\mathbf{S}$ must also be self-adjoint. The invertibility follows directly from $A\,\mathbf{I} \leq \mathbf{S} \leq B\,\mathbf{I}$ with $A > 0$.

Besides the frame operator $\mathbf{S} = \mathbf{T}^*\mathbf{T}$, also the composite operator $\mathbf{T}\mathbf{T}^*$ is important. This operator will be discussed in section 4.9.

4.3 The Dual Frame

We have seen above that $\mathbf{S}$ is invertible on $\mathcal{H}$ with bounded inverse $\mathbf{S}^{-1}$ satisfying

$$\frac{1}{B}\mathbf{I} \leq \mathbf{S}^{-1} \leq \frac{1}{A}\mathbf{I}. \tag{4.15}$$

Theorem 4.2: If $\{g_j(t)\}$ is a frame with frame bounds A and B, then the family $\{\tilde{g}_j(t)\}$ given by

$$\tilde{g}_j(t) = (\mathbf{S}^{-1}g_j)(t) \tag{4.16}$$

is a frame with bounds $\tilde{A} = \frac{1}{B}$ and $\tilde{B} = \frac{1}{A}$. The associated operator which assigns to each $x(t) \in \mathcal{H}$ the sequence $\langle x, \tilde{g}_j\rangle$ is given by

$$\tilde{\mathbf{T}} = \mathbf{T}\mathbf{S}^{-1} = \mathbf{T}(\mathbf{T}^*\mathbf{T})^{-1}$$

where

$$\tilde{\mathbf{T}} : \mathcal{H} \to l^2(J)$$
$$\tilde{\mathbf{T}} : x \to \langle x, \tilde{g}_j\rangle_{j\in J} \quad \text{or equivalently } (\tilde{\mathbf{T}}x)_j = \langle x, \tilde{g}_j\rangle.$$

We shall call $\{\tilde{g}_j(t)\}$ the *dual frame* associated to the frame $\{g_j(t)\}$.

Proof [28]: From (4.15) it follows that $\mathbf{S}^{-1}$ is positive definite and thus self-adjoint. Hence we have $\langle x, \tilde{g}_j\rangle = \langle x, \mathbf{S}^{-1}g_j\rangle = \langle \mathbf{S}^{-1}x, g_j\rangle$ for all $x(t) \in \mathcal{H}$. Thus, using (4.11), we obtain

$$\begin{aligned}\sum_j |\langle x, \tilde{g}_j\rangle|^2 &= \sum_j |\langle x, \mathbf{S}^{-1}g_j\rangle|^2 = \sum_j |\langle \mathbf{S}^{-1}x, g_j\rangle|^2 \\ &= \langle \mathbf{S}(\mathbf{S}^{-1}x), \mathbf{S}^{-1}x\rangle = \langle x, \mathbf{S}^{-1}x\rangle = \langle \mathbf{S}^{-1}x, x\rangle.\end{aligned}$$

So we can conclude from (4.15) that

$$\frac{1}{B}\|x\|^2 \leq \sum_j |\langle x, \tilde{g}_j\rangle|^2 \leq \frac{1}{A}\|x\|^2, \tag{4.17}$$

i.e., the $\tilde{g}_j(t)$ constitute a frame with frame bounds $\tilde{A} = \frac{1}{B}$ and $\tilde{B} = \frac{1}{A}$. It remains to show that $\tilde{\mathbf{T}} = \mathbf{T}\mathbf{S}^{-1}$:

$$(\tilde{\mathbf{T}}x)_j = \langle x, \tilde{g}_j\rangle = \langle x, \mathbf{S}^{-1}g_j\rangle = \langle \mathbf{S}^{-1}x, g_j\rangle = (\mathbf{T}\mathbf{S}^{-1}x)_j. \quad \square$$

For the dual frame too, it is convenient to introduce the *dual frame operator*:

Definition 4.5: The frame operator associated to the dual frame,

$$\tilde{\mathbf{S}} = \tilde{\mathbf{T}}^*\tilde{\mathbf{T}}, \quad \text{where} \quad (\tilde{\mathbf{S}}x)(t) = \sum_j \langle x, \tilde{g}_j\rangle \tilde{g}_j(t) \tag{4.18}$$

will be called the *dual frame operator.*

Lemma 4.1: The dual frame operator $\tilde{\mathbf{S}}$ satisfies

$$\tilde{\mathbf{S}} = \mathbf{S}^{-1}. \tag{4.19}$$

Proof: For any $x(t) \in \mathcal{H}$, we have

$$\begin{aligned}(\tilde{\mathbf{S}}x)(t) &= \sum_j \langle x, \tilde{g}_j\rangle \tilde{g}_j(t) = \sum_j \langle x, \mathbf{S}^{-1}g_j\rangle (\mathbf{S}^{-1}g_j)(t) \\ &= \left(\mathbf{S}^{-1}\sum_j \langle \mathbf{S}^{-1}x, g_j\rangle g_j\right)(t) = (\mathbf{S}^{-1}\mathbf{S}\mathbf{S}^{-1}x)(t) = (\mathbf{S}^{-1}x)(t). \quad \square\end{aligned}$$

We shall next consider the operator which provides the orthogonal projection of an arbitrary sequence onto the range of the operator $\mathbf{T}$.

Theorem 4.3: The operator[8]

$$\mathbf{P} : l^2(J) \to \mathrm{Ran}(\mathbf{T}) \subseteq l_2(J)$$

defined as

$$\mathbf{P} = \tilde{\mathbf{T}}\mathbf{T}^* = \mathbf{T}\mathbf{S}^{-1}\mathbf{T}^* = \mathbf{T}\tilde{\mathbf{T}}^*$$

is the orthogonal projection operator on $\mathrm{Ran}(\mathbf{T})$.

Note that $\mathbf{P}$ is defined for all sequences $c_j \in l^2(J)$, and

$$(\mathbf{P}c)_j = (\tilde{\mathbf{T}}\mathbf{T}^*c)_j = \left(\tilde{\mathbf{T}}\sum_k c_k g_k(t)\right)_j = \left\langle \sum_k c_k g_k, \tilde{g}_j \right\rangle$$

or equivalently

$$(\mathbf{P}c)_j = (\mathbf{T}\tilde{\mathbf{T}}^*c)_j = \left(\mathbf{T}\sum_k c_k \tilde{g}_k(t)\right)_j = \left\langle \sum_k c_k \tilde{g}_k, g_j \right\rangle.$$

Proof [28]: We have to show that

1. $\mathbf{P}$ is the identity operator $\mathbf{I}$ on $\mathrm{Ran}(\mathbf{T})$.

[8] $\mathrm{Ran}(\mathbf{T})$ denotes the range of the operator $\mathbf{T}$. The range of a linear operator $\mathbf{T} : \mathcal{X} \to \mathcal{Y}$ is $\mathrm{Ran}(\mathbf{T}) = \{y \mid y = \mathbf{T}x \quad \text{with} \quad x \in \mathcal{X}\}$

2. $\mathbf{P}$ is the zero operator on $\{\text{Ran}(\mathbf{T})\}^{\perp}$.

We first prove that $\mathbf{P}$ is the identity operator for $c \in \text{Ran}(\mathbf{T})$. If $c \in \text{Ran}(\mathbf{T})$, $c = \mathbf{T}x$, then $\mathbf{P}c = \tilde{\mathbf{T}}\mathbf{T}^*c = \mathbf{T}\mathbf{S}^{-1}\mathbf{T}^*c = \mathbf{T}\mathbf{S}^{-1}\mathbf{T}^*\mathbf{T}x = \mathbf{T}\mathbf{S}^{-1}\mathbf{S}x = \mathbf{T}x = c$.

The second part of the proof is based on the fact (shown in [98], p. 358) that $c \in \{\text{Ran}(\mathbf{T})\}^{\perp}$ if and only if $\mathbf{T}^*c = 0$. Assuming now that $c \in \{\text{Ran}(\mathbf{T})\}^{\perp}$ one obtains

$$\mathbf{P}c = \tilde{\mathbf{T}}\mathbf{T}^*c = 0$$

since $\mathbf{T}^*c = 0$ for $c \in \{\text{Ran}(\mathbf{T})\}^{\perp}$. □

4.4 Signal Expansions

The following theorem can be considered the *central result of the theory of frames.* It states that a frame is a *complete* set of functions, i.e., any signal $x(t) \in \mathcal{H}$ can be expanded into a frame. The expansion coefficients can be chosen as the inner products of $x(t)$ with the dual frame functions. The question whether or not these coefficients are *unique* will be addressed in Section 4.6.

Theorem 4.4: Let $\{g_j(t)\}$ and $\{\tilde{g}_j(t)\}$ be dual frames in $\mathcal{H}$. Any signal $x(t) \in \mathcal{H}$ can be expressed as

$$x(t) = (\mathbf{T}^*\tilde{\mathbf{T}}x)(t) = \sum_j \langle x, \tilde{g}_j \rangle \, g_j(t) \tag{4.20}$$

$$x(t) = (\tilde{\mathbf{T}}^*\mathbf{T}x)(t) = \sum_j \langle x, g_j \rangle \, \tilde{g}_j(t). \tag{4.21}$$

Note that, equivalently,

$$\mathbf{T}^*\tilde{\mathbf{T}} = \tilde{\mathbf{T}}^*\mathbf{T} = \mathbf{I}, \tag{4.22}$$

where $\mathbf{I}$ is the identity operator on $\mathcal{H}$.

(4.20) and (4.21) are "completeness relations," since they can also be written as

$$\sum_j g_j(t)\, \tilde{g}_j^*(t') = \sum_j \tilde{g}_j(t)\, g_j^*(t') = I(t-t'), \tag{4.23}$$

where $I(t, t')$ is the kernel of the identity operator $\mathbf{I}$ on $\mathcal{H}$.

Proof: We have

$$\begin{aligned}(\mathbf{T}^*\tilde{\mathbf{T}}x)(t) &= \sum_j \langle x, \tilde{g}_j \rangle g_j(t) = \sum_j \langle x, \mathbf{S}^{-1}g_j \rangle g_j(t) \\ &= \sum_j \langle \mathbf{S}^{-1}x, g_j \rangle g_j(t) = (\mathbf{S}\mathbf{S}^{-1}x)(t) = x(t),\end{aligned}$$

which proves $\mathbf{T}^*\tilde{\mathbf{T}} = \mathbf{I}$ or, equivalently, (4.20). The expansion (4.21) can be proved in the same manner. □

The duality of the functions $g_j(t)$ and $\tilde{g}_j(t)$ is also expressed by the following corollary.

Corollary 4.1: For any $x(t), y(t) \in \mathcal{H}$ we have

$$\langle x, y\rangle = \langle \mathbf{T}x, \tilde{\mathbf{T}}y\rangle = \sum_j \langle x, g_j\rangle \langle \tilde{g}_j, y\rangle \tag{4.24}$$

and

$$\langle x, y\rangle = \langle \tilde{\mathbf{T}}x, \mathbf{T}y\rangle = \sum_j \langle x, \tilde{g}_j\rangle \langle g_j, y\rangle. \tag{4.25}$$

Proof: With $\tilde{\mathbf{T}}^*\mathbf{T} = \mathbf{I}$ and $\mathbf{T}^*\tilde{\mathbf{T}} = \mathbf{I}$, we have

$$\langle x, y\rangle = \langle \mathbf{I}x, y\rangle = \langle \tilde{\mathbf{T}}^*\mathbf{T}, y\rangle = \langle \mathbf{T}x, \tilde{\mathbf{T}}y\rangle = \sum_j \langle x, g_j\rangle\langle y, \tilde{g}_j\rangle^*$$

$$\langle x, y\rangle = \langle \mathbf{I}x, y\rangle = \langle \mathbf{T}^*\tilde{\mathbf{T}}x, y\rangle = \langle \tilde{\mathbf{T}}x, \mathbf{T}y\rangle = \sum_j \langle x, \tilde{g}_j\rangle\langle y, g_j\rangle^*. \quad \square$$

Now there arises the question how the dual frame $\{\tilde{g}_j(t)\}$ can be obtained from $\{g_j(t)\}$. Due to Theorem 4.2, $\tilde{g}_j(t) = (\mathbf{S}^{-1}g_j)(t)$. The following theorem presents a series expansion of $\mathbf{S}^{-1}$.

Theorem 4.5: Let $\{g_j(t)\}$ and $\{\tilde{g}_j(t)\}$ be dual frames in $\mathcal{H}$, and let A and B be the frame bounds of $\{g_j(t)\}$. We then have

$$\mathbf{S}^{-1} = \frac{2}{A+B}\sum_{n=0}^{\infty}\left(\mathbf{I} - \frac{2}{A+B}\mathbf{S}\right)^n \tag{4.26}$$

or equivalently

$$\tilde{g}_j(t) = \frac{2}{A+B}\sum_{n=0}^{\infty}\left(\mathbf{I} - \frac{2}{A+B}\mathbf{S}\right)^n g_j(t). \tag{4.27}$$

The von Neumann series (4.26) is guaranteed to converge uniformly [98]. This convergence is governed by the frame bounds A, B according to

$$\left\|\mathbf{I} - \frac{2}{A+B}\mathbf{S}\right\| \le \frac{B-A}{B+A} < 1.$$

Proof [28]: Recall that $\tilde{g}_j(t) = (\mathbf{S}^{-1}g_j)(t)$. We can write $\mathbf{S}^{-1}$ as $\frac{2}{A+B}\left[\mathbf{I} - (\mathbf{I} - \frac{2}{A+B}\mathbf{S})\right]^{-1}$. When the von Neumann series expansion of $\left[\mathbf{I} - (\mathbf{I} - \frac{2}{A+B}\mathbf{S})\right]^{-1}$ converges uniformly, we obtain

$$\mathbf{S}^{-1} = \frac{2}{A+B}\left[\mathbf{I} - \left(\mathbf{I} - \frac{2}{A+B}\mathbf{S}\right)\right]^{-1} = \frac{2}{A+B}\sum_{n=0}^{\infty}\left(\mathbf{I} - \frac{2}{A+B}\mathbf{S}\right)^n.$$

The von Neumann series converges uniformly if $\|\mathbf{I} - \frac{2}{A+B}\mathbf{S}\| < 1$. From (4.13) it follows that

$$\begin{array}{rcll} -2B\,\mathbf{I} \le & -2\mathbf{S} & \le -2A\,\mathbf{I} & \quad | + (A+B)\mathbf{I} \\ (A-B)\,\mathbf{I} \le & (A+B)\mathbf{I} - 2\mathbf{S} & \le (B-A)\,\mathbf{I} & \\ -\dfrac{B-A}{A+B}\mathbf{I} \le & \mathbf{I} - \frac{2}{A+B}\mathbf{S} & \le \dfrac{B-A}{B+A}\mathbf{I}. & \end{array}$$

So we conclude that

$$\left\| \mathbf{I} - \frac{2}{A+B}\mathbf{S} \right\| \leq \frac{B-A}{B+A} < 1, \qquad \text{since } 0 < A \leq B. \quad \square \tag{4.28}$$

The closer A to B, the better the convergence. Frames with $A \approx B$ are called *snug frames.* A special (optimum) case is $A = B$:

Definition 4.6: A frame $\{g_j(t)\}$ with frame bounds $A = B$ is called a *tight frame.*

Note that, for a tight frame,

$$\langle \mathbf{S}x, x \rangle = \sum_j |\langle x, g_j \rangle|^2 = A\|x\|^2. \tag{4.29}$$

Corollary 4.2: If $\{g_j(t)\}$ is a tight frame in $\mathcal{H}$, then

$$\mathbf{S} = A\mathbf{I} \tag{4.30}$$

or equivalently

$$x(t) = \frac{1}{A}(\mathbf{S}x)(t) = \frac{1}{A}\sum_j \langle x, g_j \rangle \, g_j(t) \qquad \forall \, x(t) \in \mathcal{H}. \tag{4.31}$$

Proof: Combining $A\,\mathbf{I} \leq \mathbf{S} \leq B\,\mathbf{I}$ and $A = B$, we obtain $\mathbf{S} = A\mathbf{I}$ and furthermore

$$x(t) = (\mathbf{I}x)(t) = \frac{1}{A}(\mathbf{S}x)(t) = \frac{1}{A}\sum_j \langle x, g_j \rangle \, g_j(t). \quad \square$$

From Corollary 4.2 we conclude that if $\{g_j(t)\}$ is a tight frame, then the dual frame $\{\tilde{g}_j(t)\}$ is tight as well. This can be seen by noting that $\mathbf{S} = A\mathbf{I}$ implies that

$$\tilde{\mathbf{S}} = \mathbf{S}^{-1} = \frac{1}{A}\mathbf{I}.$$

Furthermore we have

$$\sum_j |\langle x, \tilde{g}_j \rangle|^2 = \langle \tilde{\mathbf{S}}x, x \rangle = \langle \mathbf{S}^{-1}x, x \rangle = \frac{1}{A}\|x\|^2.$$

We can see that tight frames provide an easy way of reconstruction, because we need not calculate the dual frame. It is evident that every orthonormal system is a tight frame with $A = 1$. Note, however, that conversely a tight frame (even with $A = 1$) need not be an orthonormal or orthogonal system. An interesting special case is considered in the next theorem.

Theorem 4.6: A tight frame with $A = 1$ and $\|g_j\|^2 = 1$ for all $j \in J$ is an orthonormal system.

Proof [28]: Combining

$$\langle \mathbf{S}g_k, g_k\rangle = A\|g_k\|^2 = \|g_k\|^2$$

and

$$\langle \mathbf{S}g_k, g_k\rangle = \sum_j |\langle g_k, g_j\rangle|^2 = \|g_k\|^4 + \sum_{j\neq k} |\langle g_k, g_j\rangle|^2,$$

we obtain

$$\|g_k\|^4 + \sum_{j\neq k} |\langle g_k, g_j\rangle|^2 = \|g_k\|^2.$$

Since $\|g_k\|^2 = 1$ for all k, it follows that $\sum_{j\neq k} |\langle g_k, g_j\rangle|^2 = 0$. This implies that the functions $g_j(t)$ are orthogonal to each other, which completes the proof. □

As mentioned above, the convergence speed of the expansion (4.26) depends on the frame bounds A and B. Let us consider the extreme case of retaining only the first term (corresponding to $n = 0$) in the series expansion (4.27), i.e. the dual frame functions $\tilde{g}_j(t)$ are approximated by

$$\tilde{g}_{j,0}(t) = \frac{2}{A+B}\, g_j(t).$$

Note that this approximation of $\tilde{g}_j(t)$ corresponds to an approximation of the reconstructed signal $x(t) = \sum_j \langle x, \tilde{g}_j\rangle\, g_j(t)$ by

$$x_0(t) = \frac{2}{A+B} \sum_j \langle x, g_j\rangle\, g_j(t).$$

The following theorem states a result on the error incurred when using this crude approximation.

Theorem 4.7: The norm of the error signal

$$R(t) = x(t) - x_0(t)$$

incurred when approximating $x(t)$ by

$$x_0(t) = \frac{2}{A+B} \sum_j \langle x, g_j\rangle\, g_j(t)$$

is bounded as

$$\|R\| \leq \frac{\frac{B}{A} - 1}{\frac{A}{B} + 1}\, \|x\|. \tag{4.32}$$

Proof: From Theorem 4.4 we have

$$\begin{aligned} x(t) &= \sum_j \langle x, g_j\rangle\, \tilde{g}_j(t) = \sum_j \langle x, g_j\rangle\, \frac{2}{A+B} \sum_{n=0}^{\infty} \left(\mathbf{I} - \frac{2}{A+B}\mathbf{S}\right)^n g_j(t) \\ &= \sum_j \langle x, g_j\rangle\, \frac{2}{A+B} \left[g_j(t) + \sum_{n=1}^{\infty} \left(\mathbf{I} - \frac{2}{A+B}\mathbf{S}\right)^n g_j(t) \right]. \end{aligned}$$

Hence we obtain

$$
\begin{aligned}
R(t) &= x(t) - \sum_j \langle x, g_j\rangle \frac{2}{A+B} g_j(t) \\
&= \sum_j \langle x, g_j\rangle \frac{2}{A+B} \sum_{n=1}^{\infty} \left(\mathbf{I} - \frac{2}{A+B}\mathbf{S}\right)^n g_j(t)
\end{aligned}
$$

and furthermore

$$
\begin{aligned}
\|R\| &= \left\| \sum_j \langle x, g_j\rangle \frac{2}{A+B} \sum_{n=1}^{\infty} \left(\mathbf{I} - \frac{2}{A+B}\mathbf{S}\right)^n g_j \right\| \\
&= \frac{2}{A+B} \left\| \sum_{n=1}^{\infty} \left(\mathbf{I} - \frac{2}{A+B}\mathbf{S}\right)^n \sum_j \langle x, g_j\rangle g_j \right\| \\
&\leq \frac{2}{A+B} \left\| \sum_{n=1}^{\infty} \left(\mathbf{I} - \frac{2}{A+B}\mathbf{S}\right)^n \right\| \left\| \sum_j \langle x, g_j\rangle g_j \right\| \\
&= \frac{2}{A+B} \left\| \sum_{n=1}^{\infty} \left(\mathbf{I} - \frac{2}{A+B}\mathbf{S}\right)^n \right\| \|\mathbf{S}x\| \\
&\leq \frac{2}{A+B} \sum_{n=1}^{\infty} \left\| \mathbf{I} - \frac{2}{A+B}\mathbf{S} \right\|^n \|\mathbf{S}x\| \\
&\leq \frac{2}{A+B} \sum_{n=1}^{\infty} \left\| \mathbf{I} - \frac{2}{A+B}\mathbf{S} \right\|^n \|\mathbf{S}\| \, \|x\|.
\end{aligned}
$$

In the proof of Theorem 4.5, we have shown that

$$
\left\| \mathbf{I} - \frac{2}{A+B}\mathbf{S} \right\| \leq \frac{B-A}{B+A}.
$$

Hence

$$
\begin{aligned}
\frac{2}{A+B} \sum_{n=1}^{\infty} \left\| \mathbf{I} - \frac{2}{A+B}\mathbf{S} \right\|^n \|\mathbf{S}\| \, \|x\| &\leq \frac{2}{A+B} \sum_{n=1}^{\infty} \left(\frac{B-A}{B+A}\right)^n \|\mathbf{S}\| \, \|x\| \\
&= \frac{2}{A+B} \left| \frac{\frac{B-A}{B+A}}{1 - \frac{B-A}{B+A}} \right| \|\mathbf{S}\| \, \|x\|.
\end{aligned}
$$

With $\|\mathbf{S}\| \leq B$, we finally obtain

$$
\|R\| \leq \frac{\frac{B}{A} - 1}{\frac{A}{B} + 1} \|x\|. \quad \square
$$

We can also see that the reconstruction error is small when $B \simeq A$. Thus, for snug frames $x_0(t)$ gives a good approximation of $x(t)$. For tight frames, $\|R\| = 0$ and (4.31) is valid.

We shall now formulate an *iterative algorithm* for the reconstruction of the signal $x(t)$ from its expansion coefficients $\langle x, g_j\rangle$ [99]. This algorithm is known to converge slowly. Better algorithms were recently proposed in [99].

Corollary 4.3: Let $\{g_j(t)\}$ be a frame for the Hilbert space $\mathcal{H}$ with frame bounds A and B. Then every signal $x(t) \in \mathcal{H}$ can be reconstructed from the coefficients $\langle x, g_j \rangle, j \in J$ by the recursion

$$x_n(t) = x_{n-1}(t) + \frac{2}{A+B}(\mathbf{S}(x - x_{n-1}))(t), \quad n \geq 1 \tag{4.33}$$

initialized by $x_0(t) = 0$. That is, $x(t) = \lim_{n\to\infty} x_n(t)$ with the error bound

$$\|x - x_n\| \leq \left(\frac{B-A}{B+A}\right)^n \|x\|.$$

The information needed for the iterative reconstruction of the signal $x(t)$ is $(\mathbf{S}x)(t) = \sum_j \langle x, g_j \rangle g_j(t)$. This requires the knowledge of the expansion coefficients $\langle x, g_j \rangle$ and the frame functions $g_j(t)$. We shall now give the proof of Corollary 4.3.

