Characterizing Degrees of Freedom through Additive Combinatorics

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***Abstract*—We establish a formal connection between the problem of characterizing degrees of freedom (DoF) in constant single-antenna interference channels (ICs) with general channel matrix and the field of additive combinatorics. The theory we develop is based on a recent breakthrough result by Hochman in fractal geometry [2]. Our first main contribution is an explicit condition on the channel matrix to admit full, i.e., $K/2$ DoF; this condition is satisfied for almost all channel matrices. We also provide a construction of corresponding full DoF-achieving input distributions. The second main result is a new DoF-formula exclusively in terms of Shannon entropy. This formula is more amenable to both analytical statements and numerical evaluations than the DoF-formula by Wu et al. [3], which is in terms of Rényi information dimension. We then use the new DoF-formula to shed light on the hardness of finding the exact number of DoF in ICs with rational channel coefficients, and to improve the best known bounds on the DoF of a well-studied channel matrix.**

I. Introduction

A breakthrough finding in network information theory established that $K/2$ degrees of freedom (DoF) can be achieved in K-user single-antenna interference channels (ICs) [4]–[6]. The corresponding transmit/receive scheme, known as interference alignment, exploits time-frequency selectivity of the channel to align interference at the receivers into low-dimensional subspaces.

Characterizing the DoF in ICs under various assumptions on the channel matrix has become a heavily researched topic. A particularly surprising result states that $K/2$ DoF can be achieved in single-antenna K-user ICs with constant channel matrix [7], [8], i.e., in channels that do not exhibit any selectivity. This result was shown to hold for (Lebesgue) almost all[1] channel matrices [7, Thm. 1]. Instead of exploiting channel selectivity, here interference alignment happens on a number-theoretic level. The technical arguments—from Diophantine approximation theory—employed in the proof of [7, Thm. 1] do not seem to allow an explicit characterization of the "almost-all set" of full-DoF admitting channel matrices. What is known, though, is that channel matrices with all entries rational admit strictly less than $K/2$ DoF [8] and hence belong to the set of exceptions relative to the "almost-all result" in [7].

Recently, Wu et al. [3] developed a general framework, based on (Rényi) information dimension, for characterizing the DoF in constant single-antenna ICs. While this general and elegant theory allows to recover, inter alia, the "almost-all result" from [7], it does not provide insights into the structure of the set of channel matrices admitting $K/2$ DoF. In addition, the DoF-formula in [3] is in terms of information dimension, which can be difficult to evaluate.

Contributions: Our first main contribution is to complement the results in [3], [7], [8] by providing *explicit and almost surely satisfied* conditions on the IC matrix to admit full, i.e., $K/2$ DoF. The conditions we find essentially require that the set of all monomials[2] in the channel coefficients be linearly independent over the rational numbers. The proof of this result is based on a recent breakthrough in fractal geometry [2], which allows us to compute the information dimension of self-similar distributions under conditions much milder than the open set condition [9] required in [3]. For channel matrices satisfying our explicit and almost sure conditions, we furthermore present an explicit construction of full DoF-achieving input distributions. The basic idea underlying this construction has roots in the field of additive combinatorics [10] and essentially ensures that the set-sum of signal and interference exhibits extremal cardinality properties. We also show that our sufficient conditions for $K/2$ DoF are not necessary. This is accomplished by constructing examples of channel matrices that admit $K/2$ DoF but do not satisfy the sufficient conditions we identify. The set of all such channel matrices, however, necessarily has Lebesgue measure 0.

Etkin and Ordentlich [8] discovered that tools from additive combinatorics can be applied to characterize DoF in ICs where the off-diagonal entries in the channel matrix are rational numbers and the diagonal entries are either irrational algebraic[3] or rational numbers. Our second main contribution is to establish a formal connection between additive combinatorics and the characterization of DoF in single-antenna ICs with *arbitrary* constant channel matrices. Specifically, we show how the DoF-characterization in terms of information dimension, discovered in [3], can be translated, again based on [2], into an alternative characterization exclusively involving Shannon entropy. The resulting new DoF-formula is more amenable to both analytical statements and numerical evaluation. To

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[1]Throughout the paper "almost all" is to be understood with respect to Lebesgue measure and "almost sure" is with respect to a probability distribution that is absolutely continuous with respect to Lebesgue measure.