Proof [99]: From (4.28) we know that

$$\left\| \mathbf{I} - \frac{2}{A+B}\mathbf{S} \right\| \leq \frac{B-A}{B+A} < 1.$$

Using (4.33) we can write

$$\begin{aligned} x(t) - x_n(t) &= x(t) - x_{n-1}(t) - \frac{2}{A+B}(\mathbf{S}(x - x_{n-1}))(t) \\ &= \left(\left(\mathbf{I} - \frac{2}{A+B}\mathbf{S}\right)(x - x_{n-1})\right)(t). \end{aligned}$$

Iterating this recursion gives

$$x(t) - x_n(t) = \left(\left(\mathbf{I} - \frac{2}{A+B}\mathbf{S}\right)^n (x - x_0)\right)(t).$$

Taking the norm yields

$$\begin{aligned} \|x - x_n\| &\leq \left\| \left(\mathbf{I} - \frac{2}{A+B}\mathbf{S}\right)^n \right\| \|x - x_0\| \\ &\leq \left\| \mathbf{I} - \frac{2}{A+B}\mathbf{S} \right\|^n \|x - x_0\|. \end{aligned}$$

With (4.28) and $x_0(t) = 0$ it follows that

$$\|x - x_n\| \leq \left(\frac{B-A}{B+A}\right)^n \|x\|, \tag{4.34}$$

which guarantees the convergence of the sequence $x_n(t)$ towards $x(t)$ as n goes to infinity. □

4.5 Exact Frames and Biorthogonality

A set of functions $\{g_j(t)\}$ is complete in a Hilbert space $\mathcal{H}$ if $\langle x, g_j\rangle = 0$ for all $j \in J$ and with $x(t) \in \mathcal{H}$ implies $x(t) = 0$, i.e., the only function in $\mathcal{H}$ which is orthogonal to every $g_j(t)$ is $x(t) = 0$. Every $x(t) \in \mathcal{H}$ can be expanded into a complete set $\{g_j(t)\}$. Obviously, frames are complete sets of functions (cf. (4.20)). On the other hand, the frame signals $g_j(t)$ need not be linearly independent, i.e., the expansion coefficients c_j occurring in the expansion $x(t) = \sum_j c_j g_j(t)$ need not be unique. Frames with linearly independent frame signals $g_j(t)$ are called *exact*.

We shall call a frame "inexact" if the removal of an arbitrary frame signal $g_m(t)$ leaves a set $\{g_j(t)\}_{j\neq m}$ that is again a frame. For example, extending a frame $\{g_j(t)\}$ by adding a linear combination of frame signals $g_j(t)$ yields another frame, which is obviously "inexact." The opposite of an inexact frame is an *exact* frame:

> **Definition 4.7:** Frames $\{g_j(t)\}$ which become incomplete sets when an arbitrary function $g_m(t)$ is removed are called *exact*.

In Section 4.6, we shall show that the expansion of a signal into an *exact* frame is *unique*. In order to give a condition under which a frame is exact, we need two lemmas. The first lemma states that among all possible expansion coefficient sequences c_j satisfying $x(t) = \sum_j c_j g_j(t)$, the frame coefficients $c_j = \langle x, \tilde{g}_j\rangle$ have minimum l^2 norm.

> **Lemma 4.2:** Given a frame $\{g_j(t)\}$ and given $x(t) \in \mathcal{H}$, let $a_j = \langle x, \tilde{g}_j\rangle$ so that $x(t) = \sum_j a_j\, g_j(t)$. If it is possible to find other scalars c_j such that $x(t) = \sum_j c_j\, g_j(t)$, then we must have
>
> $$\sum_j |c_j|^2 = \sum_j |a_j|^2 + \sum_j |a_j - c_j|^2. \tag{4.35}$$

Note that this implies $\sum_j |c_j|^2 > \sum_j |a_j|^2$, i.e., the coefficients c_j have larger l^2 norm. This statement will be reconsidered from a different point of view in Section 4.8.

Proof [30]: We have

$$a_j = \langle x, \tilde{g}_j\rangle = \langle x, \mathbf{S}^{-1} g_j\rangle = \langle \mathbf{S}^{-1} x, g_j\rangle = \langle \tilde{x}, g_j\rangle$$

with $\tilde{x}(t) = (\mathbf{S}^{-1}x)(t)$. Therefore,

$$\langle x, \tilde{x}\rangle = \left\langle \sum_j a_j g_j, \tilde{x}\right\rangle = \sum_j a_j \langle g_j, \tilde{x}\rangle = \sum_j a_j a_j^* = \sum_j |a_j|^2$$

and

$$\langle x, \tilde{x}\rangle = \left\langle \sum_j c_j g_j, \tilde{x}\right\rangle = \sum_j c_j \langle g_j, \tilde{x}\rangle = \sum_j c_j a_j^*.$$

We conclude that

$$\sum_j |a_j|^2 = \sum_j c_j a_j^* = \sum_j c_j^* a_j.$$

Hence,

$$\begin{aligned}\sum_j |a_j|^2 + \sum_j |a_j - c_j|^2 &= \sum_j |a_j|^2 + \sum_j (a_j - c_j)(a_j^* - c_j^*) \\ &= \sum_j |a_j|^2 + \sum_j |a_j|^2 - \sum_j a_j c_j^* - \sum_j a_j^* c_j + \sum_j |c_j|^2 \\ &= \sum_j |c_j|^2. \quad \square\end{aligned}$$

Lemma 4.3: Let $\{g_j(t)\}$ be a frame. Then for each m we have

$$\sum_{j\neq m} |\langle g_m, \tilde{g}_j\rangle|^2 = \frac{1 - |\langle g_m, \tilde{g}_m\rangle|^2 - |1 - \langle g_m, \tilde{g}_m\rangle|^2}{2}.$$

Proof: There is obviously $g_m(t) = \sum_j c_j g_j(t)$, where $c_m = 1$ and $c_j = 0$ for $j \neq m$, so that $\sum_j |c_j|^2 = 1$. Furthermore let $a_j = \langle g_m, \tilde{g}_j\rangle$. We have

$$\sum_j |a_j - c_j|^2 = |a_m - c_m|^2 + \sum_{j\neq m} |a_j - c_j|^2 = |\langle g_m, \tilde{g}_m\rangle - 1|^2 + \sum_{j\neq m} |\langle g_m, \tilde{g}_j\rangle|^2,$$

and (4.35) yields

$$\begin{aligned}1 &= \sum_j |c_j|^2 = \sum_j |a_j|^2 + \sum_j |a_j - c_j|^2 \\ &= \sum_j |a_j|^2 + |a_m - c_m|^2 + \sum_{j\neq m} |a_j - c_j|^2 \\ &= \sum_j |\langle g_m, \tilde{g}_j\rangle|^2 + |\langle g_m, \tilde{g}_m\rangle - 1|^2 + \sum_{j\neq m} |\langle g_m, \tilde{g}_j\rangle|^2 \\ &= 2\sum_{j\neq m} |\langle g_m, \tilde{g}_j\rangle|^2 + |\langle g_m, \tilde{g}_m\rangle|^2 + |1 - \langle g_m, \tilde{g}_m\rangle|^2\end{aligned}$$

and hence

$$\sum_{j\neq m} |\langle g_m, \tilde{g}_j\rangle|^2 = \frac{1 - |\langle g_m, \tilde{g}_m\rangle|^2 - |1 - \langle g_m, \tilde{g}_m\rangle|^2}{2}. \quad \square$$

We are now able to formulate a condition for a frame to be exact:

Theorem 4.8: The removal of a function $g_m(t)$ from a frame $\{g_j(t)\}$ leaves either a frame or an incomplete set. In fact,

$$\begin{aligned}&\langle g_m, \tilde{g}_m\rangle = 1 \quad \text{for arbitrary } m \quad \Leftrightarrow \quad \{g_j(t)\} \text{ is exact.} \\ &\langle g_m, \tilde{g}_m\rangle \neq 1 \quad \text{for arbitrary } m \quad \Leftrightarrow \quad \{g_j(t)\} \text{ is inexact.}\end{aligned}$$

Proof [30]: We first show that $\langle g_m, \tilde{g}_m\rangle = 1$ implies that $\{g_j(t)\}_{j\neq m}$ is incomplete and hence $\{g_j(t)\}$ is an exact frame. From Lemma 4.3 we have

$$\sum_{j\neq m} |\langle g_m, \tilde{g}_j\rangle|^2 = \frac{1 - |\langle g_m, \tilde{g}_m\rangle|^2 - |1 - \langle g_m, \tilde{g}_m\rangle|^2}{2}.$$

Suppose now that $\langle g_m, \tilde{g}_m\rangle = 1$. Then $\sum_{j\neq m} |\langle g_m, \tilde{g}_j\rangle|^2 = 0$, so $\langle g_m, \tilde{g}_j\rangle = \langle \tilde{g}_m, g_j\rangle = 0$ for $j \neq m$. That is, $\tilde{g}_m(t)$ is orthogonal to $g_j(t)$ for every $j \neq m$. But $\tilde{g}_m(t) \neq 0$ since $\langle \tilde{g}_m, g_m\rangle = 1 \neq 0$. Therefore $\{g_j(t)\}_{j\neq m}$ is incomplete, because $\tilde{g}_m(t)$ is orthogonal to every function of the family $\{g_j(t)\}_{j\neq m}$.

We next show that $\langle g_m, \tilde{g}_m\rangle \neq 1$ implies that $\{g_j(t)\}_{j\neq m}$ is still a frame. We can always write

$$g_m(t) = \sum_j \langle g_m, \tilde{g}_j\rangle g_j(t) = \langle g_m, \tilde{g}_m\rangle g_m(t) + \sum_{j\neq m} \langle g_m, \tilde{g}_j\rangle g_j(t).$$

If $\langle g_m, \tilde{g}_m\rangle \neq 1$, this can be written as

$$g_m(t) = \frac{1}{1 - \langle g_m, \tilde{g}_m\rangle} \sum_{j\neq m} \langle g_m, \tilde{g}_j\rangle g_j(t),$$

and for $x(t) \in \mathcal{H}$ we have

$$\begin{aligned} |\langle x, g_m\rangle|^2 &= \left|\frac{1}{1 - \langle g_m, \tilde{g}_m\rangle}\right|^2 \left|\sum_{j\neq m} \langle g_m, \tilde{g}_j\rangle \langle x, g_j\rangle\right|^2 \\ &\leq \frac{1}{|1 - \langle g_m, \tilde{g}_m\rangle|^2} \left[\sum_{j\neq m} |\langle g_m, \tilde{g}_j\rangle|^2\right] \left[\sum_{j\neq m} |\langle x, g_j\rangle|^2\right]. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_j |\langle x, g_j\rangle|^2 &= |\langle x, g_m\rangle|^2 + \sum_{j\neq m} |\langle x, g_j\rangle|^2 \\ &\leq \frac{1}{|1 - \langle g_m, \tilde{g}_m\rangle|^2} \left[\sum_{j\neq m} |\langle g_m, \tilde{g}_j\rangle|^2\right] \left[\sum_{j\neq m} |\langle x, g_j\rangle|^2\right] + \sum_{j\neq m} |\langle x, g_j\rangle|^2 \\ &= \sum_{j\neq m} |\langle x, g_j\rangle|^2 \left[1 + \frac{1}{|1 - \langle g_m, \tilde{g}_m\rangle|^2} \sum_{j\neq m} |\langle g_m, \tilde{g}_j\rangle|^2\right] \\ &= C \sum_{j\neq m} |\langle x, g_j\rangle|^2 \end{aligned}$$

or equivalently

$$\frac{1}{C} \sum_j |\langle x, g_j\rangle|^2 \leq \sum_{j\neq m} |\langle x, g_j\rangle|^2$$

with

$$C = 1 + \frac{1}{|1 - \langle g_m, \tilde{g}_m\rangle|^2} \sum_{j\neq m} |\langle g_m, \tilde{g}_j\rangle|^2.$$

With (4.5) it follows that

$$\frac{A}{C}\|x\|^2 \leq \frac{1}{C}\sum_j |\langle x, g_j\rangle|^2 \leq \sum_{j\neq m} |\langle x, g_j\rangle|^2 \leq \sum_j |\langle x, g_j\rangle|^2 \leq B\|x\|^2,$$

so $\{g_j(t)\}_{j\neq m}$ is a frame with bounds $\frac{A}{C}, B$.

To see that, conversely, an exact $\{g_j(t)\}$ implies that $\langle g_m, \tilde{g}_m\rangle = 1$ for all m, we suppose that $\{g_j(t)\}$ is exact and $\langle g_m, \tilde{g}_m\rangle \neq 1$. But the condition $\langle g_m, \tilde{g}_m\rangle \neq 1$ implies that $\{g_j(t)\}$ is inexact, which is a contradiction. It remains to show that an inexact $\{g_j(t)\}$ implies $\langle g_m, \tilde{g}_m\rangle \neq 1$ for all m. Suppose that $\{g_j(t)\}$ is inexact and $\langle g_m, \tilde{g}_m\rangle = 1$. The condition $\langle g_m, \tilde{g}_m\rangle = 1$ implies that $\{g_j(t)\}$ is exact, which is a contradiction. □

Corollary 4.4: If $\{g_j(t)\}$ is an exact frame, then $\{g_j(t)\}$ and $\{\tilde{g}_j(t)\}$ are biorthogonal, i.e.,

$$\langle g_m, \tilde{g}_j\rangle = \delta_{mj} = \begin{cases} 1, & \text{if } j = m \\ 0, & \text{if } j \neq m \end{cases}$$

Conversely, if $\{g_j(t)\}$ and $\{\tilde{g}_j(t)\}$ are biorthogonal, then $\{g_j(t)\}$ is exact.

Proof [30]: If $\{g_j(t)\}$ is exact, then by Theorem 4.8 we must have $\langle g_m, \tilde{g}_m\rangle = 1$ for every m, and hence by Lemma 4.3 we have $\sum_{j\neq m} |\langle g_m, \tilde{g}_j\rangle|^2 = 0$ and thus also $\langle g_m, \tilde{g}_j\rangle = 0$ for all $j \neq m$, as claimed. It remains to show that, conversely, the biorthogonality of $\{g_j(t)\}$ and $\{\tilde{g}_j(t)\}$ implies that the frame $\{g_j(t)\}$ is exact. For biorthogonal functions $g_j(t)$ and $\tilde{g}_j(t)$ we have $\langle g_m, \tilde{g}_j\rangle = \delta_{mj}$ and hence $\langle g_m, \tilde{g}_m\rangle = 1$ for all m. So by Theorem 4.8 we conclude that $\{g_j(t)\}$ is exact. □

Corollary 4.5 A frame $\{g_j(t)\}$ is exact if and only if the dual frame $\{\tilde{g}_j(t)\}$ is exact.

Proof: This follows immediately from Corollary 4.4 and the symmetry inherent in the inner product $\langle g_m, \tilde{g}_j\rangle$. □

The next theorem establishes bounds on the energy of $g_j(t)$.

Theorem 4.9: Let $\{g_j(t)\}$ be a frame with bounds A, B. Then

$$\|g_j\|^2 \leq B.$$

Furthermore if $\{g_j(t)\}$ is exact, then

$$\|g_j\|^2 \geq A.$$

Proof:

1. For m fixed we have

$$\begin{aligned}\|g_m\|^4 &= |\langle g_m, g_m\rangle|^2 \leq |\langle g_m, g_m\rangle|^2 + \sum_{j\neq m} |\langle g_m, g_j\rangle|^2 \\ &= \sum_j |\langle g_m, g_j\rangle|^2 \leq B\|g_m\|^2,\end{aligned}$$

so $\|g_m\|^2 \leq B$.

2. If $\{g_j(t)\}$ is an exact frame then $\{g_j(t)\}$ and $\{\tilde{g}_j(t)\}$ are biorthogonal by Corollary 4.4. Therefore for m fixed we have

$$A\|\tilde{g}_m\|^2 \leq \sum_j |\langle \tilde{g}_m, g_j\rangle|^2 = |\langle \tilde{g}_m, g_m\rangle|^2 \leq \|\tilde{g}_m\|^2\|g_m\|^2,$$

so $\|g_m\|^2 \geq A$. □

4.6 Frames and Bases

We shall now give some results about the relation between *frames* and *bases*. The central result is that an exact frame is a basis, i.e. the expansion of a signal into an exact frame is *unique*. We first give three definitions.

Definition 4.8: A set of signals $\{g_j(t)\}$ in a Hilbert space $\mathcal{H}$ is a *basis* for $\mathcal{H}$ if for every $x(t) \in \mathcal{H}$ there exist unique scalars c_j such that

$$x(t) = \sum_j c_j\, g_j(t).$$

Definition 4.9: A basis $g_j(t)$ in a Hilbert space $\mathcal{H}$ is *bounded* if $0 < \|g_j\| < \infty$ for every j.

Definition 4.10: A basis $\{g_j(t)\}$ in a Hilbert space $\mathcal{H}$ is *unconditional* if the series $\sum_j c_j\, g_j(t)$ converges unconditionally, that is, every permutation of the series converges.

We are now able to state

Theorem 4.10: A set of signals $\{g_j(t)\}$ in a Hilbert space $\mathcal{H}$ is an *exact frame* for $\mathcal{H}$ if and only if it is a *bounded unconditional basis* for $\mathcal{H}$.

Proof [30]: Assume $\{g_j(t)\}$ is an exact frame with bounds A, B. Then from Theorem 4.9 we have

$$\|g_m\|^2 \geq A \quad \text{and} \quad \|g_m\|^2 \leq B.$$

We have thus shown that the set of signals $\{g_j(t)\}$ is bounded in the sense of Definition 4.9. By Theorem 4.4 we have

$$x(t) = \sum_j c_j g_j(t) \quad \text{with } c_j = \langle x, \tilde{g}_j \rangle$$

for any $x(t) \in \mathcal{H}$. We now have to prove that this representation is unique. Assume that there is a representation of the form $x(t) = \sum_j a_j\, g_j(t)$ with $a_j \neq c_j$. Then,

$$c_l = \langle x, \tilde{g}_l \rangle = \Big\langle \sum_j a_j g_j, \tilde{g}_l \Big\rangle = \sum_j a_j \langle g_j, \tilde{g}_l \rangle = a_l$$

where we have used the biorthogonality of $\{g_j(t)\}$ and $\{\tilde{g}_j(t)\}$. So it follows that $a_l = c_l$ and we conclude that the representation is unique. Thus $\{g_j(t)\}$ is a basis for $\mathcal{H}$, and since every permutation of a frame is also a frame, we conclude that the basis is unconditional. We have shown that the exactness of $\{g_j(t)\}$ implies that $\{g_j(t)\}$ is a bounded unconditional basis.

It remains to prove that the converse is also true. Assume that $\{g_j(t)\}$ is a bounded unconditional basis for $\mathcal{H}$. In Hilbert spaces, all bounded unconditional bases are equivalent to orthonormal bases, in the sense that if $\{g_j(t)\}$ is a bounded unconditional basis, then there exist an orthonormal basis $\{e_j(t)\}$ and a topological isomorphism[9] $\mathbf{U} : \mathcal{H} \to \mathcal{H}$ such that $g_j(t) = (\mathbf{U}e_j)(t)$ for all j [100]. Given $x(t) \in \mathcal{H}$, we therefore have

$$\sum_j |\langle x, g_j \rangle|^2 = \sum_j |\langle x, \mathbf{U}e_j \rangle|^2 = \sum_j |\langle \mathbf{U}^* x, e_j \rangle|^2 = \|\mathbf{U}^* x\|^2 .$$

But on the other hand

$$\|x\| = \|\mathbf{I}x\| = \left\|\mathbf{U}^{*^{-1}} \mathbf{U}^* x\right\| \leq \left\|\mathbf{U}^{*-1}\right\| \|\mathbf{U}^* x\|$$

which implies that

$$\frac{\|x\|}{\|\mathbf{U}^{*-1}\|} \leq \|\mathbf{U}^* x\|$$

and hence

$$\frac{\|x\|^2}{\|\mathbf{U}^{*-1}\|^2} \leq \|\mathbf{U}^* x\|^2 = \sum_j |\langle x, g_j \rangle|^2.$$

Furthermore we have

$$\sum_j |\langle x, g_j \rangle|^2 = \|\mathbf{U}^* x\|^2 \leq \|\mathbf{U}^*\|^2 \|x\|^2 = \|\mathbf{U}\|^2 \|x\|^2.$$

Thus, $\{g_j(t)\}$ is a frame with frame bounds

$$A = \frac{1}{\|\mathbf{U}^{*-1}\|^2} \quad \text{and} \quad B = \|\mathbf{U}\|^2 .$$

[9]A topological isomorphism is a continuous linear transformation of $\mathcal{H}$ onto $\mathcal{H}$ such that the inverse transformation exists and is continuous.

It is clearly exact since the removal of any vector from a basis leaves an incomplete set. □

Theorem 4.11: Inexact frames are not bases.

Thus, inexact frames are complete (any signal $x(t) \in \mathcal{H}$ can be expanded as $x(t) = \sum_j c_j\, g_j(t)$) but *the expansion coefficients c_j are not unique.*

Proof [101]: Assume $\{g_j(t)\}$ is an inexact frame. Then $\{g_j(t)\}_{j\neq m}$ is a frame for some m. By Theorem 4.4 we have $g_m(t) = \sum\limits_j a_j\, g_j(t) = a_m\, g_m(t) + \sum_{j\neq m} a_j\, g_j(t)$, where $a_j = \langle g_m, \tilde{g}_j \rangle$. But we also have $g_m(t) = \sum\limits_j c_j\, g_j(t)$, where $c_j = 1$ for $j = m$ and $c_j = 0$ for $j \neq m$. By Theorem 4.8 we must have $a_m = \langle g_m, \tilde{g}_m \rangle \neq 1$, i.e., $a_m \neq c_m$. Thus we have two different representations for $g_m(t)$ in terms of $\{g_j(t)\}$, and hence $\{g_j(t)\}$ is not a basis. □

Theorem 4.12: A frame is tight and exact if and only if it is an orthogonal basis with $\|g_j\|^2 = A$.

Proof: For an exact frame we have $\langle g_j, \tilde{g}_m \rangle = \delta_{jm}$. If the frame is tight we have $\tilde{g}_j(t) = \frac{1}{A} g_j(t)$. Thus we conclude that $\langle g_j, g_m \rangle = A\,\delta_{jm}$, which proves that $\{g_j(t)\}$ is an orthogonal basis. To see that the converse is also true one has to recall that every orthogonal basis with $\langle g_j, g_m \rangle = A\,\delta_{jm}$ is a tight frame with frame bound $A = B$. Thus we have $\tilde{g}_j(t) = \frac{1}{A} g_j(t)$ and we obtain the biorthogonality relation $\langle g_j, \tilde{g}_m \rangle = \delta_{jm}$ which implies the exactness of the frame. □

An exact frame is a set of *linearly independent* functions.[10] Since an exact frame $\{g_j(t)\}$ is a bounded unconditional basis for the corresponding Hilbert space, the expansion coefficients c_j in $x(t) = \sum_j c_j g_j(t)$ are *unique.* Due to [98] (pp.178, Theorem 4.6.5) the set of signals $g_j(t)$ is linearly independent if and only if the expansion coefficients c_j are unique. Hence we conclude that an exact frame is a set of linearly independent functions. Furthermore an inexact frame is a set of linear dependent functions because the expansion coefficients c_j are not unique (see Theorem 4.11).

4.7 Transformation of Frames

We shall next characterize frame-preserving mappings[11]. Starting from a frame $\{g_j(t)\}$ for a space $\mathcal{H}_1$, we want to find frames $\{h_j(t)\}$ for some other space $\mathcal{H}_2$. One possible

[10] A set $\{g_j(t)\}$ of signals in a Hilbert space $\mathcal{H}$ is linearly independent if and only if for each finite subset $\{g_1(t), ..., g_n(t)\}$ the only n-tuple of scalars satisfying the equation $a_1 g_1(t) + ... + a_n g_n(t) = 0$ is the trivial solution $a_1 = ... = a_n = 0$.

[11] The results of this section have partly been taken from [102]. We also present extensions of these results.

approach is to construct a set of functions $h_j(t) = (\mathbf{U}g_j)(t)$, where $\mathbf{U}$ is a bounded, linear operator from $\mathcal{H}_1$ into $\mathcal{H}_2$ with $\operatorname{Ran}\{\mathbf{U}\} = \mathcal{H}_2$. The following theorem states a necessary and sufficient condition on the operator $\mathbf{U}$ in order for $h_j(t)$ to be a frame.

Theorem 4.13: Let $\{g_j(t)\}$ be a frame for $\mathcal{H}_1$ with bounds A, B. Let $\mathbf{U}$ be a bounded, linear operator mapping $\mathcal{H}_1$ into $\mathcal{H}_2$ and $\operatorname{Ran}\{\mathbf{U}\} = \mathcal{H}_2$. Then $\{h_j(t)\} = \{(\mathbf{U}g_j)(t)\}$ is a frame for $\mathcal{H}_2$ if and only if the adjoint operator $\mathbf{U}^*$ is bounded below, i.e. if there exists a positive constant δ such that the adjoint operator $\mathbf{U}^*$ satisfies

$$\|\mathbf{U}^* y\|^2 \geq \delta \|y\|^2 \qquad \forall\, y(t) \in \mathcal{H}_2. \tag{4.36}$$

Frame bounds for $\{h_j(t)\}$ are given by $C = \delta A$ and $D = B\|U\|^2$.

Proof: We first prove that (4.36) is a *sufficient* condition. Given $y(t) \in \mathcal{H}_2$, we have $(\mathbf{U}^* y)(t) \in \mathcal{H}_1$ and thus, by (4.5),

$$\sum_j |\langle y, \mathbf{U}g_j\rangle|^2 = \sum_j |\langle \mathbf{U}^* y, g_j\rangle|^2 \geq A\|\mathbf{U}^* y\|^2 \geq A\,\delta \|y\|^2 \quad \forall y(t) \in \mathcal{H}_2.$$

So we have found a lower frame bound $C = \delta A$ for $\{(\mathbf{U}g_j)(t)\}$. The next step is to compute an upper frame bound for $\{(\mathbf{U}g_j)(t)\}$. From (4.5) we have

$$\begin{aligned}\sum_j |\langle y, \mathbf{U}g_j\rangle|^2 &= \sum_j |\langle \mathbf{U}^* y, g_j\rangle|^2 \leq B\|\mathbf{U}^* y\|^2 \\ &\leq B\|\mathbf{U}^*\|^2 \|y\|^2 = B\|\mathbf{U}\|^2\|y\|^2 \quad \forall y(t) \in \mathcal{H}_2,\end{aligned}$$

where the last equation follows from the fact that $\|\mathbf{U}^*\| = \|\mathbf{U}\|$. We conclude that an upper frame bound is $D = B\|\mathbf{U}\|^2$, which completes the proof of the sufficiency.

We shall next prove that (4.36) is a *necessary* condition. Assume that $\{(\mathbf{U}g_j)(t)\}$ is a frame for $\mathcal{H}_2$ with frame bounds C and D,

$$C\|y\|^2 \leq \sum_j |\langle y, \mathbf{U}g_j\rangle|^2 \leq D\|y\|^2.$$

Again using (4.5), it follows that

$$C\|y\|^2 \leq \sum_j |\langle y, \mathbf{U}g_j\rangle|^2 = \sum_j |\langle \mathbf{U}^* y, g_j\rangle|^2 \leq B\|\mathbf{U}^* y\|^2$$

where B is the upper frame bound of $\{g_j(t)\}$, and hence

$$\|\mathbf{U}^* y\|^2 \geq \frac{C}{B}\|y\|^2,$$

which is (4.36) with $\delta = \frac{C}{B}$. The constant δ is greater than zero because $B < \infty$. □

Corollary 4.6: If $\{g_j(t)\}$ is a frame for $\mathcal{H}_1$, $\mathbf{U}$ maps $\mathcal{H}_1$ into $\mathcal{H}_2$, and $\{h_j(t)\} = \{(\mathbf{U}g_j)(t)\}$ is a frame for $\mathcal{H}_2$, then the frame operator $\mathbf{S}_h$ of $\{h_j(t)\}$ is

$$\mathbf{S}_h = \mathbf{U}\mathbf{S}\mathbf{U}^*,$$

where $\mathbf{S}$ is the frame operator of $\{g_j(t)\}$. Furthermore if $\mathbf{U}$ is an invertible mapping of $\mathcal{H}_1$ onto $\mathcal{H}_2$, the dual frame $\{\tilde{h}_j(t)\}$ is given by

$$\tilde{h}_j(t) = \mathbf{U}^{*^{-1}}\tilde{g}_j(t).$$

Proof: We first show that the frame operator of $\{h_j(t)\}$ can be expressed as $\mathbf{S}_h = \mathbf{U}\mathbf{S}\mathbf{U}^*$:

$$\begin{aligned}
(\mathbf{S}_h x)(t) &= \sum_j \langle x, h_j\rangle\, h_j(t) = \sum_j \langle x, \mathbf{U}g_j\rangle\, (\mathbf{U}g_j)(t) \\
&= \sum_j \langle \mathbf{U}^* x, g_j\rangle (\mathbf{U}g_j)(t) = \left(\mathbf{U}\sum_j \langle \mathbf{U}^* x, g_j\rangle\, g_j\right)(t) \\
&= (\mathbf{U}\mathbf{S}\mathbf{U}^* x)(t).
\end{aligned}$$

Furthermore we have to show that $\{\tilde{h}_j(t)\} = \{\mathbf{U}^{*^{-1}} g_j(t)\}$. Using the previous result about the frame operator $\mathbf{S}_h$, we obtain

$$\begin{aligned}
\tilde{h}_j(t) &= (\mathbf{S}_h^{-1} h_j)(t) = (\mathbf{S}_h^{-1}\mathbf{U}\, g_j)(t) = \left((\mathbf{U}\mathbf{S}\mathbf{U}^*)^{-1}\mathbf{U}\, g_j\right)(t) \\
&= (\mathbf{U}^{*^{-1}}\mathbf{S}^{-1}\mathbf{U}^{-1}\mathbf{U}\, g_j)(t) = (\mathbf{U}^{*^{-1}}\tilde{g}_j)(t),
\end{aligned}$$

where $\mathbf{S}$ is the frame operator of $\{g_j(t)\}$. □

An important special case is an operator $\mathbf{U}$ that is *unitary*, i.e. $\mathbf{U}^{*^{-1}} = \mathbf{U}$. In that case we have

$$\tilde{h}_j(t) = (\mathbf{U}\tilde{g}_j)(t),$$

i.e., the dual frames $\{\tilde{g}_j(t)\}$ and $\{\tilde{h}_j(t)\}$ are related by the same mapping (namely, $\mathbf{U}$) as the frames $g_j(t)$ and $h_j(t)$.

We shall now discuss a frame-preserving mapping of particular importance, namely $\mathbf{U} = \mathbf{S}^{-1/2}$, which maps $\mathcal{H}_1$ onto $\mathcal{H}_1$ (i.e., $\mathcal{H}_2 = \mathcal{H}_1$).

Corollary 4.7: Let $\{g_j(t)\}$ be a frame with frame operator $\mathbf{S}$. Then $\{(\mathbf{S}^{-1/2}g_j)(t)\}$ is a tight frame with $A = 1$. Moreover if $\{g_j(t)\}$ is exact, then $\{(\mathbf{S}^{-1/2}g_j)(t)\}$ is an orthonormal basis for $\mathcal{H}$.

Proof [101]: We know from Theorem 4.13 that $\{(\mathbf{S}^{-1/2}g_j)(t)\}$ is a frame, because $\|\mathbf{S}^{-1/2^*}x\|^2 \geq \delta\|x\|^2$ with $\delta > 0$. We shall first show that $\|\mathbf{S}^{-1/2^*}x\|^2 = \|\mathbf{S}^{-1/2}x\|^2 \geq 0$.

This follows from $\langle \mathbf{S}^{-1}x, x\rangle = \langle \mathbf{S}^{-1/2}x, \mathbf{S}^{-1/2}x\rangle = \|\mathbf{S}^{-1/2}x\|^2 \geq \frac{1}{B}\|x\|^2$. We are now able to show that $\{(\mathbf{S}^{-1/2}g_j)(t)\}$ is tight with $A = 1$. According to Corollary 4.2, we have to show that

$$x(t) = \sum_j \langle x, \mathbf{S}^{-1/2}g_j\rangle\,(\mathbf{S}^{-1/2}g_j)(t).$$

Indeed, the right-hand side of this equation is

$$\begin{aligned}\sum_j \langle x, \mathbf{S}^{-1/2}g_j\rangle(\mathbf{S}^{-1/2}g_j)(t) &= \mathbf{S}^{-1/2}\sum_j \langle \mathbf{S}^{-1/2}x, g_j\rangle\, g_j(t)\\ &= (\mathbf{S}^{-1/2}\mathbf{S}\mathbf{S}^{-1/2}x)(t) = (\mathbf{I}x)(t) = x(t).\end{aligned}$$

For the proof of the second statement, we have to consider the inner product

$$\langle \mathbf{S}^{-1/2}g_j, \mathbf{S}^{-1/2}g_m\rangle = \langle g_j, \mathbf{S}^{-1}g_m\rangle = \langle g_j, \tilde{g}_m\rangle.$$

Since $\{g_j(t)\}$ is exact we have

$$\langle g_j, \tilde{g}_m\rangle = \delta_{mj},$$

and hence we conclude that $\{(\mathbf{S}^{-1/2}g_j)(t)\}$ is an orthonormal basis for $\mathcal{H}$. □

We shall next ask whether the orthogonal projection of a frame $\{g_j(t)\}$ for $\mathcal{H}$ into a subspace $\mathcal{H}_1 \subset \mathcal{H}$ yields a frame for $\mathcal{H}_1$. The following theorem gives an answer.

> **Theorem 4.14:** Let $\mathcal{H}_2 \subset \mathcal{H}_1$ be a subspace of $\mathcal{H}_1$, and let $\{g_j(t)\}$ be a frame for $\mathcal{H}_1$. Let $\mathbf{P}$ denote the orthogonal projection operator from $\mathcal{H}_1$ into $\mathcal{H}_2$. Then $\{(\mathbf{P}g_j)(t)\}$ and $\{(\mathbf{P}\tilde{g}_j)(t)\}$ are dual frames of $\mathcal{H}_2$. Moreover, the frame bounds A and B of $\{g_j(t)\}$ are also frame bounds for $\{(\mathbf{P}g_j)(t)\}$.

Proof [102]: The orthogonal projection operator $\mathbf{P}$ satisfies

$$\|\mathbf{P}^*y\| = \|\mathbf{P}y\|^2 = \|y\|^2 \qquad \forall y(t) \in \mathcal{H}_2.$$

We thus have the conditions of Theorem 4.13 with $\delta = 1$. It follows that $\{(\mathbf{P}g_j)(t)\}$ is a frame for $\mathcal{H}_2$. To obtain the dual frame we write $x(t)$ in terms of the frame $\{g_j(t)\}$,

$$x(t) = \sum_j \langle x, \tilde{g}_j\rangle\, g_j(t).$$

Furthermore we use the fact that $y(t) = (\mathbf{P}y)(t)$ for $y(t) \in \mathcal{H}_2$ to obtain

$$y(t) = (\mathbf{P}y)(t) = \sum_j \langle \mathbf{P}y, \tilde{g}_j\rangle\, g_j(t) = \sum_j \langle y, \mathbf{P}\tilde{g}_j\rangle\, g_j(t), \qquad \forall y(t) \in \mathcal{H}_2$$

and further

$$y(t) = (\mathbf{P}y)(t) = \left(\mathbf{P}\sum_j \langle y, \mathbf{P}\tilde{g}_j\rangle\, g_j\right)(t) = \sum_j \langle y, \mathbf{P}\tilde{g}_j\rangle(\mathbf{P}g_j)(t) \qquad \forall y(t) \in \mathcal{H}_2.$$

From this equation, A. Aldroubi [102] concludes[12] that $\{(\mathbf{P}g_j)(t)\}$ and $\{(\mathbf{P}\tilde{g}_j)(t)\}$ are dual frames in $\mathcal{H}_2$.

To complete the proof, we have to show that the frame bounds of $\{g_j(t)\}$ are also frame bounds of $\{(\mathbf{P}g_j)(t)\}$. Since $\mathcal{H}_2 \subset \mathcal{H}_1$, it follows from (4.5) that

$$A\|y\|^2 \leq \sum_j |\langle y, g_j\rangle|^2 \leq B\|y\|^2 \qquad \forall y(t) \in \mathcal{H}_2.$$

Furthermore $y(t) = (\mathbf{P}y)(t)$ for all $y(t) \in \mathcal{H}_2$ and hence

$$A\|y\|^2 \leq \sum_j |\langle \mathbf{P}y, g_j\rangle|^2 \leq B\|y\|^2, \qquad \forall y(t) \in \mathcal{H}_2,$$

which implies that

$$A\|y\|^2 \leq \sum_j |\langle y, \mathbf{P}g_j\rangle|^2 \leq B\|y\|^2 \qquad \forall y(t) \in \mathcal{H}_2. \quad \square$$

4.8 Frames and Pseudoinverses

Consider an inexact frame $\{g_j(t)\}$ for the Hilbert space $\mathcal{H}$. Due to Theorem 4.4, any signal $x(t) \in \mathcal{H}$ can be represented as $x(t) = \sum_j \langle x, \tilde{g}_j\rangle\, g_j(t)$, where $\{\tilde{g}_j(t)\}$ is the dual frame. A consequence of the "overcompleteness" of an inexact frame $\{g_j(t)\}$ is that the "frame coefficients" $\langle x, \tilde{g}_j\rangle$ do not constitute the only sequence c_j satisfying

$$x(t) = \sum_j c_j\, g_j(t). \tag{4.37}$$

However, Lemma 4.2 states that the coefficients $a_j = \langle x, \tilde{g}_j\rangle$ have *minimum norm* among all possible sequences c_j. We shall now consider this property from a different point of view.

Let us reformulate the problem. We consider the operator $\mathbf{T}^*$ (Definition 4.3) which acts on a sequence $c_j \in l^2(J)$ as

$$(\mathbf{T}^* c_j)(t) = \sum_j c_j\, g_j(t).$$

Due to (4.37), the coefficients c_j satisfy the linear equation

$$(\mathbf{T}^* c_j)(t) = x(t).$$

As stated further above, it follows from the inexactness of the frame $\{g_j(t)\}$ that this equation has more than one solution c_j. Hence, the equation $\mathbf{T}^* c_j = x$ is *underdetermined.* We are interested in the solution a_j with minimum norm, i.e.

$$a_j = \arg\min_{c_j} \|c_j\|.$$

[12] In our opinion it is doubtful whether this conclusion can really be drawn, since the expansion coefficients c_j in $x(t) = \sum_j c_j (\mathbf{P}g_j)(t)$ are not unique in general.

According to [103] (pp. 161-165), this sequence is given by the *pseudoinverse* $(\mathbf{T}^*)^\dagger = \mathbf{T}(\mathbf{T}^*\mathbf{T})^{-1}$ of the operator $\mathbf{T}^*$,

$$a_j = (\mathbf{T}^*)^\dagger x = \mathbf{T}(\mathbf{T}^*\mathbf{T})^{-1}x = \mathbf{T}\mathbf{S}^{-1}x = \tilde{\mathbf{T}}x.$$

Thus, the operator $\tilde{\mathbf{T}} = \mathbf{T}\mathbf{S}^{-1}$ defined in Theorem 4.2, assigning to each signal $x(t)$ the frame coefficient sequence $a_j = \langle x, \tilde{g}_j\rangle$, is the pseudoinverse of $\mathbf{T}^*$ which assigns to each signal $x(t) \in \mathcal{H}$ the coefficient sequence with minum norm:

$$\tilde{\mathbf{T}} = \mathbf{T}\mathbf{S}^{-1} = (\mathbf{T}^*)^\dagger.$$

4.9 Frames and the Gram Matrix

The *Gram matrix* $\mathbf{G}$ of a set of signals $\{g_j(t)\}$ is a Hermitian, positive semidefinite matrix whose elements are the inner products of the signals $g_j(t)$,

$$G_{ij} = \langle g_i, g_j\rangle.$$

Given a frame $\{g_j(t)\}$, consider the operator $\mathbf{T}\mathbf{T}^*$ mapping $l^2(J)$ into $l^2(J)$ according to

$$\begin{aligned}(\mathbf{T}\mathbf{T}^*c)_j &= \left(\mathbf{T}\sum_k c_k g_k(t)\right)_j = \left\langle \sum_k c_k g_k, g_j\right\rangle \\ &= \sum_k c_k\langle g_k, g_j\rangle = \sum_k G_{kj}c_k = \sum_k (\mathbf{G}^T)_{jk}\, c_k = (\mathbf{G}^T c)_j.\end{aligned}$$

We thus have shown

Theorem 4.15: The operator

$$\begin{aligned}&\mathbf{T}\mathbf{T}^* : \ l^2(J) \to l^2(J) \\ &(\mathbf{T}\mathbf{T}^*c)_j = \sum_k \langle g_k, g_j\rangle\, c_k\end{aligned}$$

is equal to the transposed Gram matrix of the frame $\{g_j(t)\}$,

$$\mathbf{T}\mathbf{T}^* = \mathbf{G}^T.$$

The following theorem relates the eigenvalues and eigenfuntions of the frame operator $\mathbf{S}$ to the eigenvalues and eigenvectors of the Gram matrix $\mathbf{G}$.

Theorem 4.16: The eigenvalues of the frame operator $\mathbf{S} = \mathbf{T}^*\mathbf{T}$ equal the eigenvalues of the Gram matrix $\mathbf{G}$. The eigenfunctions $u(t)$ of the frame operator $\mathbf{S}$ are related to the eigenvectors v of the Gram matrix $\mathbf{G}$ according to

$$v_j = \langle u, g_j\rangle^*, \quad j \in J,$$

where v_j denotes the jth component of the eigenvector v.

Proof: Let λ and $u(t)$ be an eigenvalue and eigenfunction, respectively, of the frame operator $\mathbf{S}$. From the eigenequation of $\mathbf{S}$,

$$(\mathbf{S}u)(t) = \lambda\, u(t),$$

we obtain

$$\langle \mathbf{S}u, g_j \rangle = \lambda \, \langle u, g_j \rangle. \tag{4.38}$$

On the other hand,

$$\begin{aligned} \langle \mathbf{S}u, g_j \rangle &= \Big\langle \sum_k \langle u, g_k \rangle \, g_k, g_j \Big\rangle = \sum_k \langle u, g_k \rangle \langle g_k, g_j \rangle \\ &= \sum_k \langle g_j, g_k \rangle^* \langle u, g_k \rangle = \sum_k G_{jk}^* \, \langle u, g_k \rangle. \end{aligned}$$

Combining with (4.38) yields

$$\sum_k G_{jk} \, \langle u, g_k \rangle^* = \lambda^* \, \langle u, g_j \rangle^*.$$

Defining the vector v with components $v_j = \langle u, g_j \rangle^*$, and using the fact that $\lambda^* = \lambda$ due to the self-adjointness of $\mathbf{S}$, this can be written as

$$\mathbf{G}v = \lambda\, v,$$

which shows that λ and v are an eigenvalue and eigenvector, respectively, of the Gram matrix. □

The importance of Theorem 4.16 lies in the fact that the tightest possible frame bounds $A = \lambda_{inf}$ and $B = \lambda_{sup}$ can be obtained by calculating the infimum and supremum, respectively, of the eigenvalues of a *matrix*, instead of the frame operator $\mathbf{S}$. Furthermore, the rank of the frame operator $\mathbf{S}$ equals the number of nonzero eigenvalues of the matrix $\mathbf{G}$.

4.10 Examples

In order to illuminate the theory of frames discussed so far, we shall now give two examples. We consider vectors $x \in \mathbb{R}^2$ instead of signals $x(t)$. The underlying Hilbert space $\mathcal{H}$ is two-dimensional (the plane $\mathbb{R}^2$).

Example 1: The four vectors

$$g_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad g_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad g_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad g_4 = \begin{pmatrix} -1 \\ -1/2 \end{pmatrix}$$

depicted in *Fig. 6* constitute a frame for $\mathbb{R}^2$. Note that this frame consists of an orthonormal basis (g_1 and g_2) augmented by two vectors g_3, g_4 which are obviously linearly dependent with respect to g_1, g_2. Thus this frame is *inexact.*

We shall first compute estimates for the frame bounds. Simple analytical manipulations and the application of the Schwarz inequality yield

$$\|x\|^2 \leq \sum_{j=1}^{4} |\langle x, g_j\rangle|^2 \leq \frac{17}{4}\|x\|^2,$$

where the norm of the vector x is defined as $\|x\| = \sqrt{x_1^2 + x_2^2}$. It follows that frame bounds are given by $A = 1, B = \frac{17}{4}$. We emphasize that these values do not represent the tightest frame bounds that can be found. The tightest frame bounds are given by the minimum and the maximum of the eigenvalues of the frame operator.

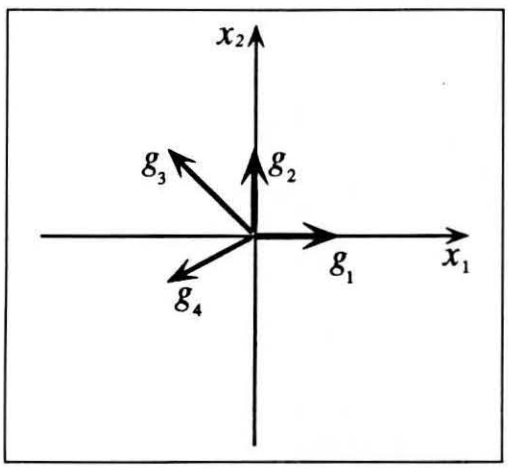

Fig. 6: Frame for $\mathbb{R}^2$.