[2]A monomial in the variables $x_1, ..., x_n$ is an expression of the form $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$, with $k_i \in \mathbb{N}$.

[3]A real number is called algebraic if it is the zero of a polynomial with integer coefficients. In particular, all rational numbers are algebraic.

support this statement, we show how the alternative DoF-formula can be used to explain why determining the exact number of DoF for channel matrices with rational entries, even for simple examples, has remained elusive so far. Specifically, we establish that DoF-characterization for rational channel matrices is equivalent to very hard open problems in additive combinatorics. Finally, we exemplify the quantitative applicability of the new DoF-formula by improving the best-known bounds on the DoF of a particular channel matrix studied in [3].

Notation: Random variables are represented by uppercase letters from the end of the alphabet. Lowercase letters are used exclusively for deterministic quantities. Boldface uppercase letters indicate matrices. Sets are denoted by uppercase calligraphic letters. For $x \in \mathbb{R}$, we write $\lfloor x \rfloor$ for the largest integer not exceeding x. All logarithms are taken to the base 2. $\mathbb{E}[\cdot]$ denotes the expectation operator. $H(\cdot)$ stands for entropy and $h(\cdot)$ for differential entropy. For a measurable real-valued function f and a measure[4] μ on its domain, the push-forward of μ by f is $(f_*\mu)(\mathcal{A}) = \mu(f^{-1}(\mathcal{A}))$ for Borel sets $\mathcal{A}$.

We will also need the Hausdorff dimension of sets $\mathcal{A} \subseteq \mathbb{R}$ and hence recall the corresponding definition. Let $\mathrm{diam}(\mathcal{A}) := \sup_{x,y\in\mathcal{A}} |x-y|$ and define, for $s \geqslant 0$, the s-dimensional Hausdorff measure by $\mu_{\mathrm{H}}^s(\mathcal{A}) = \lim_{\delta\to 0} h_\delta^s(\mathcal{A})$, where

$$h_\delta^s(\mathcal{A}) := \inf\Bigg\{ \sum_{i=1}^{\infty} \mathrm{diam}(\mathcal{U}_i)^s \ : \ \mathcal{A} \subseteq \bigcup_{i=1}^{\infty} \mathcal{U}_i \text{ and } \mathrm{diam}(\mathcal{U}_i) \leqslant \delta, \text{ for all } i \Bigg\}.$$

The Hausdorff dimension of $\mathcal{A}$ is now defined as $\dim_{\mathrm{H}}(\mathcal{A}) := \inf\{s \geqslant 0 \ : \ \mu_{\mathrm{H}}^s(\mathcal{A}) = 0\}$.

Outline of the paper: In Section II, we introduce the system model for constant single-antenna ICs. Section III contains our first main result, Theorem 1, providing explicit and almost surely satisfied conditions on channel matrices to admit full, i.e., $K/2$ DoF. In Section IV, we review the basic material on information dimension, self-similar distributions, and additive combinatorics needed in the paper. Section V is devoted to sketching the ideas underlying the proof of Theorem 1 in an informal fashion and to introducing the recent result by Hochman [2] that both our main results rely on. In Section VI, we formally prove Theorem 1. In Section VII, we establish that our sufficient conditions for $K/2$ DoF are not necessary. Our second main result, Theorem 3, which provides a DoF-characterization exclusively in terms of Shannon entropy, is presented, along with its proof, in Section VIII. Finally, in Section IX we discuss the formal connection between DoF and sumset theory, a branch of additive combinatorics, and we apply the new DoF-formula to channel matrices with rational entries.

II. System model

We consider a single-antenna K-user IC with constant channel matrix $\mathbf{H} = (h_{ij})_{1\leqslant i,j\leqslant K} \in \mathbb{R}^{K\times K}$ and input-output relation

$$Y_i = \sqrt{\mathsf{snr}} \sum_{j=1}^{K} h_{ij} X_j + Z_i, \quad i = 1, ..., K, \tag{1}$$

where $X_i \in \mathbb{R}$ is the input at the i-th transmitter, $Y_i \in \mathbb{R}$ is the output at the i-th receiver, and $Z_i \in \mathbb{R}$ is noise of absolutely continuous distribution such that $h(Z_i) > -\infty$ and $H(\lfloor Z_i \rfloor) < \infty$. The input signals are independent across transmitters and noise is i.i.d. across users and channel uses.