In our example the frame operator is a 2×2 matrix given by (cf. (4.10))

$$\mathbf{S} = \sum_{j=1}^{4} g_j g_j^H = \begin{pmatrix} 3 & -1/2 \\ -1/2 & 9/4 \end{pmatrix},$$

where gh^H denotes the outer (dyadic) product of the vectors g and h. The tightest frame bounds are obtained as the minimum and the maximum of the eigenvalues of the matrix $\mathbf{S}$, which are

$$A = \lambda_{min} = 2, \qquad B = \lambda_{max} = \frac{13}{4}.$$

The inverse frame operator (matrix) is obtained by inverting the matrix $\mathbf{S}$,

$$\mathbf{S}^{-1} = \frac{1}{26}\begin{pmatrix} 9 & 2 \\ 2 & 12 \end{pmatrix}.$$

The dual frame $\{\tilde{g}_j\}$ is obtained by applying the inverse frame operator to the vectors g_j:

$$\tilde{g}_1 = \mathbf{S}^{-1}g_1 = \frac{1}{26}\begin{pmatrix} 9 \\ 2 \end{pmatrix} \qquad \tilde{g}_2 = \mathbf{S}^{-1}g_2 = \frac{1}{26}\begin{pmatrix} 2 \\ 12 \end{pmatrix}$$

$$\tilde{g}_3 = \mathbf{S}^{-1}g_3 = \frac{1}{26}\begin{pmatrix} -7 \\ 10 \end{pmatrix} \qquad \tilde{g}_4 = \mathbf{S}^{-1}g_4 = \frac{1}{26}\begin{pmatrix} -10 \\ -8 \end{pmatrix}.$$

Any vector $x \in \mathbb{R}^2$ can be reconstructed from the expansion coefficients $\langle x, \tilde{g}_j \rangle = \tilde{g}_j^H x$ as

$$x = \sum_{j=1}^{4} \langle x, \tilde{g}_j \rangle \, g_j.$$

Example 2 [28]: We shall now consider a tight frame for $\mathbb{R}^2$. The frame vectors are

$$g_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad g_2 = \begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \end{pmatrix} \quad g_3 = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}$$

(see *Fig. 7*). Obviously, these vectors must be linearly dependent, and thus the frame is again inexact. The frame operator is given by

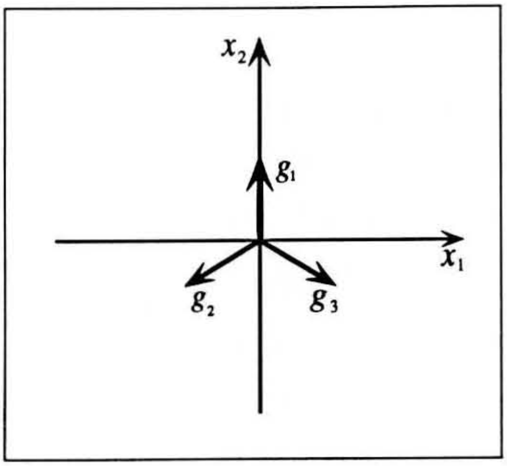

Fig. 7: Tight frame for $\mathbb{R}^2$.

$$\mathbf{S} = \frac{3}{2}\mathbf{I},$$

where $\mathbf{I}$ is the 2×2 identity matrix. The frame bounds are $A = B = \frac{3}{2}$. Since the frame vectors all have length 1, the frame bound $\frac{3}{2}$ can be interpreted as a "redundancy factor" (We have 3 vectors in a 2-dimensional space). The inverse frame operator is given by

$$\mathbf{S}^{-1} = \frac{2}{3}\mathbf{I},$$

and the dual frame vectors are $\tilde{g}_j = \frac{2}{3}\, g_j$.

5 Weyl-Heisenberg Frames

There are two important classes of frames that have been studied extensively, namely Weyl-Heisenberg (or Gabor) frames and affine (or wavelet) frames. Weyl-Heisenberg frames consist of functions which are time-shifted and modulated versions of an arbitrary signal $g(t)$. Affine frames are obtained by time-shifting and dilating an arbitrary function $g(t)$. We shall now study the properties of Weyl-Heisenberg frames.

5.1 Definition and Properties

The Weyl operator [83, 104] $\mathbf{W}_{\tau,\nu}$ is defined as

$$\begin{aligned} &\mathbf{W}_{\tau,\nu} : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \\ &(\mathbf{W}_{\tau,\nu}\, x)(t) = e^{-j\pi\tau\nu}\, e^{2\pi j\nu t}\, x(t-\tau). \end{aligned} \tag{5.1}$$

This definition of the Weyl operator comes from quantum mechanics. In the context of the Gabor expansion, the following definition of the Weyl operator is preferred:

Definition 5.1: The Weyl operator $\mathbf{W}_{m,n}^{(T,F)}$ is defined as

$$\left(\mathbf{W}_{m,n}^{(T,F)} g\right)(t) = g_{m,n}(t) = e^{2\pi jnFt}\, g(t-mT), \tag{5.2}$$

where $m, n \in \mathbb{Z}$, and $T > 0$ and $F > 0$ are fixed time and frequency parameters, respectively.

This means that $\mathbf{W}_{m,n}^{(T,F)}$ is a discretized version of $\mathbf{W}_{\tau,\nu}$ with $\tau = mT$ and $\nu = nF$, and furthermore the phase factor $e^{-j\pi\tau\nu}$ is omitted in our definition. We shall use the definition (5.2) throughout the text. The Weyl operator is a TF shift operator. It produces a time shift by $\tau = mT$ and a frequency shift by $\nu = nF$. We shall usually omit the superscript (T, F) in order to simplify the notation. We now discuss some important properties of the Weyl operator $\mathbf{W}_{m,n}$.

Theorem 5.1 [104]: The composition of the operator $\mathbf{W}_{m,n}$ with the operator $\mathbf{W}_{k,l}$ yields the operator $\mathbf{W}_{m+k,n+l}$ up to a phase factor,

$$\mathbf{W}_{m,n}\mathbf{W}_{k,l} = e^{-2\pi j(mT)(lF)}\,\mathbf{W}_{m+k,n+l}. \tag{5.3}$$

Proof:

$$\begin{aligned} (\mathbf{W}_{m,n}\mathbf{W}_{k,l}\, x)(t) &= \mathbf{W}_{m,n}\left[e^{2\pi jlFt}\, x(t-kT)\right] \\ &= e^{2\pi jnFt}\, e^{2\pi jlF(t-mT)}\, x(t-(m+k)T) \\ &= e^{-2\pi jlmTF}\, e^{2\pi j(n+l)Ft}\, x(t-(m+k)T) \\ &= e^{-2\pi jlmTF}\,(\mathbf{W}_{m+k,n+l}\, x)(t). \quad \square \end{aligned}$$

Corollary 5.1: The Weyl operator is unitary on $L_2(\mathbb{R})$, i.e.

$$\mathbf{W}_{m,n}\mathbf{W}^*_{m,n} = \mathbf{W}^*_{m,n}\mathbf{W}_{m,n} = \mathbf{I},$$

where $\mathbf{I}$ is the identity operator on $L_2(\mathbb{R})$. The adjoint operator $\mathbf{W}^*_{m,n}$ is given by

$$\mathbf{W}^*_{m,n} = e^{-2\pi jmnTF}\,\mathbf{W}_{-m,-n}. \tag{5.4}$$

Proof: The adjoint operator $\mathbf{W}^*_{m,n}$ is defined by the relation

$$\langle \mathbf{W}_{m,n}\,x, y\rangle = \langle x, \mathbf{W}^*_{m,n}\,y\rangle.$$

With

$$\begin{aligned}
\langle \mathbf{W}_{m,n}\,x, y\rangle &= \int_{-\infty}^{\infty} e^{2\pi jnFt}\,x(t-mT)\,y^*(t)\,dt \\
&= e^{2\pi jmnTF}\int_{-\infty}^{\infty} x(t)\,y^*(t+mT)\,e^{2\pi jnFt}\,dt = e^{2\pi jmnTF}\langle x, \mathbf{W}_{-m,-n}\,y\rangle
\end{aligned}$$

it follows that the adjoint operator is given by $\mathbf{W}^*_{m,n} = e^{-2\pi jmnTF}\,\mathbf{W}_{-m,-n}$. We are now able to prove the unitarity of the Weyl operator. With (5.3) we obtain

$$\mathbf{W}_{m,n}\mathbf{W}^*_{m,n} = e^{-2\pi jmnTF}\,\mathbf{W}_{m,n}\mathbf{W}_{-m,-n} = \mathbf{I},$$

and equivalently

$$\mathbf{W}^*_{m,n}\,\mathbf{W}_{m,n} = e^{-2\pi jmnTF}\,\mathbf{W}_{-m,-n}\,\mathbf{W}_{m,n} = \mathbf{I},$$

which completes the proof. □

From the unitarity of the Weyl operator, it follows immediately that the inverse Weyl operator $\mathbf{W}^{-1}_{m,n}$ is equal to the adjoint Weyl operator $\mathbf{W}^*_{m,n}$,

$$\mathbf{W}^{-1}_{m,n} = \mathbf{W}^*_{m,n} = e^{-2\pi jmnTF}\,\mathbf{W}_{-m,-n}.$$

Furthermore the unitarity of the Weyl operator implies that $\langle \mathbf{W}_{m,n}x, \mathbf{W}_{m,n}y\rangle = \langle x, y\rangle$.

We are now ready to consider Weyl-Heisenberg frames, whose definition is based on the Weyl operator:

Definiton 5.2: When the set of functions $\{g_{m,n}(t) = (\mathbf{W}_{m,n}\,g)(t)\}$ with $m, n \in \mathbb{Z}$ and $T, F > 0$ is a frame for $L_2(\mathbb{R})$[13], it is called a *Weyl-Heisenberg (WH) frame.*

[13]In this section we shall only consider frames for $L^2(\mathbb{R})$.

Due to this definition, the functions of a WH frame are simply TF shifted versions of a basic function $g(t)$.

The next theorem considers the case when $g(t)$ is time-limited or band-limited.

Theorem 5.2: Let $g(t)$ be a time-limited function with duration $T_0 < T$ or a band-limited function with bandwidth $F_0 < F$. Then $\{g_{m,n}(t)\}$ is not a WH frame.

Proof: We shall give the proof for a time-limited function $g(t)$. Let the support of $g(t)$ be restricted to the interval $0 \le t \le T_0 < T$. Consider $x(t) \in L_2(\mathbb{R})$ such that the support of $x(t)$ is restricted to the interval $[T_0, T)$. Then $\langle x, g_{m,n}\rangle = 0$ for all $m, n \in \mathbb{Z}$. Hence the set of functions $\{g_{m,n}(t)\}$ is not a frame for any F. The proof for band-limited $g(t)$ is analogous. □

The following theorem states one of the most important results in the theory of WH frames.

Theorem 5.3 [28]: Both the frame operator $\mathbf{S}$ and its inverse $\mathbf{S}^{-1}$ commute with the Weyl operator $\mathbf{W}_{m,n}$,

$$\begin{aligned} \mathbf{W}_{m,n}\,\mathbf{S} &= \mathbf{S}\,\mathbf{W}_{m,n} && (5.5)\\ \mathbf{W}_{m,n}\,\mathbf{S}^{-1} &= \mathbf{S}^{-1}\,\mathbf{W}_{m,n}. && (5.6) \end{aligned}$$

Proof:

$$\begin{aligned} (\mathbf{W}_{m,n}\,\mathbf{S}x)(t) &= \mathbf{W}_{m,n}\sum_k\sum_l \langle x, \mathbf{W}_{k,l}\,g\rangle(\mathbf{W}_{k,l}\,g)(t)\\ &= \sum_k\sum_l \langle x, \mathbf{W}_{k,l}\,g\rangle(\mathbf{W}_{m,n}\mathbf{W}_{k,l}\,g)(t)\\ &= \sum_k\sum_l \langle x, \mathbf{W}_{k,l}\,g\rangle e^{-2\pi jmTlF}(\mathbf{W}_{m+k,n+l}\,g)(t). \end{aligned}$$

On the other hand we have

$$\begin{aligned} (\mathbf{S}\mathbf{W}_{m,n}\,x)(t) &= \sum_k\sum_l \langle \mathbf{W}_{m,n}x, \mathbf{W}_{k,l}\,g\rangle(\mathbf{W}_{k,l}\,g)(t)\\ &= \sum_k\sum_l \langle x, \mathbf{W}^*_{m,n}\mathbf{W}_{k,l}\,g\rangle(\mathbf{W}_{k,l}\,g)(t)\\ &= \sum_k\sum_l e^{2\pi j(n-l)mTF}\langle x, \mathbf{W}_{k-m,l-n}g\rangle(\mathbf{W}_{k,l}\,g)(t)\\ &= \sum_k\sum_l \langle x, \mathbf{W}_{k,l}\,g\rangle\, e^{-2\pi jmTlF}(\mathbf{W}_{m+k,n+l}g)(t), \end{aligned}$$

where (5.3) and (5.4) have been used. This proves (5.5). The statement (5.6) results from (5.5) by multiplying both sides of (5.5) by $\mathbf{S}^{-1}$ from the left and from the right. □

An application of Theorem 5.3 is the following Corollary.

Corollary 5.2 [28]: Let $\{g_{m,n}(t) = (\mathbf{W}_{m,n}g)(t)\}$ be a WH frame. Then, the dual frame is again a WH frame,

$$\tilde{g}_{m,n}(t) = (\mathbf{W}_{m,n}\,\tilde{g})(t) \quad \text{with} \quad \tilde{g}(t) = (\mathbf{S}^{-1}g)(t).$$

Proof: Using (5.6) we obtain

$$\tilde{g}_{m,n}(t) = (\mathbf{S}^{-1}g_{m,n})(t) = (\mathbf{S}^{-1}\mathbf{W}_{m,n}\,g)(t) = (\mathbf{W}_{m,n}\,\mathbf{S}^{-1}g)(t) = (\mathbf{W}_{m,n}\,\tilde{g})(t). \quad \square$$

This means that the dual functions $\tilde{g}_{m,n}(t)$ are obtained by applying the Weyl operator $\mathbf{W}_{m,n}$ to the dual function $\tilde{g}(t) = (\mathbf{S}^{-1}g)(t)$. The WH frame functions $g_{m,n}(t)$ are TF-shifted versions of the signal $g(t)$ and the dual frame functions $\tilde{g}_{m,n}(t)$ are TF-shifted versions of $\tilde{g}(t)$.

As will be seen in the next section, the Zak transform (ZT) is of fundamental importance in the theory of WH frames. We shall therefore consider the composition of the ZT operator with the Weyl operator. In the following, we assume that the sampling period T used in the ZT equals the time period T used for the definition of the Weyl operator.

Theorem 5.4: The composition of the ZT operator with the Weyl operator yields a frequency-shifted version of the ZT operator multiplied by a phase factor, i.e.

$$(\mathbf{Z}\mathbf{W}_{m,n}\,g)(t,f) = \mathcal{Z}_{g_{m,n}}(t,f) = e^{2\pi jmnTF}\,e^{2\pi j(nFt-mTf)}\,\mathcal{Z}_g(t,f-nF). \quad (5.7)$$

Proof:

$$\begin{aligned}
(\mathbf{Z}\mathbf{W}_{m,n}g)(t,f) &= \mathcal{Z}_{g_{m,n}}(t,f) = \sum_{k=-\infty}^{\infty} g(t+(k-m)T)\,e^{2\pi jnF(t+kT)}\,e^{-2\pi jkTf} \\
&= e^{2\pi jmnTF}\,e^{2\pi j(nFt-mTf)} \sum_{k=-\infty}^{\infty} g(t+kT)\,e^{-2\pi jkTf}\,e^{2\pi jknTF} \\
&= e^{2\pi jmnTF}\,e^{2\pi j(nFt-mTf)}\,\mathcal{Z}_g(t,f-nF). \quad \square
\end{aligned}$$

Note that the composition of the ZT operator with the adjoint Weyl operator is given by

$$(\mathbf{Z}\mathbf{W}^{*}_{m,n}\,g)(t,f) = e^{-2\pi j(nFt-mTf)}\,\mathcal{Z}_g(t,f+nF).$$

The following theorem gives an alternative expression for the frame operator.

Theorem 5.5 [105]: The frame operator for a WH frame $\{g_{m,n}(t)\}$ can be written as

$$(\mathbf{S}x)(t) = \frac{1}{F}\sum_{n=-\infty}^{\infty} x\left(t-\frac{n}{F}\right) r_g\left(t,t-\frac{n}{F}\right), \quad (5.8)$$

with

$$r_g(t_1,t_2) = \sum_{m=-\infty}^{\infty} g(t_1 - mT)\, g^*(t_2 - mT). \tag{5.9}$$

Proof [105]: The WH frame operator $\mathbf{S}$ is given by

$$\begin{aligned}
(\mathbf{S}x)(t) &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \langle x, g_{m,n}\rangle\, g_{m,n}(t) \\
&= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \left[\int_{t'} x(t')\, g^*(t'-mT)\, e^{-2\pi jnFt'}\, dt'\right] g(t-mT)\, e^{2\pi jnFt} \\
&= \int_{-\infty}^{\infty} x(t')\, r_g(t,t') \underbrace{\sum_{n=-\infty}^{\infty} e^{2\pi jnF(t-t')}}_{\frac{1}{F}\sum_{n=-\infty}^{\infty}\delta\left(t-t'-\frac{n}{F}\right)} dt' \\
&= \frac{1}{F}\sum_{n=-\infty}^{\infty} x\left(t-\frac{n}{F}\right) r_g\left(t, t-\frac{n}{F}\right). \quad \square
\end{aligned}$$

We now have to answer the question for which T and F the family $\{g_{m,n}(t)\}$ is a frame. There are three different cases, namely

1. $TF = 1$, the *critical case*
2. $TF < 1$, the *oversampled case*
3. $TF > 1$, the *undersampled case.*

Note that TF is the area of the fundamental rectangle in the TF-plane as depicted in Fig. 1. The above three cases will be discussed in the next three subsections.

5.2 Critical Sampling

We first consider the *critical case* $TF = 1$. The corresponding lattice in the TF plane is called the *Von-Neumann Lattice.* The next subsection shows the usefulness of the ZT in the context of WH frames in the critical case $TF = 1$.

5.2.1 Critical Weyl-Heisenberg Frames and the Zak Transform

In the critical case, the ZT of the signal $g_{m,n}(t) = (\mathbf{W}_{m,n}\, g)(t)$ is given by (5.7) for $TF = 1$ as

$$(\mathbf{Z}\mathbf{W}_{m,n}\, g)(t,f) = \mathcal{Z}_{g_{m,n}}(t,f) = e^{2\pi j(n\frac{t}{T} - mTf)}\, \mathcal{Z}_g(t,f), \tag{5.10}$$

where $\mathcal{Z}_g(t,f)$ denotes the ZT of the signal $g(t)$. Note that $F = \frac{1}{T}$ and hence

$$\mathcal{Z}_g(t, f - nF) = \mathcal{Z}_g\left(t, f - \frac{n}{T}\right) = \mathcal{Z}_g(t,f).$$

The next theorem states that applying the WH-frame operator to an arbitrary signal $x(t)$ corresponds to a multiplication of the ZT of $x(t)$ by $|\mathcal{Z}_g(t,f)|^2$ in the ZT domain.

Theorem 5.6 [95]: Let $\mathbf{S}$ denote the frame operator of a WH frame $\{g_{m,n}(t)\}$ in the critical case. Then the ZT of $(\mathbf{S}x)(t)$ equals the ZT of $x(t)$ multiplied by $T|\mathcal{Z}_g(t,f)|^2$,

$$(\mathbf{ZS}x)(t,f) = T\mathcal{Z}_x(t,f)\,|\mathcal{Z}_g(t,f)|^2. \tag{5.11}$$

Proof:

$$\begin{aligned}(\mathbf{ZS}x)(t,f) &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\langle x, g_{m,n}\rangle\, \mathcal{Z}_{g_{m,n}}(t,f) \\ &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\langle x, g_{m,n}\rangle\, e^{2\pi jn\frac{t}{T}}\, e^{-2\pi jmTf}\, \mathcal{Z}_g(t,f) \\ &= \mathcal{Z}_g(t,f)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} A_{x,g}^{(a)}\left(mT,\frac{n}{T}\right) e^{2\pi jn\frac{t}{T}}\, e^{-2\pi jmTf},\end{aligned}$$

where (5.10) has been used. With (3.26) we obtain

$$(\mathbf{ZS}x)(t,f) = T\mathcal{Z}_x(t,f)\,|\mathcal{Z}_g(t,f)|^2. \quad \Box$$

Corollary 5.3: The inner product $\langle \mathbf{S}x, x\rangle$ can be written as

$$\begin{aligned}\langle \mathbf{S}x, x\rangle &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}|\langle x, g_{m,n}\rangle|^2 = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left|A_{x,g}^{(a)}\left(mT,\frac{n}{T}\right)\right|^2 \\ &= T^2\int_0^T\int_0^{1/T}|\mathcal{Z}_x(t,f)|^2\,|\mathcal{Z}_g(t,f)|^2\,dt\,df.\end{aligned}$$

Proof: Starting from (3.19) and using Theorem 5.6, we obtain

$$\begin{aligned}\langle \mathbf{S}x, x\rangle &= T\int_0^T\int_0^{1/T}\mathcal{Z}_{\mathbf{S}x}(t,f)\,\mathcal{Z}_x^*(t,f)\,dt\,df \\ &= T^2\int_0^T\int_0^{1/T}|\mathcal{Z}_x(t,f)|^2\,|\mathcal{Z}_g(t,f)|^2\,dt\,df. \quad \Box\end{aligned}$$

Theorem 5.7 [95]: The set $\{g_{m,n}(t)\}$ is a frame for $L^2(\mathbb{R})$ with bounds A, B if and only if

$$A \le T|\mathcal{Z}_g(t,f)|^2 \le B \tag{5.12}$$

with $A > 0$ and $B < \infty$.

Proof: We shall first show that $A \le T|\mathcal{Z}_g(t,f)|^2 \le B$ implies that $\{g_{m,n}(t)\}$ is a frame with frame bounds A and B. Multiplying both sides of (5.12) by $|\mathcal{Z}_x(t,f)|^2$ we obtain

$$A|\mathcal{Z}_x(t,f)|^2 \le T|\mathcal{Z}_x(t,f)|^2\,|\mathcal{Z}_g(t,f)|^2 \le B|\mathcal{Z}_x(t,f)|^2. \tag{5.13}$$

This inequality can be integrated and we finally have

$$A\int_0^T\int_0^{1/T}|\mathcal{Z}_x(t,f)|^2 \le T\int_0^T\int_0^{1/T}|\mathcal{Z}_x(t,f)|^2\,|\mathcal{Z}_g(t,f)|^2 \le B\int_0^T\int_0^{1/T}|\mathcal{Z}_x(t,f)|^2. \tag{5.14}$$

Using Corollary 5.3 and (3.21), this can be written as

$$A\|x\|^2 \le \langle \mathbf{S}x, x\rangle \le B\|x\|^2,$$

which shows that $\{g_{m,n}(t)\}$ is a frame with frame bounds A, B.

I. Daubechies shows in [95] that the converse is also true. Starting from $A\|x\|^2 \le \langle \mathbf{S}x, x\rangle \le B\|x\|^2$ and using Corollary 5.3 and the unitarity of the ZT she concludes that

$$A \le T|\mathcal{Z}_g(t,f)|^2 \le B. \quad \square$$

Due to Theorem 5.7, a necessary condition for $\{g_{m,n}(t)\}$ to be a WH frame is that the ZT does not have a zero. Note that this implies (see Theorem 3.1) that the ZT may not be continuous.

Corollary 5.4: In the critical case, $\{g_{m,n}(t)\}$ is a tight frame with $A = B$ if and only if $|\mathcal{Z}_g(t,f)|^2$ is constant,

$$T|\mathcal{Z}_g(t,f)|^2 = A.$$

Proof: The proof follows immediately from (5.12). $\square$

Theorem 5.8 [95]: Let $\{g_{m,n}(t)\}$ be a WH frame in the critical case. Then the ZT of the dual function $\tilde{g}(t)$ is given by

$$\mathcal{Z}_{\tilde{g}}(t,f) = \frac{1}{T\mathcal{Z}_g^*(t,f)}. \tag{5.15}$$

Proof: With $(\mathbf{S}\tilde{g})(t) = g(t)$ we obtain

$$(\mathbf{ZS}\tilde{g})(t,f) = \mathcal{Z}_g(t,f).$$

Due to Theorem 5.6 we have further

$$T\mathcal{Z}_{\tilde{g}}(t,f)|\mathcal{Z}_g(t,f)|^2 = \mathcal{Z}_g(t,f).$$

Since $\{g_{m,n}(t)\}$ is a frame, there is $|\mathcal{Z}_g(t,f)|^2 \ge \frac{A}{T} > 0$ and hence we can divide both sides by $\mathcal{Z}_g(t,f)$ to obtain

$$\mathcal{Z}_{\tilde{g}}(t,f) = \frac{1}{T\mathcal{Z}_g^*(t,f)}. \quad \square$$

We are now able to formulate the following important theorem.

Theorem 5.9 [30]: Let $\{g_{m,n}(t)\}$ be a WH frame in the critical case $TF = 1$. Then $\{g_{m,n}(t)\}$ is exact, i.e. $\{g_{m,n}(t)\}$ and $\{\tilde{g}_{m,n}(t)\}$ are biorthogonal,

$$\langle g_{m,n}, \tilde{g}_{m',n'} \rangle = \delta_{m-m'} \, \delta_{n-n'} \, .$$

Proof: To prove the exactness we need only show that $\{g_{m,n}(t)\}_{(m,n)\neq(k,l)}$ is incomplete for every (k,l). We here assume $k = l = 0$; analogous derivations can be done for all other choices of (k,l). Using Theorem 5.8 and (5.10), we have for $(m,n) \neq (0,0)$

$$\begin{aligned} \langle \tilde{g}, g_{m,n} \rangle &= T \langle \mathcal{Z}_{\tilde{g}}, \mathcal{Z}_{g_{m,n}} \rangle \\ &= T \int_0^T \int_0^{1/T} \frac{1}{T \mathcal{Z}_g^*(t,f)} \, e^{-2\pi j n \frac{t}{T}} \, e^{2\pi j m T f} \, \mathcal{Z}_g^*(t,f) \, dt \, df \\ &= \int_0^T \int_0^{1/T} e^{-2\pi j n \frac{t}{T}} \, e^{2\pi j m T f} \, dt \, df = 0. \end{aligned}$$

Thus $\tilde{g}(t)$ is orthogonal to every $g_{m,n}(t)$ with $(m,n) \neq (0,0)$, but we have $\tilde{g}(t) \neq 0$. Hence $\{g_{m,n}(t)\}_{(m,n)\neq(0,0)}$ is incomplete. □

We shall now formulate a theorem on the eigenfunctions and the eigenvalues of the WH-frame operator.

Theorem 5.10: Let $\mathbf{S}$ be the frame operator of a critical WH frame $\{g_{m,n}(t)\}$. Let furthermore $u(t)$ be an eigenfunction of the frame operator $\mathbf{S}$ and λ the corresponding eigenvalue. Then the following equation holds:

$$\mathcal{Z}_u(t,f) \left[|\mathcal{Z}_g(t,f)|^2 - \frac{\lambda}{T} \right] = 0. \tag{5.16}$$

Proof: The eigenequation of the frame operator reads

$$(\mathbf{S}u)(t) = \lambda \, u(t).$$

Applying the ZT to both sides of the eigenequation we obtain

$$T \mathcal{Z}_u(t,f) |\mathcal{Z}_g(t,f)|^2 = \lambda \, \mathcal{Z}_u(t,f). \quad \square$$

Theorem 5.10 implies that in those regions of the TF plane where the ZT of the eigenfunction $u(t)$ is not vanishing, $T|\mathcal{Z}_g(t,f)|^2$ is constant and equals the eigenvalue λ: $\mathcal{Z}_u(t,f) \neq 0 \Rightarrow T|\mathcal{Z}_g(t,f)|^2 = \lambda$. Furthermore we can conclude that the ZTs of two eigenfunctions corresponding to different eigenvalues are non-overlapping. Indeed, if the supports of the ZTs of two different eigenfunctions $u_1(t), u_2(t)$ overlapped (i.e., if there was a TF region where $\mathcal{Z}_{u_1}(t,f) \neq 0$ and $\mathcal{Z}_{u_2}(t,f) \neq 0$), then $T|\mathcal{Z}_g(t,f)|^2$ would take on two different values λ_1, λ_2 in the region of overlap, which is obviously impossible.

5.2.2 The Balian-Low Theorem

We shall see that in the critical case $TF = 1$ only a restricted class of functions $g(t)$ can give rise to a WH frame. The *Balian-Low theorem* states that only windows $g(t)$ which are either not very smooth or decay slowly are allowed. If $g(t)$ is not very smooth, then its Fourier transform decays slowly, so *when a function $g(t)$ gives rise to a WH frame in the critical case, it may not have good TF concentration.* For the proof of the Balian-Low theorem we need the following theorem.

Theorem 5.11 [106]: Let $\{g_{m,n}(t)\}$ be a WH frame in the critical case $TF = 1$, and let $\{\tilde{g}_{m,n}(t)\}$ denote the dual frame. If $tg(t)$ is square-integrable, then $f\tilde{G}(f)$ is not square-integrable,[14]

$$\int_{-\infty}^{\infty} t^2 \, |g(t)|^2 \, dt < \infty \quad \Rightarrow \quad \int_{-\infty}^{\infty} f^2 \, |\tilde{G}(f)|^2 \, df = \infty.$$

If $fG(f)$ is square-integrable, then $t\tilde{g}(t)$ is not square-integrable,

$$\int_{-\infty}^{\infty} f^2 \, |G(f)|^2 \, df < \infty \quad \Rightarrow \quad \int_{-\infty}^{\infty} t^2 \, |\tilde{g}(t)|^2 \, dt = \infty.$$

The proof of this theorem is due to G. Battle [106] and can be found in Appendix 5.A. Note that Theorem 5.11 states that if $g(t)$ is well localized in time, then the dual function $\tilde{g}(t)$ is poorly localized in the frequency domain, and if $g(t)$ is well localized in the frequency domain, then $\tilde{g}(t)$ is poorly localized in the time domain. We are now able to formulate the Balian-Low theorem.

Theorem 5.12 (Balian-Low): If $\{g_{m,n}(t)\}$ is a WH frame for $L^2(\mathbb{R})$ in the critical case $TF = 1$, then either $tg(t) \notin L^2(\mathbb{R})$ or $fG(f) \notin L^2(\mathbb{R})$, i.e.

$$\int_{-\infty}^{\infty} t^2 \, |g(t)|^2 \, dt = \infty \quad \text{or} \quad \int_{-\infty}^{\infty} f^2 \, |G(f)|^2 \, df = \infty.$$

We shall here present a combination of the proof given by I. Daubechies and A.J.E.M. Janssen [107] and that given by Battle [106].

Proof: Let us define the operators $\mathbf{X}$ and $\mathbf{P}$ as $(\mathbf{X}g)(t) = tg(t)$ and $(\mathbf{P}g)(t) = \frac{1}{2\pi j}\frac{d}{dt}g(t)$. Note that the Fourier transform of $(\mathbf{P}g)(t)$ is $fG(f)$.

We first need the following two results about the relation between the ZT and the operators $\mathbf{X}$ and $\mathbf{P}$, defined above:

$$(\mathbf{ZX}g)(t,f) = (\mathbf{Z}\,tg)(t,f) = t\,\mathcal{Z}_g(t,f) - \frac{1}{2\pi j}\frac{\partial}{\partial f}\mathcal{Z}_g(t,f)$$

$$(\mathbf{ZP}g)(t,f) = \frac{1}{2\pi j}(\mathbf{Z}g')(t,f) = \frac{1}{2\pi j}\frac{\partial}{\partial t}\mathcal{Z}_g(t,f).$$

---

[14] $\tilde{G}(f)$ denotes the Fourier transform of the dual function $\tilde{g}(t)$.

From the unitarity of the ZT it follows that $(\mathbf{X}g)(t) \in L^2(\mathbb{R})$ if and only if $\frac{\partial \mathcal{Z}_g(t,f)}{\partial f} \in L^2([0,T) \times [0,\frac{1}{T}))$ and $(\mathbf{P}g)(t) \in L^2(\mathbb{R})$ if and only if $\frac{\partial \mathcal{Z}_g(t,f)}{\partial t} \in L^2([0,T) \times [0,\frac{1}{T}))$. Note that $t\mathcal{Z}_g(t,f)$ is square-integrable on the fundamental rectangle because $\mathcal{Z}_g(t,f)$ is bounded (see Theorem 5.7).

We now assume that $(\mathbf{X}g)(t)$ and $(\mathbf{P}g)(t)$ are *both* square-integrable. Then it follows that $\frac{\partial \mathcal{Z}_g(t,f)}{\partial f}$ and $\frac{\partial \mathcal{Z}_g(t,f)}{\partial t}$ are square-integrable on the fundamental rectangle. From Theorem 5.8 we know that

$$\mathcal{Z}_{\tilde{g}}(t,f) = \frac{1}{T\mathcal{Z}_g^*(t,f)}.$$

So it follows that

$$\frac{\partial}{\partial t}\mathcal{Z}_{\tilde{g}}(t,f) = \frac{1}{T}\left(\frac{\partial}{\partial t}\frac{1}{\mathcal{Z}_g(t,f)}\right)^* = -\frac{1}{T}\left(\frac{1}{[\mathcal{Z}_g(t,f)]^2}\frac{\partial}{\partial t}\mathcal{Z}_g(t,f)\right)^*$$

and furthermore

$$\begin{aligned}
\int_0^T\int_0^{1/T}\left|\frac{\partial}{\partial t}\mathcal{Z}_{\tilde{g}}(t,f)\right|^2 dt\,df &\leq \frac{1}{T^2}\int_0^T\int_0^{1/T}\left|\frac{1}{\mathcal{Z}_g(t,f)}\right|^4\left|\frac{\partial}{\partial t}\mathcal{Z}_g(t,f)\right|^2 dt\,df \\
&\leq \frac{1}{(AT)^2}\int_0^T\int_0^F\left|\frac{\partial}{\partial t}\mathcal{Z}_g(t,f)\right|^2 dt\,df,
\end{aligned}$$

where we have used Theorem 5.7 which states that $\mathcal{Z}_g(t,f)$ is bounded below as $|\mathcal{Z}_g(t,f)|^2 \geq \frac{A}{T}$. So from $\frac{\partial}{\partial t}\mathcal{Z}_g(t,f) \in L^2([0,T) \times [0,\frac{1}{T}))$ it follows that $\frac{\partial}{\partial t}\mathcal{Z}_{\tilde{g}}(t,f) \in L^2([0,T) \times [0,\frac{1}{T}))$ and hence $\frac{d}{dt}\tilde{g}(t) \in L^2(\mathbb{R})$. The same argument can be used to show that $t\,\tilde{g}(t)$ is square-integrable. This means that $(\mathbf{X}g)(t) \in L^2(\mathbb{R})$ and $(\mathbf{P}g)(t) \in L^2(\mathbb{R})$ implies that $(\mathbf{X}\tilde{g})(t) \in L^2(\mathbb{R})$ and $(\mathbf{P}\tilde{g})(t) \in L^2(\mathbb{R})$. But this is impossible due to Theorem 5.11. So we have a contradiction and the proof is complete. □

5.3 Oversampling

Let us next consider the *oversampled case* $TF < 1$, where $\frac{1}{TF} > 1$ is the oversampling factor.

5.3.1 Frame Conditions

Even for $TF < 1$, a given window $g(t)$ gives rise to a WH frame *only* if T and F are properly chosen. The following theorem of I. Daubechies [28] says that under certain restrictions on the window function $g(t)$ and the time-shift parameter T, a frequency-shift parameter F_0 can be found such that, for $0 < F < F_0$, the triple $(g(t),T,F)$ gives rise to a frame. We recall the definition of $r_g(t_1,t_2)$,

$$r_g(t_1,t_2) = \sum_{m=-\infty}^{\infty} g(t_1 - mT)\,g^*(t_2 - mT).$$

Theorem 5.13 [28]: Let

$$m(g(t);T) = \min_{t\in[0,T)} r_g(t,t) \tag{5.17}$$
$$M(g(t);T) = \max_{t\in[0,T)} r_g(t,t) \tag{5.18}$$

be the minimum and maximum of the T-periodic function $r_g(t,t) = \sum_m |g(t-mT)|^2$, respectively, and assume $m(g(t);T) > 0$ and $M(g(t);T) < \infty$. Furthermore, assume that for all $\tau \in \mathcal{R}$

$$\max_{\tau\in\mathcal{R}} (1+\tau^2)^{\frac{1+\epsilon}{2}} \beta(\tau) = C_\epsilon < \infty \quad \text{for some } \epsilon > 0 \tag{5.19}$$

with

$$\beta(\tau) = \max_{t\in[0,T)} \sum_m |g(t-mT)|\,|g(t+\tau-mT)|. \tag{5.20}$$

Then there exists an $F_0 > 0$ such that for every $F \in (0, F_0), \{g_{m,n}(t)\}$ associated with $(g(t), T, F)$ is a frame with frame bounds

$$A \geq \frac{1}{F}\left[m(g(t);T) - \sum_{n\neq 0}^{\infty} \sqrt{\beta\left(\frac{n}{F}\right)\beta\left(-\frac{n}{F}\right)}\right] \tag{5.21}$$
$$B \leq \frac{1}{F}\left[M(g(t);T) + \sum_{n\neq 0}^{\infty} \sqrt{\beta\left(\frac{n}{F}\right)\beta\left(-\frac{n}{F}\right)}\right]. \tag{5.22}$$

The proof of this theorem is due to I. Daubechies [28] and can be found in Appendix 5.B.

The condition $m\,(g(t);T) > 0$ implies that there may not be gaps between translates $g(t-mT)$ of $g(t)$. The condition $M\,(g(t);T) < \infty$ is satisfied if $g(t)$ decays sufficiently at $|t| \to \infty$. For $|g(t)| \leq C(1+t^2)^{-\frac{3}{2}}$, the condition $M\,(g(t);T) < \infty$ is always satisfied.

We shall now discuss the special case of a WH frame built from a window $g(t)$ with finite support.

Corollary 5.5: Let $\{g_{m,n}(t)\}$ be a WH frame in the oversampled case $TF < 1$. If the support of the window function $g(t)$ is restricted to an interval of length T_0 with $T \leq T_0 \leq \frac{1}{F}$, the frame operator is a multiplication operator,

$$(\mathbf{S}x)(t) = \frac{1}{F}\, x(t)\, r_g(t,t), \tag{5.23}$$

with $r_g(t,t) = \sum_{m=-\infty}^{\infty} |g(t-mT)|^2$.

Proof: The expression (5.8) can be written as

$$\begin{aligned}(\mathbf{S}x)(t) &= \frac{1}{F}\sum_{n=-\infty}^{\infty} x\left(t-\frac{n}{F}\right) r_g\left(t, t-\frac{n}{F}\right)\\ &= \frac{1}{F}\, x(t)\, r_g(t,t) + \frac{1}{F}\sum_{n\neq 0} x\left(t-\frac{n}{F}\right) r_g\left(t,t-\frac{n}{F}\right).\end{aligned}$$

If $T_0 < \frac{1}{F}$, then $g(t)\, g\left(t - \frac{n}{F}\right) = 0$ for $n \neq 0$ and further $r_g\left(t, t - \frac{n}{F}\right) = 0$ for all $n \neq 0$, so that we finally obtain

$$(\mathbf{S}x)(t) = \frac{1}{F}\, x(t)\, r_g(t,t).$$

The condition $T_0 \geq T$ follows from Theorem 5.2. □

In the case of window functions $g(t)$ with duration $T_0 < \frac{1}{F}$, it follows from Corollary 5.5 that the inversion of the frame operator $\mathbf{S}$ is trivial.

Corollary 5.6: Let $\{g_{m,n}(t)\}$ be a WH frame in the oversampled case $TF < 1$. If the window function $g(t)$ has finite duration T_0 with $T < T_0 < \frac{1}{F}$, and $m(g(t); T) = \min_{t \in [0,T]} r_g(t,t) > 0$, the inversion of the frame operator reduces to a division by the periodic function $\frac{1}{F}\, r_g(t,t)$,

$$x(t) = F\, \frac{(\mathbf{S}x)(t)}{r_g(t,t)}.$$

Proof: This follows directly from Corollary 5.5. Note that one of our requirements is that the periodic function $r_g(t,t) = \sum_{m=-\infty}^{\infty} |g(t - mT)|^2$ is bounded below, so that the division is allowed. □

We are now able to formulate conditions on the window function $g(t)$ which guarantee that the WH frame obtained will be tight.

Corollary 5.7: Let $\{g_{m,n}(t)\}$ be a WH frame in the oversampled case $TF < 1$. Let the support of $g(t)$ be of length T_0 with $T < T_0 < \frac{1}{F}$ and let $r_g(t,t) = \alpha$, where $\alpha > 0$. Then $\{g_{m,n}(t)\}$ is a tight frame with frame bound $A = \frac{\alpha}{F}$.

Proof: With $r_g(t,t) = \alpha$, (5.23) reduces to

$$(\mathbf{S}x)(t) = \frac{\alpha}{F}\, x(t),$$

which proves that $\{g_{m,n}(t)\}$ is a tight frame with frame bound $A = \frac{\alpha}{F}$. □

5.3.2 Biorthogonality on the Dual Grid

In the critical case $TF = 1$, we have shown in Section 5.2.1 that a WH frame $\{g_{m,n}(t)\}$ is always exact, which implies that the frame and its dual $\{\tilde{g}_{m,n}(t)\}$ are biorthogonal,

$$\langle g_{m,n}, \tilde{g}_{k,l} \rangle = \delta_{k-m}\, \delta_{l-n} = \begin{cases} 1, & m = k,\ n = l \\ 0, & \text{otherwise.} \end{cases}$$

This equation is not true for the oversampled case $TF < 1$, but we shall see that a biorthogonality relation exists with respect to a "dual grid" determined by time shift $T_d = \frac{1}{F}$ and frequency shift $F_d = \frac{1}{T}$. This biorthogonality relation has been found by J. Wexler and S. Raz in [26].

Theorem 5.14 [26]: Let $g(t)$ and $h(t)$ be such that the completeness relation

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g_{m,n}(t)\, h^*_{m,n}(t') = \delta(t-t') \tag{5.24}$$

is satisfied, i.e., all $x(t) \in L_2(\mathbb{R})$ can be expanded as

$$x(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle x, h_{m,n}\rangle\, g_{m,n}(t).$$

Then, the following biorthogonality relation holds:

$$\langle g^d_{m,n}, h^d_{k,l}\rangle = TF\, \delta_{k-m}\, \delta_{l-n}, \tag{5.25}$$

where

$$g^d_{m,n}(t) = g(t - mT_d)\, e^{2\pi j n F_d t} = g\Big(t - m\frac{1}{F}\Big)\; e^{2\pi j n \frac{t}{T}}.$$

Conversely if functions $g(t), h(t)$ can be found such that the biorthogonality relation (5.25) is satisfied, then the completeness relation (5.24) holds.

Proof [26]: Starting from the completeness relation (5.24), we obtain

$$\begin{aligned}
\delta(t-t') &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g_{m,n}(t)\, h^*_{m,n}(t') = \sum_{m=-\infty}^{\infty} g(t-mT)\, h^*(t'-mT) \underbrace{\sum_{n=-\infty}^{\infty} e^{2\pi j n F(t-t')}}_{\frac{1}{F}\sum_{n=-\infty}^{\infty} \delta\left(t-t'-n\frac{1}{F}\right)} \\
&= \frac{1}{F} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g(t-mT)\, h^*\left(t - n\frac{1}{F} - mT\right)\, \delta\left(t - t' - n\frac{1}{F}\right) \\
&= \frac{1}{F} \sum_{n=-\infty}^{\infty} \delta\Big(t - t' - n\frac{1}{F}\Big) \sum_{m=-\infty}^{\infty} g(t-mT)\, h^*\left(t - n\frac{1}{F} - mT\right).
\end{aligned}$$

Using the Poisson sum formula

$$\sum_{m=-\infty}^{\infty} y(t-mT) = \frac{1}{T} \sum_{m=-\infty}^{\infty} Y\left(\frac{m}{T}\right)\, e^{2\pi j m \frac{t}{T}}$$

with $y(t) = g(t)\, h^*\left(t - n\frac{1}{F}\right)$, we obtain

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} g(t-mT)\, h^*\Big(t - n\frac{1}{F} - mT\Big) &= \frac{1}{T} \sum_{m=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(\hat{t})\, h^*\Big(\hat{t} - n\frac{1}{F}\Big)\; e^{-2\pi j m \frac{\hat{t}}{T}}\, d\hat{t}\right] e^{2\pi j m \frac{t}{T}} \\
&= \frac{1}{T} \sum_{m=-\infty}^{\infty} \langle g, h^d_{n,m}\rangle\, e^{2\pi j m \frac{t}{T}}.
\end{aligned}$$

With $m \to n$ and $n \to m$ we obtain further

$$\begin{aligned}
\delta(t-t') &= \frac{1}{TF} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle g, h^d_{m,n}\rangle\, e^{2\pi j n \frac{t}{T}}\, \delta\Big(t - t' - m\frac{1}{F}\Big) \\
&= \left[\frac{1}{TF} \sum_{n=-\infty}^{\infty} \langle g, h^d_{0,n}\rangle\, e^{2\pi j n \frac{t}{T}}\right] \delta\,(t-t') \\
&\quad + \sum_{m\neq 0} \left[\frac{1}{TF} \sum_{n=-\infty}^{\infty} \langle g, h^d_{m,n}\rangle\, e^{2\pi j n \frac{t}{T}}\right] \delta\left(t - t' - m\frac{1}{F}\right).
\end{aligned}$$

This implies that $\sum_{n=-\infty}^{\infty} \langle g, h^d_{m,n}\rangle\, e^{2\pi j n \frac{t}{T}}$ has to be equal to TF for $m = 0$ and zero otherwise, i.e.

$$\sum_{n=-\infty}^{\infty} \langle g, h^d_{m,n}\rangle\, e^{2\pi j n \frac{t}{T}} = TF\delta_m. \tag{5.26}$$

The expression $\sum_{n=-\infty}^{\infty} \langle g, h^d_{m,n}\rangle e^{2\pi j n \frac{t}{T}}$ is the Fourier series expansion of a T-periodic signal. Thus, $\sum_{n=-\infty}^{\infty} \langle g, h^d_{0,n}\rangle\, e^{2\pi j n \frac{t}{T}} = TF$ if and only if $\langle g, h^d_{0,n}\rangle = TF\,\delta_n$, and $\sum_{n=-\infty}^{\infty} \langle g, h^d_{m,n}\rangle\, e^{2\pi j n \frac{t}{T}} = 0$ for $m \neq 0$ if and only if $\langle g, h^d_{m,n}\rangle = 0$ for $m \neq 0$. Combining, it follows that $\sum_{n=-\infty}^{\infty} \langle g, h^d_{m,n}\rangle\, e^{2\pi j n \frac{t}{T}} = TF\,\delta_m$ if and only if

$$\langle g, h^d_{m,n}\rangle = TF\,\delta_m\,\delta_n,$$

which further implies

$$\langle g^d_{m,n}, h^d_{k,l}\rangle = e^{2\pi j (n-l) m T_d F_d}\, \underbrace{\langle g, h^d_{k-m,l-n}\rangle}_{TF\,\delta_{k-m}\,\delta_{l-n}} = TF\,\delta_{k-m}\,\delta_{l-n}, \tag{5.27}$$

which is the biorthogonality relation to be proved.

We shall now show that, conversely, biorthogonality implies completeness. Further above, we have shown that

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g_{m,n}(t)\, h^*_{m,n}(t') = \frac{1}{TF} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \langle g, h^d_{m,n}\rangle\, e^{2\pi j n \frac{t}{T}}\, \delta\left(t - t' - m\frac{1}{F}\right).$$

Inserting the biorthogonality relation $\langle g, h^d_{m,n}\rangle = TF\,\delta_m\,\delta_n$, we obtain the completeness relation (5.24). □

We can thus draw the following conclusions:

- In the oversampled case $TF < 1$, biorthogonality of $\{g^d_{m,n}(t)\}$ and $\{h^d_{m,n}(t)\}$ (i.e. biorthogonality on the dual grid) and completeness as stated by (5.24) are equivalent.

- In the critical case $TF = 1$, the dual grid equals the original grid. Thus, conventional biorthogonality of $\{g_{m,n}(t)\}$ and $\{h_{m,n}(t)\}$

  $$\langle g_{m,n}, h_{k,l}\rangle = \delta_{m-k}\,\delta_{n-l}$$

  and completeness as stated by (5.24) are equivalent.

Note that we did not assume that $\{g_{m,n}(t)\}$ is a WH frame. Since every WH frame $\{g_{m,n}(t)\}$ and the dual frame $\{h_{m,n}(t) = \tilde{g}_{m,n}(t)\}$ satisfy the completeness relation (5.24), the first part of Theorem 5.14 also holds for WH frames: if $\{g_{m,n}(t)\}$ is a WH frame, then

$$\langle g^d_{m,n}, \tilde{g}^d_{k,l}\rangle = TF\,\delta_{k-m}\,\delta_{l-n}.$$

In this context, we emphasize that if $\{g_{m,n}(t)\}$ is a WH frame for $TF < 1$, then $\{g^d_{m,n}(t)\}$ is not a WH frame. Indeed, $TF < 1$ implies $T_dF_d = \frac{1}{TF} > 1$, so that the dual grid corresponds to the case of undersampling discussed in Section 5.4.

5.3.3 Oversampling by Rational Factors and the Zak Transform

In this section we shall discuss WH frames which are oversampled by rational factors, i.e. $TF = p/q < 1$ with $p, q \in \mathbb{N}$. In this context, the ZT again plays an important role. We shall see that especially for oversampling by integer factors, i.e. $TF = 1/N$ with $N \in \mathbb{N}$, the ZT is of fundamental importance in WH-frame theory. Again, we assume that the sampling period T used in the ZT equals the time period T used in the WH frame $\{g_{m,n}(t)\}$. The following approach is due to M. Zibulski and Y. Zeevi [105]. We shall first formulate

Theorem 5.15 [105]: Let $\{g_{m,n}(t)\}$ be a WH frame in the oversampled case, with $TF = p/q$ a rational number ($p, q \in \mathbb{N}$). The composition of the ZT operator and the frame operator then yields

$$(\mathbf{ZS}x)(t,f) = \frac{1}{qF}\sum_{l=0}^{p-1}\sum_{i=0}^{q-1} \mathcal{Z}_x\left(t + l\frac{q}{p}T, f\right)\, \mathcal{Z}_g\left(t, f - \frac{i}{qT}\right)\, \mathcal{Z}_g^*\left(t + l\frac{q}{p}T, f - \frac{i}{qT}\right).$$

Proof [105]: From Theorem 5.5 we know that the signal $(\mathbf{S}x)(t)$ can be written as

$$(\mathbf{S}x)(t) = \frac{1}{F}\sum_{n=-\infty}^{\infty} x\left(t - \frac{n}{F}\right) r_g\left(t, t - \frac{n}{F}\right).$$

Taking the ZT yields

$$\begin{aligned}(\mathbf{ZS}x)(t,f) &= \frac{1}{F}\sum_{k=-\infty}^{\infty} e^{-2\pi jkTf}\sum_{n=-\infty}^{\infty} x\left(t + kT - \frac{n}{F}\right) r_g\left(t + kT, t + kT - \frac{n}{F}\right) \\ &= \frac{1}{F}\sum_{k=-\infty}^{\infty} e^{-2\pi jkTf}\sum_{n=-\infty}^{\infty} x\left(t + \left(k - n\frac{q}{p}\right)T\right) r_g\left(t + kT, t + kT - n\frac{q}{p}T\right),\end{aligned}$$

where $\frac{1}{F} = \frac{q}{p}T$ has been used. Since $r_g(t_1 + kT, t_2 + kT) = r_g(t_1, t_2)$, we obtain

$$(\mathbf{ZS}x)(t,f) = \frac{1}{F}\sum_{k=-\infty}^{\infty} e^{-2\pi jkTf}\sum_{n=-\infty}^{\infty} x\left(t + \left(k - n\frac{q}{p}\right)T\right) r_g\left(t, t - n\frac{q}{p}T\right).$$

Substituting $n = n'p - l$ with $n' \in \mathbb{Z}$ and $0 \le l \le p-1$, we have

$$(\mathbf{ZS}x)(t,f) = \frac{1}{F}\sum_{k=-\infty}^{\infty} e^{-2\pi jkTf}\sum_{n'=-\infty}^{\infty}\sum_{l=0}^{p-1} x\left(t + kT - n'qT + l\frac{q}{p}T\right) r_g\left(t, t - n'qT + l\frac{q}{p}T\right).$$

With $k' = k - n'q$ we obtain further

$$
\begin{aligned}
(\mathbf{ZS}x)(t,f) &= \frac{1}{F}\sum_{l=0}^{p-1}\sum_{k'=-\infty}^{\infty} e^{-2\pi jk'Tf}\, x\left(t + k'T + l\frac{q}{p}T\right) \\
&\quad \sum_{n'=-\infty}^{\infty} r_g\left(t, t - n'qT + l\frac{q}{p}T\right) e^{-2\pi jn'qTf} \\
&= \frac{1}{F}\sum_{l=0}^{p-1} \mathcal{Z}_x\left(t + l\frac{q}{p}T, f\right) \sum_{n=-\infty}^{\infty} r_g\left(t, t - nqT + l\frac{q}{p}T\right) e^{-2\pi jnqTf}
\end{aligned}
$$

We shall now elaborate the expression

$$
\sum_{n=-\infty}^{\infty} r_g\left(t, t - nqT + l\frac{q}{p}T\right) e^{-2\pi jnqTf}.
$$

We have

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty} r_g\left(t, t - nqT + l\frac{q}{p}T\right) e^{-2\pi jnqTf} \\
&= \sum_{n=-\infty}^{\infty} r_g\left(t, t - nT + l\frac{q}{p}T\right) e^{-2\pi jnTf} \sum_{k=-\infty}^{\infty} \delta_{n-kq} \\
&= \sum_{n=-\infty}^{\infty} r_g\left(t, t - nT + l\frac{q}{p}T\right) e^{-2\pi jnTf} \left[\frac{1}{q}\sum_{i=0}^{q-1} e^{2\pi jn\frac{i}{q}}\right] \\
&= \frac{1}{q}\sum_{i=0}^{q-1}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g(t - mT)\, g^*\left(t - nT + l\frac{q}{p}T\right) e^{-2\pi jnT(f-\frac{i}{qT})} e^{2\pi jmT(f-\frac{i}{qT})} \\
&= \frac{1}{q}\sum_{i=0}^{q-1}\left[\sum_{m=-\infty}^{\infty} g(t + mT) e^{-2\pi jmT(f-\frac{i}{qT})}\right] \sum_{n=-\infty}^{\infty} g^*\left(t + nT + l\frac{q}{p}T\right) e^{2\pi jnT(f-\frac{i}{qT})} \\
&= \frac{1}{q}\sum_{i=0}^{q-1} \mathcal{Z}_g\left(t, f - \frac{i}{qT}\right) \mathcal{Z}_g^*\left(t + l\frac{q}{p}T, f - \frac{i}{qT}\right). \quad \square
\end{aligned}
$$

In the following, we shall restrict ourselves to the case of oversampling by integer factors, i.e. $TF = 1/N$ or $p = 1, q = N$. The general case $TF = p/q$ is discussed in detail in [105].

Corollary 5.8: Let $\{g_{m,n}(t)\}$ be a WH frame in the oversampled case $TF = 1/N$. Then the composition of the ZT operator and the frame operator yields

$$
\begin{aligned}
(\mathbf{ZS}x)(t,f) &= \frac{1}{NF}\, \mathcal{Z}_x(t,f) \sum_{i=0}^{N-1} \left|\mathcal{Z}_g\left(t, f - \frac{i}{NT}\right)\right|^2 \\
&= T\mathcal{Z}_x(t,f) Y_g(t,f), \qquad (5.28)
\end{aligned}
$$

with

$$
Y_g(t,f) = \sum_{i=0}^{N-1} \left|\mathcal{Z}_g\left(t, f - \frac{i}{NT}\right)\right|^2. \qquad (5.29)
$$

Proof: This follows immediately from Theorem 5.15 with $p = 1$ and $q = N$. □

We note that $Y_g(t,f) = |\mathcal{Z}_g(t,f)|^2$ for $N = 1$. Thus, in the case of critical sampling, (5.28) duly reduces to (5.11). In the case of oversampling by integer factors, the application of the frame operator to an arbitrary signal $x(t)$ corresponds to multiplication of $\mathcal{Z}_x(t,f)$ by $Y_g(t,f) = \sum_{i=0}^{N-1} \left|\mathcal{Z}_g\left(t, f - \frac{i}{NT}\right)\right|^2$ in the ZT domain.

We shall now formulate some important theorems, which make use of the previous results.

Theorem 5.16 [105]: In the oversampled case $TF = 1/N$, the set $\{g_{m,n}(t)\}$ is a frame for $L^2(\mathbb{R})$ with bounds A, B if and only if

$$A \leq T\,Y_g(t,f) \leq B. \tag{5.30}$$

Proof: The proof is analogous to that of Theorem 5.7. The function $|\mathcal{Z}_g(t,f)|^2$ in Theorem 5.7 has to be replaced by $Y_g(t,f) = \sum_{i=0}^{N-1} |\mathcal{Z}_g(t, f - iF)|^2$. □

Corollary 5.9 [105]: In the oversampled case $TF = 1/N$, the set $\{g_{m,n}(t)\}$ is a tight frame with $A = B$ if and only if

$$T\,Y_g(t,f) = A.$$

Proof: The proof follows immediately from Theorem 5.16.

Theorem 5.17 [105]: Let $\{g_{m,n}(t)\}$ be a WH frame in the oversampled case $TF = 1/N$. Then the ZT of the dual function $\tilde{g}(t)$ is given by

$$\mathcal{Z}_{\tilde{g}}(t,f) = \frac{\mathcal{Z}_g(t,f)}{T\,Y_g(t,f)}. \tag{5.31}$$

Proof: Starting from

$$(\mathbf{ZS}\tilde{g})(t,f) = T\mathcal{Z}_{\tilde{g}}(t,f)\,Y_g(t,f)$$

and using $(\mathbf{ZS}\tilde{g})(t,f) = (\mathbf{Z}g)(t,f) = \mathcal{Z}_g(t,f)$, we obtain

$$\mathcal{Z}_{\tilde{g}}(t,f) = \frac{\mathcal{Z}_g(t,f)}{T\,Y_g(t,f)}. \quad \square$$

In the critical case $N = 1$, we have $Y_g(t,f) = |\mathcal{Z}_g(t,f)|^2$ and (5.31) gives

$$\mathcal{Z}_{\tilde{g}}(t,f) = \frac{\mathcal{Z}_g(t,f)}{T|\mathcal{Z}_g(t,f)|^2} = \frac{1}{T\mathcal{Z}_g^*(t,f)},$$

which is (5.15).

We can now draw the following conclusion. In the case of integer oversampling, there exists a simple expression for the ZT of the dual window; the only difference from the critical case is that the function $Y_g(t,f) = \sum_{i=0}^{N-1} |\mathcal{Z}_g(t, f-iF)|^2$ plays the role of the function $|\mathcal{Z}_g(t,f)|^2$ relevant to the critical case. Thus, the ZT is a powerful tool for calculating the dual function, since it can be implemented efficiently using FFT methods. In the general rational case $TF = p/q$, it is shown in [105] that the calculation of the dual function requires the solution of a system of linear equations.

5.4 Undersampling

In the *undersampled* case $TF > 1$, the TF grid defined by T and F is "too loose," i.e. the $g_{m,n}(t)$ cannot span $L^2(\mathbb{R})$. This fact is formulated in the following theorem.

> **Theorem 5.18:** If $TF > 1$ and $g(t) \in L^2(\mathbb{R})$, then $\{g_{m,n}(t)\}$ is incomplete in $L^2(\mathbb{R})$.

This means that if $TF > 1$, $\{g_{m,n}(t)\}$ cannot be a frame for $L^2(\mathbb{R})$.

Proof: The proof is given for the case $TF = N$, with $N \in \mathbb{N}$ and $N > 1$. It can be extended to the cases where TF is rational or irrational. The proof for irrational TF is rather involved however.

For $TF = N$, $N \in \mathbb{N}$, $N > 1$ we have

$$\mathcal{Z}_{g_{m,n}}(t,f) = e^{2\pi jnN\frac{t}{T}}\, e^{-2\pi jmTf}\, \mathcal{Z}_g(t,f).$$

Using this relation, we calculate the inner products $\langle x, g_{m,n}\rangle$, where $x(t) \in L^2(\mathbb{R})$ is an arbitrary signal that is not identically zero:

$$\begin{aligned}
\langle x, g_{m,n}\rangle &= T\langle \mathcal{Z}_x, \mathcal{Z}_{g_{m,n}}\rangle \\
&= T\int_0^T\int_0^{1/T} \mathcal{Z}_x(t,f)\, \mathcal{Z}_g^*(t,f)\, e^{-2\pi jnN\frac{t}{T}}\, e^{2\pi jmTf}\, dt\, df.
\end{aligned} \tag{5.32}$$

We can see that the inner products $\langle x, g_{m,n}\rangle$ are up to a constant factor the Fourier coefficients of the periodic function $\mathcal{Z}_x(t,f)\mathcal{Z}_g^*(t,f)$ with respect to the basis $\{b_{m,n}(t,f) = e^{2\pi jnN\frac{t}{T}}\, e^{-2\pi jmTf}\}$. It is evident that for $N > 1$ this basis is not complete in $L^2([0,T) \times [0,1/T))$.

Indeed, we can construct a signal $x(t) \in L_2(\mathbb{R})$ for which $\langle x, g_{m,n}\rangle = 0$ for all $m,n \in \mathbb{Z}$. Consider the signal $x(t)$ defined by the following Fourier series representation of the periodic function $\mathcal{Z}_x(t,f)\, \mathcal{Z}_g^*(t,f)$:

$$\mathcal{Z}_x(t,f)\mathcal{Z}_g^*(t,f) = \sum_k \sum_l \alpha_{k,l}\, e^{2\pi jl\frac{t}{T}}\, e^{-2\pi jkTf} \tag{5.33}$$

with $\alpha_{k,l} \neq 0$ for $l \neq iN$ and $\alpha_{k,l} = 0$ for $l = iN$. Due to the unitarity of the ZT, it is always possible to find an $x(t)$ for given $g(t)$ such that (5.33) (with $\alpha_{k,l}$ as defined above) holds. Inserting (5.33) into (5.32), we obtain

$$
\begin{aligned}
\langle x, g_{m,n}\rangle &= T\int_0^T\int_0^{1/T} \mathcal{Z}_x(t,f)\,\mathcal{Z}_g^*(t,f)\,e^{-2\pi jnN\frac{t}{T}}\,e^{2\pi jmTf}\,dt\,df \\
&= T\sum_k\sum_l \alpha_{k,l}\left[\int_0^T\int_0^{1/T} e^{2\pi j(l-nN)\frac{t}{T}}\,e^{2\pi j(m-k)Tf}\,dt\,df\right] \\
&= T\sum_k\sum_l \alpha_{k,l}\,\delta_{l-nN}\,\delta_{m-k} = T\alpha_{m,nN} = 0 \quad \forall m,n \in \mathbb{Z}.
\end{aligned}
$$

Hence, $\langle x, g_{m,n}\rangle = 0$ for all m,n although $x(t) \neq 0$. It follows that $\{g_{m,n}(t)\}$ is incomplete for $TF = N$. $\square$

Appendix 5.A: Proof of Theorem 5.11

Using the operators $\mathbf{X}$ and $\mathbf{P}$ defined as $(\mathbf{X}g)(t) = tg(t)$ and $(\mathbf{P}g)(t) = \frac{1}{2\pi j}\frac{d}{dt}g(t)$, where the Fourier transform of $(\mathbf{P}g)(t)$ is $fG(f)$, we have to show that

1. $(\mathbf{X}g)(t)$ and $(\mathbf{P}\tilde{g})(t)$ cannot both be in $L^2(\mathbb{R})$,

2. $(\mathbf{P}g)(t)$ and $(\mathbf{X}\tilde{g})(t)$ cannot both be in $L^2(\mathbb{R})$.

In the critical case $TF = 1$, WH frames are always exact (see Theorem 5.7). So $\{g_{m,n}(t)\}$ and $\{\tilde{g}_{m,n}(t)\}$ are biorthogonal, i.e.

$$\langle g_{m,n}, \tilde{g}_{m',n'}\rangle = \delta_{m-m'}\,\delta_{n-n'}. \tag{A.1}$$

We assume that $(\mathbf{X}g)(t)$ and $(\mathbf{P}\tilde{g})(t)$ are both square-integrable. From Corollary 4.1, we then have

$$\langle \mathbf{X}g, \mathbf{P}\tilde{g}\rangle = \sum_m \sum_n \langle \mathbf{X}g, \tilde{g}_{m,n}\rangle\langle g_{m,n}, \mathbf{P}\tilde{g}\rangle.$$

We first consider the inner product $\langle \mathbf{X}g, \tilde{g}_{m,n}\rangle$:

$$\begin{aligned}
\langle \mathbf{X}g, \tilde{g}_{m,n}\rangle &= \int_{-\infty}^{\infty} t\,g(t)\,e^{-2\pi jnFt}\,\tilde{g}^*(t-mT)\,dt \\
&= \int_{-\infty}^{\infty} t\,g(t+mT)\,e^{-2\pi jnFt}\,\tilde{g}^*(t)\,dt \;+\; mT\int_{-\infty}^{\infty} g(t+mT)\,e^{-2\pi jnFt}\,\tilde{g}^*(t)\,dt \\
&= \langle g_{-m,-n}, \mathbf{X}\tilde{g}\rangle + mT\langle g_{-m,-n}, \tilde{g}\rangle \;=\; \langle g_{-m,-n}, \mathbf{X}\tilde{g}\rangle + mT\delta_m\delta_n \\
&= \langle g_{-m,-n}, \mathbf{X}\tilde{g}\rangle.
\end{aligned}$$

Next, applying Parseval's theorem to the inner product $\langle g_{m,n}, \mathbf{P}\tilde{g}\rangle$ yields

$$\begin{aligned}
\langle g_{m,n}, \mathbf{P}\tilde{g}\rangle &= \int_{-\infty}^{\infty} e^{2\pi jnFt}\,g(t-mT)\left(\frac{1}{2\pi j}\frac{d}{dt}\tilde{g}(t)\right)^*dt \\
&= \int_{-\infty}^{\infty} G(f-nF)\,e^{-2\pi jmT(f-nF)}\,[f\tilde{G}(f)]^*\,df \\
&= \int_{-\infty}^{\infty} f\,G(f)\,e^{-2\pi jmTf}\,\tilde{G}^*(f+nF)\,df \\
&\quad + nF\int_{-\infty}^{\infty} G(f)\,e^{-2\pi jmTf}\,\tilde{G}^*(f+nF)\,df \\
&= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi j}\frac{d}{dt}\,g(t)\right]\tilde{g}^*(t+mT)\,e^{2\pi jnFt}\,dt \\
&\quad + nF\int_{-\infty}^{\infty} g(t)\,\tilde{g}^*(t+mT)\,e^{2\pi jnFt}\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \langle \mathbf{P}g, \tilde{g}_{-m,-n}\rangle + nF\langle g, \tilde{g}_{-m,-n}\rangle \\
&= \langle \mathbf{P}g, \tilde{g}_{-m,-n}\rangle + nF\delta_m\delta_n \\
&= \langle \mathbf{P}g, \tilde{g}_{-m,-n}\rangle.
\end{aligned}
$$

So we conclude that

$$
\begin{aligned}
\langle \mathbf{X}g, \mathbf{P}\tilde{g}\rangle &= \sum_m \sum_n \langle \mathbf{X}g, \tilde{g}_{m,n}\rangle \langle g_{m,n}, \mathbf{P}\tilde{g}\rangle = \sum_m \sum_n \langle g_{-m,-n}, \mathbf{X}\tilde{g}\rangle \langle \mathbf{P}g, \tilde{g}_{-m,-n}\rangle \\
&= \sum_m \sum_n \langle \mathbf{P}g, \tilde{g}_{m,n}\rangle \langle g_{m,n}, \mathbf{X}\tilde{g}\rangle = \langle \mathbf{P}g, \mathbf{X}\tilde{g}\rangle.
\end{aligned}
$$

We furthermore have

$$
\begin{aligned}
\int_{-\infty}^{\infty} t\frac{d}{dt}[g(t)\tilde{g}^*(t)]\,dt &= \int_{-\infty}^{\infty} t\,g(t)\left(\frac{d}{dt}\tilde{g}^*(t)\right) dt + \int_{-\infty}^{\infty} \left(\frac{d}{dt}\,g(t)\right) t\tilde{g}^*(t)\,dt \\
&= -2\pi j\langle \mathbf{X}g, \mathbf{P}\tilde{g}\rangle + 2\pi j\langle \mathbf{P}g, \mathbf{X}\tilde{g}\rangle = 0.
\end{aligned}
$$

But on the other hand

$$
\int_{-\infty}^{\infty} t\frac{d}{dt}[g(t)\tilde{g}^*(t)]\,dt = t\,g(t)\,\tilde{g}^*(t)\,\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g(t)\,\tilde{g}^*(t)\,dt.
$$

From the assumption that $tg(t)$ and $\tilde{g}(t)$ are both square-integrable, it follows that

$$
\lim_{t\to\pm\infty} t\,g(t)\,\tilde{g}^*(t) = 0,
$$

and we obtain

$$
0 = \int_{-\infty}^{\infty} t\frac{d}{dt}\,[g(t)\tilde{g}^*(t)]\;dt = -\int_{-\infty}^{\infty} g(t)\tilde{g}^*(t)\,dt.
$$

So we conclude that $\langle g, \tilde{g}\rangle = 0$. But from Corollary 4.4 we have $\langle g, \tilde{g}\rangle = 1$. This is a contradiction, which shows that our assumption $(\mathbf{X}g)(t)$ and $(\mathbf{P}g)(t)$ both square-integrable) must be wrong. The second statement can be proved using the same technique. □

Appendix 5.B : Proof of Theorem 5.13

We first show the lower frame bound in Theorem 5.10. Using Theorem 5.5, we have

$$\langle \mathbf{S}x, x\rangle = \sum_m \sum_n |\langle x, g_{m,n}\rangle|^2 \tag{B.1}$$

$$\begin{aligned}
&= \frac{1}{F}\sum_n \int_{-\infty}^{\infty} r_g\left(t, t-\frac{n}{F}\right) x^*(t)\, x\left(t-\frac{n}{F}\right) dt \\
&= \frac{1}{F}\int_{-\infty}^{\infty} r_g(t,t)|x(t)|^2\, dt \\
&\quad + \frac{1}{F}\sum_{n\neq 0} \int_{-\infty}^{\infty} r_g\left(t, t-\frac{n}{F}\right) x^*(t)\, x\left(t-\frac{n}{F}\right) dt \qquad \text{(B.2)} \\
&\geq \frac{1}{F}\int_{-\infty}^{\infty} r_g(t,t)\,|x(t)|^2\, dt \\
&\quad - \frac{1}{F}\sum_{n\neq 0} \left| \int_{-\infty}^{\infty} r_g\left(t, t-\frac{n}{F}\right) x^*(t)\, x\left(t-\frac{n}{F}\right) dt \right|
\end{aligned}$$

We shall now give a lower bound for the first term, $\frac{1}{F}\int_{-\infty}^{\infty} r_g(t,t)\,|x(t)|^2\, dt$. With (5.17) we have

$$\begin{aligned}
\frac{1}{F}\int_{-\infty}^{\infty} r_g(t,t)\,|x(t)|^2\, dt &\geq \frac{1}{F} m(g(t);T) \int_{-\infty}^{\infty} |x(t)|^2\, dt \\
&= \frac{1}{F} m(g(t);T)\, \|x\|^2.
\end{aligned}$$

We have thus

$$\begin{aligned}
&\sum_m \sum_n |\langle x, g_{m,n}\rangle|^2 \geq \frac{1}{F} m(g(t);T)\,\|x\|^2 \\
&-\frac{1}{F}\sum_{m=-\infty}^{\infty}\sum_{n\neq 0} \left| \int_{-\infty}^{\infty} g(t-mT)\, g^*\left(t-mT-\frac{n}{F}\right)\left(t, t-\frac{n}{F}\right) x^*(t)\, x\left(t-\frac{n}{F}\right) dt \right|.
\end{aligned}$$

The next step is to derive an upper bound for

$$\frac{1}{F}\left| \sum_m \int_{-\infty}^{\infty} g(t-mT)\, g^*\left(t-mT-\frac{n}{F}\right) x^*(t)\, x\left(t-\frac{n}{F}\right) dt \right|.$$

Setting $b_{m,n}(t) = g(t-mT)\, g^*(t-mT-\frac{n}{F})$, we have

$$\begin{aligned}
&\frac{1}{F}\left| \sum_m \int_{-\infty}^{\infty} b_{m,n}(t)\, x^*(t)\, x\left(t-\frac{n}{F}\right) dt \right| \\
&\leq \frac{1}{F}\sum_m \int_{-\infty}^{\infty} |b_{m,n}(t)|\; |x^*(t)|\; \left| x\left(t-\frac{n}{F}\right)\right| dt
\end{aligned}$$

$$= \frac{1}{F} \sum_m \int_{-\infty}^{\infty} \sqrt{|b_{m,n}(t)|}\, |x(t)| \sqrt{|b_{m,n}(t)|}\, \left| x\left(t - \frac{n}{F}\right)\right| dt$$
$$\leq \frac{1}{F} \sum_m \sqrt{\int_{-\infty}^{\infty} |b_{m,n}(t)|\, |x(t)|^2\, dt} \sqrt{\int_{-\infty}^{\infty} |b_{m,n}(t)| \left| x\left(t - \frac{n}{F}\right)\right|^2 dt}$$
$$\leq \frac{1}{F} \sqrt{\sum_m \int_{-\infty}^{\infty} |b_{m,n}(t)|\, |x^*(t)|^2\, dt} \sqrt{\sum_m \int_{-\infty}^{\infty} |b_{m,n}(t)| \left| x\left(t - \frac{n}{F}\right)\right|^2 dt}$$

where the Schwarz inequality for functions

$$\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt \leq \sqrt{\int_{-\infty}^{\infty} |x(t)|^2\, dt} \sqrt{\int_{-\infty}^{\infty} |y(t)|^2\, dt}$$

and the Schwarz inequality for sequences

$$\sum_m x_m\, y_m \leq \sqrt{\sum_m x_m^2} \sqrt{\sum_m y_m^2}$$

were used.

With

$$\sum_{m=-\infty}^{\infty} |b_{m,n}(t)| = \sum_{m=-\infty}^{\infty} \left| g(t - mT)\, g^*\left(t - mT - \frac{n}{F}\right)\right| \leq \beta\left(-\frac{n}{F}\right)$$

according to (5.20), we obtain further

$$\frac{1}{F} \sqrt{\sum_m \int_{-\infty}^{\infty} |b_{m,n}(t)|\, |x^*(t)|^2\, dt} \sqrt{\sum_m \int_{-\infty}^{\infty} |b_{m,n}(t)| \left| x\left(t - \frac{n}{F}\right)\right|^2 dt}$$
$$\leq \frac{1}{F} \sqrt{\int_{-\infty}^{\infty} \beta\left(-\frac{n}{F}\right) |x(t)|^2\, dt} \sqrt{\int_{-\infty}^{\infty} \beta\left(\frac{n}{F}\right) |x(\tau)|^2\, d\tau}$$
$$= \frac{1}{F} \sqrt{\beta\left(-\frac{n}{F}\right) \beta\left(\frac{n}{F}\right)}\, \|x\|^2. \tag{B.3}$$

We have thus shown that

$$\frac{1}{F} \left| \sum_m \int_{-\infty}^{\infty} b_{m,n}(t)\, x^*(t)\, x\left(t - \frac{n}{F}\right) dt \right| \leq \frac{1}{F} \sqrt{\beta\left(-\frac{n}{F}\right) \beta\left(\frac{n}{F}\right)}\, \|x\|^2. \tag{B.4}$$

Inserting in (B.4) yields

$$\begin{aligned} \sum_m \sum_n |\langle x, g_{m,n}\rangle|^2 &\geq \frac{1}{F}\, m(g(t);T)\, \|x\|^2 - \frac{1}{F} \sum_{n \neq 0} \sqrt{\beta\left(\frac{n}{F}\right) \beta\left(-\frac{n}{F}\right)} \|x\|^2 \\ &= \frac{1}{F} \left[m(g(t);T) - \sum_{n \neq 0} \sqrt{\beta\left(\frac{n}{F}\right) \beta\left(-\frac{n}{F}\right)} \right] \|x\|^2. \end{aligned} \tag{B.5}$$

The decay condition on $\beta(\tau)$ says that for some $\epsilon > 0$ and $C_\epsilon < \infty$ we have

$$\beta(\tau) \le \frac{C_\epsilon}{(1+\tau^2)^{\frac{1+\epsilon}{2}}} \quad \text{for all } \tau. \tag{B.6}$$

Using the Schwarz inequality for sequences it follows that

$$\begin{aligned}
&\sum_{n\neq 0} \sqrt{\beta\left(\frac{n}{F}\right)\beta\left(-\frac{n}{F}\right)} \le \sqrt{\sum_{n\neq 0}\beta\left(\frac{n}{F}\right)}\sqrt{\sum_{n\neq 0}\beta\left(-\frac{n}{F}\right)} \\
&\le \sqrt{\sum_{n\neq 0}\frac{C_\epsilon}{(1+(\frac{n}{F})^2)^{\frac{1+\epsilon}{2}}}\sum_{n\neq 0}\frac{C_\epsilon}{(1+(-\frac{n}{F})^2)^{\frac{1+\epsilon}{2}}}} = \sum_{n\neq 0}\frac{C_\epsilon}{[1+(\frac{n}{F})^2]^{\frac{1+\epsilon}{2}}} \\
&\le \sum_{n\neq 0}\frac{C_\epsilon}{\left|\frac{n}{F}\right|^{1+\epsilon}} = C_\epsilon F^{1+\epsilon}\sum_{n\neq 0}\frac{1}{|n|^{1+\epsilon}} \le C_\epsilon F^{1+\epsilon}\sum_{n\neq 0}\left|\frac{1}{n^{1+\epsilon}}\right| \\
&\le 2C_\epsilon F^{1+\epsilon}\sum_{n=1}^{\infty}\frac{1}{n^{1+\epsilon}}.
\end{aligned}$$

So the summation always converges, because the hyperharmonic sum $\sum_{n=1}^{\infty}\frac{1}{n^a}$ always converges for $a > 1$. By choosing F small enough, it follows from the above inequality that $\sum_{n\neq 0}\sqrt{\beta\left(\frac{n}{F}\right)\beta\left(-\frac{n}{F}\right)}$ can be made arbitrary small. Let us define the critical frequency F_0 such that

$$\sum_{n\neq 0}\sqrt{\beta\left(\frac{n}{F}\right)\beta\left(-\frac{n}{F}\right)} < m\,(g(t);T)$$

for $F < F_0$. It then follows that we have from (B.5)

$$A = \frac{1}{F}\left[m(g(t);T) - \sum_{n\neq 0}\sqrt{\beta\left(\frac{n}{F}\right)\beta\left(-\frac{n}{F}\right)}\right] > 0$$

for all $F \in (0, F_0)$, and hence (B.5) becomes

$$\sum_m\sum_n |\langle x, g_{m,n}\rangle|^2 \ge A\|x\|^2, \tag{B.7}$$

for $F \in (0, F_0)$, with the lower frame bound $A > 0$.

We shall now prove the upper frame bound. Indeed, an analogous development shows that

$$\sum_m\sum_n |\langle x, g_{m,n}\rangle|^2 \le B\|x\|^2 \tag{B.8}$$

where

$$B = \frac{1}{F}\left[M(g(t);T) + \sum_{n\neq 0}\sqrt{\beta\left(\frac{n}{F}\right)\beta\left(-\frac{n}{F}\right)}\right] < \infty. \tag{B.9}$$

To see this we start from (B.2) to obtain

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |\langle x, g_{m,n}\rangle|^2 = \frac{1}{F} \int_{-\infty}^{\infty} \sum_m |g(t-mT)|^2 \, |x(t)|^2 \, dt$$
$$+ \frac{1}{F} \sum_m \sum_{n\neq 0} \int_{-\infty}^{\infty} g(t-mT) \, g^* \left(t - mT - \frac{n}{F}\right) x^*(t) \, x\left(t - \frac{n}{F}\right) dt.$$

Thus, we have

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |\langle x, g_{m,n}\rangle|^2 \leq \frac{1}{F} M(g(t); T, F) \|x\|^2$$
$$+ \frac{1}{F} \sum_m \sum_{n\neq 0} \int_{-\infty}^{\infty} \left| g(t-mT) \, g^* \left(t - mT - \frac{n}{F}\right) x^*(t) \, x\left(t - \frac{n}{F}\right) dt \right| \quad \text{(B.10)}$$

In the first part of our proof (cf. (B.3)), we have shown that

$$\frac{1}{F} \left| \sum_m \int_{-\infty}^{\infty} b_{m,n}(t) \, x^*(t) \, x\left(t - \frac{n}{F}\right) dt \right| \leq \frac{1}{F} \sqrt{\beta\left(-\frac{n}{F}\right) \beta\left(\frac{n}{F}\right)} \, \|x\|^2.$$

Inserting in (B.10) gives (B.8), (B.9).

6 The Continuous Gabor Expansion

In 1946, D. Gabor suggested a signal expansion which decomposes an arbitrary function $x(t)$ into a weighted set of TF shifted versions of an elementary building block $g(t)$,

$$x(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} g_{m,n}(t) \tag{6.1}$$

with

$$g_{m,n}(t) = (\mathbf{W}_{m,n} g)(t) = g(t - mT)\, e^{2\pi j n F t}.$$

The weighting is expressed by the complex-valued *Gabor coefficients* $a_{m,n}$. This expansion is known as the Gabor expansion (GE) of a finite-energy signal $x(t) \in L^2(\mathbb{R})$. The GE is an important linear TF representation. Gabor considered only the case where $g(t)$ is a Gaussian and $TF = 1$ (i.e., critical sampling). Later, the Gabor expansion was generalized to building blocks $g(t)$ other than the Gaussian function and higher sampling densities where $TF < 1$. The GE provides a rectangular tiling of the TF plane (see Fig. 1). Under certain conditions the squared Gabor coefficients $|a_{m,n}|^2$ can be viewed as a measure for the local energy content of the signal $x(t)$ around the TF point (mT, nF).

The product TF is the area of a fundamental sampling grid rectangle; it determines the sampling density in the TF plane. The properties of the expansion depend heavily on the value of TF. These properties are the existence and the uniqueness of the expansion coefficients $a_{m,n}$ and the numerical stability of the expansion. In general one has to distinguish three cases (see Fig. 1):

- *critical sampling*, $TF = 1$
- *oversampling*, $TF < 1$
- *undersampling*, $TF > 1$.

We have already discussed this distinction in the context of Weyl-Heisenberg (WH) frames (see Section 5). WH frames play a fundamental role in the theory of the GE. In the following, we briefly summarize the main results of Section 5.

- In the case of *critical sampling*, $\{g_{m,n}(t)\} = \{(\mathbf{W}_{m,n} g)(t)\}$ is a WH frame for certain window functions $g(t)$. In these cases, the GE exists for arbitrary $x(t) \in L_2(\mathbb{R})$, and the Gabor coefficients $a_{m,n}$ can be calculated using the dual WH frame (see Section 5). A necessary and sufficient condition for the existence of a WH frame in the critical case is that the ZT of the window function $g(t)$ is finite and has no zeros in the fundamental rectangle (see Theorem 5.7). While there exist functions $g(t)$ generating a WH frame in the critical case $TF = 1$, the Balian-Low Theorem states that these functions may not be well localized in both time domain and frequency domain. Thus, the TF localization properties of the GE are poor in the critical case. On the other hand, due to the fact that WH frames are always exact in the critical case, the Gabor coefficients are unique.

- In the case of *oversampling*, $TF < 1$, window functions $g(t)$ giving rise to a frame *and* having good TF localization properties can be found. However, the Gabor coefficients are not unique for $TF < 1$. This is due to the fact that in the oversampled case, the frame $\{g_{m,n}(t)\}$ is inexact in general, i.e. the WH frame functions are not linearly independent.

- A fundamental result from the theory of WH frames is that in the case of *undersampling* the set $\{g_{m,n}(t)\} = \{(\mathbf{W}_{m,n}g)(t)\}$ is not a WH frame, and indeed is incomplete in $L_2(\mathbb{R})$. This immediately implies that for $TF > 1$ the GE cannot exist for arbitrary signals with finite energy.

One of the major problems with the GE is the calculation of the expansion coefficients. Since the functions $g_{m,n}(t)$ are typically not orthogonal, the Gabor coefficients cannot be exactly calculated as the inner product of the signal $x(t)$ with the $g_{m,n}(t)$. The theory of frames provides a suitable framework for the calculation of the Gabor coefficients. We shall now discuss several methods for calculating the Gabor coefficients for a given signal $x(t)$, making intensive use of results from frame theory.

Since the Gabor coefficients can be calculated using the dual window function $\tilde{g}(t)$, it is theoretically sufficient to consider the problem of calculating the dual window. However, there exist methods which immediately yield the Gabor coefficients without explicitly using the dual window. We shall therefore first discuss methods which yield the dual window and then consider methods which immediately yield the Gabor coefficients.

6.1 Calculation of the dual window

The ZT (see Section 3) provides a way to calculate the dual function. Let us assume that $\{g_{m,n}(t)\}$ is a WH frame in the critical case $TF = 1$. The ZT of the dual function $\tilde{g}(t)$ is given by (see Theorem 5.8)

$$\mathcal{Z}_{\tilde{g}}(t,f) = \frac{1}{T\mathcal{Z}_g^*(t,f)}.$$

Thus we can compute the dual function by means of the ZT inversion formula (3.17) as

$$\tilde{g}(t) = \int_0^{1/T} \frac{1}{\mathcal{Z}_g^*(t,f)}\,df. \tag{6.2}$$

However as we know from Theorem 5.7 the ZT of $g(t)$ is bounded as

$$A \leq T|\mathcal{Z}_g(t,f)|^2 \leq B$$

with $A > 0$ if and only if $\{g_{m,n}(t)\}$ is a frame. We conclude that the dual function can always be calculated via (6.2) if $\{g_{m,n}(t)\}$ is a frame.

We shall next consider the case of oversampling by rational factors, i.e. $TF = p/q$. In [105] it is shown that in the case of general $TF = p/q$ the calculation of the dual function via the ZT requires the solution of a linear system of equations. In the special case of oversampling by integer factors, $TF = 1/N$, however, Theorem 5.17 gives an expression for the ZT of the dual function $\tilde{g}(t)$ in terms of the ZT of $g(t)$,

$$\mathcal{Z}_{\tilde{g}}(t,f) = \frac{\mathcal{Z}_g(t,f)}{T \sum\limits_{i=0}^{N-1} |\mathcal{Z}_g(t,f-iF)|^2}.$$

Hence, for $TF = 1/N$ the dual function $\tilde{g}(t)$ is again easily calculated using the ZT. The main advantage of the ZT method relies on the fact that the ZT can be implemented using FFT methods, which makes the computation very efficient.

6.2 Calculation of the Gabor Coefficients

6.2.1 The Biorthogonal Function Method

We shall start with the *biorthogonal function method* first suggested by M.J. Bastiaans in [25]. With $\{g_{m,n}(t)\}$ denoting the set of TF shifted versions of the elementary function $g(t)$, one tries to find a function $h(t)$ such that the set $h_{m,n}(t)$ is biorthogonal to $\{g_{m,n}(t)\}$, i.e.

$$\langle g_{m,n}, h_{k,l}\rangle = \delta_{m-k}\,\delta_{n-l} = \begin{cases} 1, & m=k, n=l \\ 0, & \text{otherwise.} \end{cases} \tag{6.3}$$

If such a $h(t)$ exists, and if the GE exists for a given signal $x(t)$, then the Gabor coefficients of $x(t)$ can be calculated as the inner products of $x(t)$ with the $h_{m,n}(t)$,

$$a_{m,n} = \langle x, h_{m,n}\rangle. \tag{6.4}$$

Proof: Using the GE $x(t) = \sum\limits_{m=-\infty}^{\infty} \sum\limits_{n=-\infty}^{\infty} a_{m,n}\, g_{m,n}(t)$ and (6.3), we obtain

$$\begin{aligned} \langle x, h_{k,l}\rangle &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n}\,\langle g_{m,n}, h_{k,l}\rangle \\ &= \sum_{m=\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n}\,\delta_{m-k}\,\delta_{n-l} = a_{k,l}\,. \quad \square \end{aligned}$$

The biorthogonal function method is fully consistent with WH frame theory in the critical case (see Section 5.2). The biorthogonal functions $h_{m,n}(t)$ are the dual frame functions $\tilde{g}_{m,n}(t)$. Indeed, if $\{g_{m,n}(t)\}$ is a WH frame in the critical case, then the dual frame $\{\tilde{g}_{m,n}(t)\}$ is again a WH frame which is biorthogonal to $\{g_{m,n}(t)\}$; furthermore, according to frame theory the expansion coefficients are calculated as the inner products $\langle x, \tilde{g}_{m,n}\rangle$. This yields Bastiaans' biorthogonal function method. However, while Bastiaans had to *assume* the existence of the biorthogonal set $\{h_{m,n}(t)\}$, frame theory guarantees the existence of this set provided that the $g_{m,n}(t)$ are a frame (which implies the completeness of the $g_{m,n}(t)$), and it shows how to calculate the dual function $\tilde{g}(t)$ (recall that $\tilde{g}(t) = (\mathbf{S}^{-1}g)(t)$). In his classical paper Bastiaans calculates the Gabor coefficients via a two-dimensional deconvolution.

Hitherto only the critical case has been considered. In the oversampled case, we have seen in Section 5.3.2 that $\{g_{m,n}(t)\}$ and $\{\tilde{g}_{m,n}(t)\}$ are biorthogonal with respect to the *dual* grid (see Section 5.3.2). Using this biorthogonality property, J. Wexler and S. Raz find explicit expressions for the Gabor coefficients in the oversampled case for finite-length discrete-time signals.

6.2.2 The STFT method

The Gabor coefficients in the form $a_{m,n} = \langle x, \tilde{g}_{m,n}\rangle$ are the sampled short time Fourier transform of the signal $x(t)$ with the analysis window $\tilde{g}(t)$. The STFT of a signal $x(t)$, using an analysis window $\tilde{g}(t)$, is defined as

$$\mathrm{STFT}_x^{(\tilde{g})}(t,f) = \int_{-\infty}^{\infty} x(t')\,\tilde{g}^*(t'-t)\,e^{-2\pi j f t'}\,dt'.$$

The Gabor coefficients can then be written as

$$a_{m,n} = \langle x, \tilde{g}_{m,n}\rangle = \int_{-\infty}^{\infty} x(t)\,\tilde{g}^*(t-mT)\,e^{-2\pi j nFt}\,dt = \mathrm{STFT}_x^{(\tilde{g})}(mT, nF).$$

This can be interpreted as the Fourier transform (sampled at $f = nF$) of the windowed signal $x(t)\,\tilde{g}^*(t-mT)$. It follows that the Gabor coefficients can be calculated efficiently using FFT techniques. Note that this is true for both the critical and the oversampled case.

6.2.3 The Zak Transform Method

An alternative approach to the calculation of the Gabor coefficients in the critical case $TF = 1$ is provided by the ZT discussed in Section 3. We start from the Gabor expansion of the signal $x(t)$

$$x(t) = \sum_m \sum_n a_{m,n}\, g_{m,n}(t).$$

Taking the ZT of both sides yields

$$\mathcal{Z}_x(t,f) = \sum_m \sum_n a_{m,n}\, \mathcal{Z}_{g_{m,n}}(t,f) = \sum_m \sum_n a_{m,n}\, \mathcal{Z}_g(t,f)\, e^{2\pi j n \frac{t}{T}}\, e^{-2\pi j m f T}, \tag{6.5}$$

where Theorem 5.3 has been used. Dividing through $\mathcal{Z}_g(t,f)$, we obtain

$$\frac{\mathcal{Z}_x(t,f)}{\mathcal{Z}_g(t,f)} = \sum_m \sum_n a_{m,n}\, e^{2\pi j n \frac{t}{T}}\, e^{-2\pi j m T f}, \tag{6.6}$$

which is the Fourier series representation of the periodic function $\frac{\mathcal{Z}_x(t,f)}{\mathcal{Z}_g(t,f)}$. Hence, the Gabor coefficients can be calculated as

$$a_{m,n} = \int_0^T \int_0^{1/T} \frac{\mathcal{Z}_x(t,f)}{\mathcal{Z}_g(t,f)}\, e^{-2\pi j n \frac{t}{T}}\, e^{2\pi j m T f}\, dt\, df. \tag{6.7}$$

This method, too, can be verified by the theory of frames. From (3.27) it follows that

$$\mathcal{Z}_x(t,f)\, \mathcal{Z}_{\tilde{g}}^*(t,f) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{T} \langle x, \tilde{g}_{m,n} \rangle\, e^{2\pi j n \frac{t}{T}}\, e^{-2\pi j m \frac{f}{F}}. \tag{6.8}$$

Due to Theorem 5.6,

$$\mathcal{Z}_{\tilde{g}}(t,f) = \frac{1}{T \mathcal{Z}_g^*(t,f)}.$$

Inserting this in (6.8) yields (6.6) with $a_{m,n} = \langle x, \tilde{g}_{m,n} \rangle$. We emphasize that under the assumption that the set $\{g_{m,n}(t)\}$ constitutes a frame, $\mathcal{Z}_g(t,f)$ has no zeros in the fundamental rectangle (see Theorem 5.6) and hence the integrand of (6.7) has no poles.

This method can be extended to the case of oversampling by integer factors, $TF = 1/N$. From Theorem 5.17 and the unitarity of the ZT it follows that

$$a_{m,n} = \int_0^T \int_0^{1/T} \frac{\mathcal{Z}_x(t,f) \mathcal{Z}_g^*(t,f)}{\sum_{i=0}^{N-1} |\mathcal{Z}_g(t, f - iF)|^2}\, e^{-2\pi j n F t}\, e^{-2\pi j m T f}\, dt\, df. \tag{6.9}$$

This method is numerically efficient since the inverse frame operator need not be calculated, and furthermore the ZT can be implemented using FFT techniques.

6.2.4 The Deconvolution Method

Starting from the GE in the critical case $TF = 1$,

$$x(t) = \sum_m \sum_n a_{m,n}\, g_{m,n}(t),$$

we form the inner products

$$\langle x, g_{k,l} \rangle = \sum_m \sum_n a_{m,n} \langle g_{m,n}, g_{k,l} \rangle.$$

With $TF = 1$, $\langle g_{m,n}, g_{k,l}\rangle = \langle g, g_{k-m,l-n}\rangle$ and we obtain further

$$\langle x, g_{k,l}\rangle = \sum_m \sum_n a_{m,n} \, \langle g, g_{k-m,l-n}\rangle.$$

Setting $x_{k,l} = \langle x, g_{k,l}\rangle$ and $b_{k,l} = \langle g, g_{k,l}\rangle$, this can be written as the 2D convolution

$$x_{k,l} = \sum_m \sum_n a_{m,n} \, b_{k-m,l-n} = a_{k,l} ** b_{k,l}. \tag{6.10}$$

We note that $x_{k,l}$ is the sampled STFT of the signal $x(t)$ with window function $g(t)$ and $b_{k,l}$ is the sampled auto-ambiguity function of the window $g(t)$:

$$\begin{aligned} x_{k,l} = \langle x, g_{k,l}\rangle &= \int_t x(t)\, g^*(t-kT)\, e^{-2\pi j l F t}\, dt \\ b_{k,l} = \langle g, g_{k,l}\rangle &= \int_t g(t)\, g^*(t-kT)\, e^{-2\pi j l F t}\, dt. \end{aligned}$$

Due to (6.10) the Gabor coefficients $a_{m,n}$ can be calculated via a 2D deconvolution. A convenient way to implement the deconvolution is to perform a Fourier transformation of the 2D sequences involved. The Fourier transform of a 2D discrete-time sequency $x_{k,l}$ is defined as

$$X(e^{2\pi j\theta_1}, e^{2\pi j\theta_2}) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x_{k,l}\, e^{-2\pi j k\theta_1}\, e^{-2\pi j l\theta_2}, \tag{6.11}$$

where θ_1 and θ_2 denote normalized frequencies restricted to the fundamental period $-1/2 \le \theta \le 1/2$. Taking the Fourier transform of both sides of (6.10) yields

$$X(e^{2\pi j\theta_1}, e^{2\pi j\theta_2}) = A(e^{2\pi j\theta_1}, e^{2\pi j\theta_2})\, B(e^{2\pi j\theta_1}, e^{2\pi j\theta_2}).$$

The Fourier transform of the Gabor coefficients can now be expressed as

$$A(e^{2\pi\theta_1}, e^{2\pi j\theta_2}) = \frac{X(e^{2\pi j\theta_1}, e^{2\pi j\theta_2})}{B(e^{2\pi j\theta_1}, e^{2\pi j\theta_2})},$$

and taking the inverse Fourier transform the GCs are obtained as

$$a_{m,n} = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{X(e^{2\pi j\theta_1}, e^{2\pi j\theta_2})}{B(e^{2\pi j\theta_1}, e^{2\pi j\theta_2})}\, e^{2\pi j m\theta_1}\, e^{2\pi j n\theta_2}\, d\theta_1\, d\theta_2. \tag{6.12}$$

This method only works in the critical case.

References

[1] F. Hlawatsch and G.F. Boudreaux-Bartels, "Linear and quadratic time-frequency signal representations," *IEEE Signal Processing Mag.*, Vol. 9, April 1992, pp. 21-67.

[2] P. Flandrin, *Temps-Fréquence*. Hermès, Paris, 1993.

[3] L. Cohen, "Time-frequency distributions — A review," *Proc. IEEE*, Vol. 77, No. 7, pp. 941-981, July 1989.

[4] B. Boashash, "Time-frequency signal analysis," chapter in *Advances in Spectrum Estimation*, Vol. 1, ed. S. Haykin, Prentice-Hall, 1991, pp. 418-517.

[5] S.N. Nawab and T.F. Quatieri, "Short-time Fourier transform," chapter in *Advanced Topics in Signal Processing*, J.S. Lim and A.V. Oppenheim, eds., Prentice Hall, 1988.

[6] L.R. Rabiner and R.W. Schafer, *Digital Processing of Speech Signals*, Prentice Hall, 1978.

[7] J.B. Allen, "Short-term spectral analysis, synthesis, and modification by discrete Fourier transform," *IEEE Trans. Acoust., Speech. Signal Processing*, Vol. ASSP-25, pp. 235-238, June 1977. (Corrections: *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. ASSP-25, p. 589, Dec. 1977.)

[8] J.B. Allen and L.R. Rabiner, "A unified approach to STFT analysis and synthesis," *Proc. IEEE*, Vol. 65, pp. 1558-1564, Nov. 1977.

[9] M.R. Portnoff, "Time-frequency representations of digital signals and systems based on short-time Fourier analysis," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. 28, pp. 55-69, Feb. 1980.

[10] R.E. Crochiere and L.R. Rabiner, *Multi-Rate Digital Signal Processing*, Prentice Hall, 1983.

[11] D. Gabor, "Theory of communication," *J. of the IEE*, Vol. 93, No. 3, pp. 429-457, 1946.

[12] J. Strawn, "Analysis and synthesis of musical transitions using the discrete short-time Fourier transform," *J. Audio Eng. Soc.*, Vol. 35, pp. 3-13, Jan.-Feb. 1987.

[13] J.B. Allen and L.R. Rabiner, "Unbiased spectral estimation and system identification using short-time spectral analysis methods," *Bell Syst. Tech. J.*, Vol. 58, pp. 1743-1763, Oct. 1979.

[14] L.R. Rabiner and J.B. Allen, "On the implementation of a short-time spectral analysis method for system identification," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. 28, pp. 69-79, Feb. 1980.

[15] J.J. Wolcin, "Maximum likelihood detection of transient signals using sequenced short-time power spectra," Tech. Mem. #831138, Naval Underwater Systems Center, New London, CT, Aug. 1983.

[16] R.A. Altes, "Detection, estimation, and classification with spectrograms," *J. Acoust. Soc. Am.*, Vol. 67, No. 4, pp. 1232-1246, Apr. 1980.

[17] L. Auslander, C. Buffalano, R. Orr, and R. Tolimieri, "A comparison of the Gabor and short-time Fourier transforms for signal detection and feature extraction in noisy environments," *Proc. SPIE Conf.*, Vol. 1348, pp. 230-247, Nov. 1990.

[18] B. Porat and B. Friedlander, "Performance analysis of a class of transient detection algorithms — A unified framework," *IEEE Trans. Signal Processing*, Vol. 40, pp. 2536-2546, Oct. 1992.

[19] J.B. Allen, "Application of short-time Fourier transform to speech processing and spectral analysis," *Proc. ICASSP-82*, pp. 1012-1015, Paris, France, May 1982.

[20] M.R. Portnoff, "Time-scale modification of speech based on short-time Fourier transform," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. 29, pp. 374-390, June 1981.

[21] M.R. Portnoff, "Short-time Fourier analysis of sampled speech," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. 29, pp. 364-373, June 1981.

[22] J.S. Lim, "Enhancement and bandwidth compression of noisy speech," *Proc. IEEE*, Vol. 67, pp. 1586-1604, Dec. 1979.

[23] C.W. Helstrom, "An expansion of a signal in Gaussian elementary signals," *IEEE Trans. Inf. Th.*, Vol. IT-12, pp. 81-82, Jan. 1966.

[24] L.K. Montgomery and I.S. Reed, "A generalization of the Gabor-Helstrom Transform," *IEEE Trans. Inf. Th.*, Vol. IT-13, pp. 344-345, Apr. 1967.

[25] M.J. Bastiaans, "Gabor's expansion of a signal into Gaussian elementary signals," *Proc. IEEE*, Vol. 68, No. 4, pp. 538-539, April 1980.

[26] J. Wexler and S. Raz, "Discrete Gabor expansions," *Signal Processing*, Vol. 21, 1990, pp. 207-220.

[27] R.J. Duffin and A.C. Schaeffer, "A class of nonharmonic Fourier series," *Trans. Am. Math. Soc.*, Vol. 72, pp. 341-366, 1952.

[28] I. Daubechies, "The wavelet transform, time-frequency localization and signal analysis," *IEEE Trans. Information Theory*, Vol. 36, No. 5, Sept. 1990, pp. 961-1005.

[29] C.E. Heil, "Wavelets and frames," in *Signal Processing, Part I: Signal Processing Theory*, L. Auslander et al., eds., The IMA Volumes in Mathematics and its Applications, Vol. 22, Springer, New York, pp. 147-160, 1990.

[30] C.E. Heil and D.F. Walnut, "Continuous and discrete wavelet transforms," *SIAM Review*, Vol. 31, No. 4, pp. 628-666, Dec. 1989.

[31] M. Poize, M. Renaudin, and P. Venier, "The Gabor transform as a modulated filter bank system," *14ème Colloque GRETSI*, Juan-Les-Pins, France, Sept. 1993, pp. 351-354.

[32] J. Zak, "Finite translations in solid state physics," *Phys. Rev. Lett.*, Vol. 19, pp. 1385-1397, 1967.

[33] A.J.E.M. Janssen, "The Zak transform: A signal transform for sampled time-continuous signals," *Philips J. Res.*, Vol. 43, pp. 23-69, 1988.

[34] R.S. Orr, "The order of computation for finite discrete Gabor transforms," *IEEE Trans. Signal Processing*, Vol. 41, No. 1, pp. 122-130, Jan. 1993.

[35] M. Porat and Y.Y. Zeevi, "The generalized Gabor scheme of image representation in biological and machine vision," *IEEE Trans. Pattern Analysis and Machine Intelligence*, Vol. 10, No. 4, pp. 452-467, July 1988.

[36] A. Teuner and B.J. Hosticka, "Adaptive Gabor transformation for image processing," *IEEE Trans. Image Processing*, Vol. 2, No. 1, pp. 112-117, Jan. 1993.

[37] D.F. Dunn and W.E. Higgins, "Optimal Gabor-filter design for texture segmentation," *IEEE ICASSP-93*, Minneapolis, MN, April 1993, Vol. V, pp. 37-40.

[38] S. Farkash and S. Raz, "Time-variant filtering via the Gabor expansion," *Signal Processing V: Theories and Applications*, Elsevier, 1990, pp. 509-512.

[39] B. Friedlander and B. Porat, "Detection of transient signals by the Gabor representation," *IEEE. Trans. Acoustics, Speech, Signal Processing*, Vol. 37, No. 2, pp. 169-179, Feb. 1989.

[40] R. Kronland-Martinet, J. Morlet, and A. Grossmann, "Analysis of sound patterns through wavelet transforms," *Int. J. Pattern Recog. Art. Intell.*, Vol. 1, pp. 273-301, 1987.

[41] I. Daubechies, "Orthonormal bases of compactly supported wavelets," *Comm. on Pure and Applied Math.*, Vol. 41, No. 7, pp. 909-996, 1988.

[42] S. Mallat, "A theory for multiresolution signal decomposition: The wavelet representation," *IEEE Trans. Pat. Anal. Machine Intell.*, Vol. 11, No. 7, pp. 674-693, July 1989.

[43] S. Mallat, "Multifrequency channel decompositions of images and wavelet models," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. 37, No. 12, pp. 2091-2110, Dec. 1989.

[44] I. Daubechies, "The wavelet transform: A method for time-frequency localization," Chapter 8 of *Advances in Spectrum Analysis and Array Processing, Vol. 1*, S. Haykin, ed., pp. 366-417, Prentice Hall, 1991.

[45] O. Rioul and M. Vetterli, "Wavelets and signal processing," *IEEE Signal Processing Mag.*, Vol. 8, No. 4, pp. 14-38, Oct. 1991.

[46] R.R. Coifman, Y. Meyer, and V. Wickerhauser, "Wavelet analysis and signal processing," in *Advances in Numerical Analysis II*, W. Light (ed.), Clarendon Press, Oxford, 1992, pp. 153-178.

[47] A. Croisier, D. Esteban, and C. Galand, "Perfect channel splitting by use of interpolation, decimation, tree decomposition techniques," *Int. Conf. on Information Sciences/Systems*, Patras, pp. 443-446, Aug. 1976.

[48] R.E. Crochiere, S.A. Weber, and J.L. Flanagan, "Digital coding of speech in subbands," *Bell Syst. Tech. J.*, Vol. 55, pp. 1069-1085, Oct. 1976.

[49] D. Esteban and C. Galand, "Application of quadrature mirror filters to split-band voice coding schemes," *IEEE ICASSP-77*, Hartford, CT, pp. 191-195, May 1977.

[50] H. Malvar, *Signal Processing with Lapped Transforms*. Artech House, 1992.

[51] A.N. Akansu and R.A. Haddad, *Multiresolution Signal Decomposition*. Academic Press, 1992.

[52] F.J. Malassenet and R.M. Mersereau, "Wavelet representations and coding of self-affine signals," *IEEE ICASSP-91*, Vol. 1, pp. 677-680, Toronto, Canada, May 1991.

[53] M. Antonini, M. Barlaud, and P. Mathieu, "Image coding using lattice vector quantization of wavelet coefficients," *IEEE ICASSP-91*, pp. 2273-2276, Toronto, Canada, May 1991.

[54] S. Mallat and S. Zhong, "Characterization of signals from multiscale edges," *IEEE Trans. Pattern Anal. Machine Intell.*, Vol. 14, No. 7, 1992, pp. 710-732.

[55] P. Burt, "Multiresolution techniques for image representation, analysis and 'smart' transmission," *Proc. SPIE Conf. Visual Communication and Image Processing*, pp. 2-15, Philadelphia, PA, Nov. 1989.

[56] W. Zettler, J. Huffmann, and D. Linden, "Applications of compactly supported wavelets to image compression," *SPIE Conf. Image Proc. Alg. and Tech.*, Vol. 1244, pp. 150-160, Santa Clara, CA, Feb. 1990.

[57] N. Baazit and C. Labit, "Laplacian pyramid versus wavelet decomposition for image sequence coding," *IEEE ICASSP-90*, Albuquerque, NM, Apr. 3-6, 1990.

[58] M. Antonini, M. Barlaud, P. Mathieu, and I. Daubechies, "Image coding using vector quantization in the wavelet transform domain," *IEEE ICASSP-90*, Albuquerque, NM, Apr. 1990, pp. 2297-2300.

[59] S. Kadambe and G.F. Boudreaux-Bartels, "Application of the wavelet transform for pitch detection of speech signals," *IEEE Trans. Inform. Theory*, Vol. 38, No. 2, pp. 917-924, March 1992.

[60] M.R. Davenport and H. Garudadri, "A neural net acoustic phonetic feature extractor based on wavelets," *Proc. of the IEEE Pacific Rim Conf. on Communications, Computers and Signal Processing*, Vol. 2, Victoria, B.C., Canada, May 1991, pp. 449-452.

[61] J.S. Lienard and C. d'Allessandro, "Wavelets and granular analysis of speech," *Proc. Int. Conf. on Wavelets, Time-Frequency Methods and Phase Space*, Marseille, France, pp. 158-163, Dec. 14-18, 1987.

[62] R.K. Young, *Wavelet Theory and Its Applications*. Kluwer, 1993.

[63] K.C. Chou, S. Golden, and A.S. Willsky, "Modeling and estimation of multiscale stochastic processes," *IEEE ICASSP-91*, Toronto, Ontario, Canada, pp. 1709-1712, May 1991.

[64] P.W. Wong, "Wavelet decomposition of harmonizable random processes," *IEEE Trans. Information Theory*, Vol. 39, No. 1, January 1993, pp. 7-18.

[65] R. Dijkerman, V. Badrinath, and R.R. Mazumdar, "Multi-scale representation of stochastic processes using compactly supported wavelets," *Proc. IEEE-SP Int. Sympos. Time-Frequency and Time-Scale Analysis*, Victoria, B.C., Canada, pp. 185-188, Oct. 1992.

[66] G.W. Wornell, "A Karhunen-Loève-like expansion for $1/f$ processes via wavelets," *IEEE Trans. Inf. Theory*, Vol. 36, No. 4, 1990, pp. 859-861.

[67] P. Abry, P. Gonçalvès, and P. Flandrin, "Wavelet-based spectral analysis of $1/f$ processes," *IEEE ICASSP-93*, Minneapolis, MN, April 1993, Vol. 3, pp. 237-240.

[68] G.W. Wornell and A.V. Oppenheim, "Estimation of fractal signals from noisy measurements using wavelets," *IEEE Trans. Signal Processing*, Vol. 40, No. 3, March 1992, pp. 611-623.

[69] C.K. Chui, *An Introduction to Wavelets. Wavelet Analysis and its Applications Vol. 1*, Academic Press, 1992.

[70] C.K. Chui (ed.), *Wavelets: A Tutorial in Theory and Applications. Wavelet Analysis and its Applications Vol. 2*, Academic Press, 1992.

[71] S. Pittner, J. Schneid, and C.W. Ueberhuber, *Wavelet Literature Survey*. Institute for Applied and Numerical Mathematics, Technical University Vienna, 1993.

[72] R.R. Coifman, Y. Meyer, S. Quaker, and M.V. Wickerhauser, "Signal processing and compression with wave packets," preprint, Yale University, New Haven, CT, 1990.

[73] M.V. Wickerhauser, "Acoustic signal compression with wavelet packets," in *Wavelets: A Tutorial in Theory and Applications*, C.K. Chui, Ed. New York: Academic, 1992, pp. 679-700.

[74] R.R. Coifman and Y. Meyer, "Orthonormal wave packet bases," preprint, Yale University, New Haven, CT.

[75] K. Ramchandran and M. Vetterli, "Best wavelet packet bases in a rate-distortion sense," *IEEE Trans. Image Processing*, Vol. 2, No. 2, April 1993, pp. 160-175.

[76] C. van den Branden Lambrecht and M. Karrakchou, "Subband adaptive filtering: The mutual wavelet packets approach," *IEEE ICASSP-93*, Vol. 3, pp. 316-319, Minneapolis, MN, April 1993.

[77] R.E. Learned, W.C. Karl, and A.S. Willsky, "Wavelet packet based transient signal classification," *Proc. IEEE-SP Int. Sympos. Time-Frequency Time-Scale Analysis*, Victoria, B.C., Canada, pp. 109-112, Oct. 1992.

[78] H. Malvar, "Lapped transforms for efficient transform/subband coding," *IEEE Trans. Acoustics, Speech, Signal Processing*, Vol. 38, No. 6, June 1990, pp. 969-978.

[79] E. Wesfreid and M.V. Wickerhauser, "Signal processing via fast Malvar wavelet transform algorithm," *14ème Colloque GRETSI*, Juan-Les-Pins, France, pp. 379-382, Sept. 1993.

[80] C. Herley, J. Kovačević, K. Ramchandran, and M. Vetterli, "Tilings of the time-frequency plane: Construction of arbitrary orthogonal bases and fast tiling algorithms," *IEEE Trans. Signal Processing*, Vol. 41, No. 12, Dec. 1993, pp. 3341-3359.

[81] R.G. Baraniuk and D.L. Jones, "New orthonormal bases and frames using chirp functions," *IEEE Trans. Signal Processing*, Vol. 41, No. 12, Dec. 1993, pp. 3543-3549.

[82] R.G. Baraniuk and D.L. Jones, "New signal-space orthonormal bases via the metaplectic transform," *Proc. IEEE-SP Int. Sympos. Time-Frequency Time-Scale Analysis*, Victoria, B.C., Canada, pp. 147-150, Oct. 1992.

[83] G.B. Folland, *Harmonic Analysis in Phase Space.* Princeton University Press, 1989.

[84] D.L. Jones and R.G. Baraniuk, "A simple scheme for adapting time-frequency representations," *Proc. IEEE-SP Int. Sympos. Time-Frequency Time-Scale Analysis*, Victoria, B.C., Canada, pp. 83-86, Oct. 1992.

[85] G. Jones and B. Boashash, "Window matching in the time-frequency plane and the adaptive spectrogram," *Proc. IEEE-SP Int. Sympos. Time-Frequency Time-Scale Analysis*, Victoria, B.C., Canada, pp. 87-90, Oct. 1992.

[86] I. Sodagar, K. Nayebi, and T.P. Barnwell, III, "A class of time-varying wavelet transforms," *IEEE ICASSP-93*, Minneapolis, MN, April 1993, Vol. 3, pp. 201-204.

[87] K. Nayebi, T.P. Barnwell, and M.J.T. Smith, "Analysis-synthesis systems with time-varying filter bank structures," *IEEE ICASSP-92*, Vol. 4, pp. 617-620, San Francisco, CA, March 1992.

[88] W. Kozek, "Matched generalized Gabor expansion of nonstationary processes," *Proc. IEEE 27th Asilomar Conf.*, Nov. 1993, Pacific Grove, CA, pp. 499-503.

[89] W. Kozek, "Optimally Karhunen-Loeve like STFT expansion of nonstationary processes," *Proc. IEEE ICASSP-93*, Minneapolis, MN, April 1993, Vol. IV, pp. 428-431.

[90] S. Mallat and Z. Zhang, "Adaptive time-frequency transforms," *IEEE ICASSP-93*, Minneapolis, MN, April 1993, Vol. 3, pp. 241-244.

[91] S. Mallat and Z. Zhang, "Matching pursuits with time-frequency dictionaries," Courant Institute of Mathematical Sciences, New York University, New York.

[92] S. Qian and D. Chen, "Signal approximation via data-adaptive normalized gaussian functions and its applications for speech processing," *IEEE ICASSP-92*, San Francisco, CA, March 1992, Vol. 1, pp. 141-144.

[93] J.H. Friedman and W. Stuetzle, "Projection pursuit regression," *Journal of the American Statistical Association*, Vol. 76, pp. 817-823, 1981.

[94] A. Gersho and R.M. Gray, *Vector Quantization and Signal Compression.* Kluwer, 1992.

[95] I. Daubechies, *Ten lectures on wavelets*, SIAM, 1992.

[96] M.J. Bastiaans, "On the sliding-window spectrum in digital signal processing," *IEEE Trans. Acoustics, Speech, and Signal Processing*, Vol. 33, No. 4, Aug. 1985, pp. 868-873.

[97] L. Auslander, I.C. Gertner, and R. Tolimieri, "The discrete Zak transform application to time-frequency analysis and synthesis of nonstationary signals," *IEEE Trans. Signal Processing*, Vol. 39, No. 4, Apr. 1991, pp. 825-835.

[98] A.W. Naylor and G.R. Sell, *Linear Operator Theory in Engineering and Science*, Springer, 1982.

[99] K. Gröchenig, "Acceleration of the frame algorithm," *IEEE Trans. Signal Processing*, Vol. 41, No. 12, Dec. 1993, pp. 3331-3340.

[100] R. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, 1980.

[101] C. Heil, "Frames in Hilbert spaces," *Technical report*, 1989.

[102] A. Aldroubi, "Portrait of frames," to appear in *Proc. Amer. Math. Soc.*.

[103] D.G. Luenberger, *Optimization by Vector Space Methods*, Wiley, 1969.

[104] I. Daubechies, A. Grossmann, Y. Meyer, "Painless nonorthogonal expansions," *J. Math. Phys.* 27, May 1986.

[105] M. Zibulski and Y. Zeevi, "Oversampling in the Gabor scheme," *IEEE Trans. Signal Processing*, Vol. 41, No. 8, 1993, pp. 2679-2687.

[106] G. Battle, "Heisenberg proof of the Balian-Low theorem," *J. Math. Phys.*, 15, 1988, pp. 175-177.

[107] I. Daubechies and A.J.E.M. Janssen, "Two theorems on lattice expansions," *IEEE Trans. Information Theory*, Vol. 39, No. 1, Jan. 1993, pp. 3-6.