The channel matrix $\mathbf{H}$ is assumed to be known perfectly at all transmitters and receivers. We impose the average power constraint

$$\frac{1}{n}\sum_{k=1}^{n}\left(x_i^{(k)}\right)^2 \leqslant 1$$

on codewords $\left(x_i^{(1)} \ ... \ x_i^{(n)}\right)$ of block-length n transmitted by user $i = 1, ..., K$. The DoF of this channel are defined as

$$\mathsf{DoF}(\mathbf{H}) := \limsup_{\mathsf{snr}\to\infty} \frac{\overline{C}(\mathbf{H};\mathsf{snr})}{\frac{1}{2}\log\mathsf{snr}}, \tag{2}$$

where $\overline{C}(\mathbf{H};\mathsf{snr})$ is the sum-capacity of the IC.

III. Explicit and almost sure conditions for $K/2$ DoF

We denote the vector consisting of the off-diagonal entries of $\mathbf{H}$ by $\check{\mathbf{h}} \in \mathbb{R}^{K(K-1)}$, and let $f_1, f_2, ...$ be the monomials in $K(K-1)$ variables, i.e., $f_i(x_1, ..., x_{K(K-1)}) = x_1^{d_1} \cdots x_{K(K-1)}^{d_{K(K-1)}}$, enumerated as follows: $f_1, ..., f_{\varphi(d)}$ are the monomials of degree[5] not larger than d, where

$$\varphi(d) := \binom{K(K-1)+d}{d}.$$

The following theorem contains the first main result of the paper, namely conditions on $\mathbf{H}$ to admit $K/2$ DoF that are explicit and satisfied for almost all $\mathbf{H}$.

Theorem 1: Suppose that the channel matrix $\mathbf{H}$ satisfies the following conditions:

For each $i = 1, ..., K$, the set

$$\{f_j(\check{\mathbf{h}}) \ : \ j \geqslant 1\} \cup \{h_{ii} f_j(\check{\mathbf{h}}) \ : \ j \geqslant 1\} \tag{$*$}$$

is linearly independent over $\mathbb{Q}$.

Then, we have

$$\mathsf{DoF}(\mathbf{H}) = K/2.$$

Proof: See Section VI. ∎

We first note that, as detailed in the proof of Theorem 1, Condition $(*)$ implies that all entries of $\mathbf{H}$ must be nonzero, i.e., $\mathbf{H}$ must be fully connected in the terminology of [8]. By [11, Prop. 1] we have $\mathsf{DoF}(\mathbf{H}) \leqslant K/2$ for fully connected channel matrices. The proof of Theorem 1 is constructive in the sense of identifying input distributions that achieve this upper bound.

[4]Throughout the paper, the terms "measurable" and "measure" are to be understood with respect to the Borel σ-algebra.

[5]The "degree" of a monomial is defined as the sum of all exponents of the variables involved (sometimes called the total degree).

Let us next dissect Condition $(*)$. A set $\mathcal{S} \subseteq \mathbb{R}$ is linearly independent over $\mathbb{Q}$ if, for all $n \in \mathbb{N}$ and all pairwise distinct $v_1, ..., v_n \in \mathcal{S}$, the only solution $q_1, ..., q_n \in \mathbb{Q}$ of the equation

$$q_1 v_1 + \ldots + q_n v_n = 0 \tag{3}$$

is $q_1 = \ldots = q_n = 0$. Thus, if Condition $(*)$ is not satisfied, there exists, for at least one $i \in \{1, ..., K\}$, a nontrivial linear combination of a finite number of elements of the set

$$\{f_j(\check{\mathbf{h}}) \,:\, j \geqslant 1\} \cup \{h_{ii} f_j(\check{\mathbf{h}}) \,:\, j \geqslant 1\}$$

with rational coefficients which equals 0. In fact, this is equivalent to the existence of a nontrivial linear combination that equals 0 and has all coefficients in $\mathbb{Z}$. This can be seen by simply multiplying (3) by a common denominator of $q_1, ..., q_n$.

To show that Condition $(*)$ is satisfied for almost all channel matrices, we will argue that the condition is violated on a set of Lebesgue measure 0 with respect to $\mathbf{H}$. To this end, we first note that for fixed $d \in \mathbb{N}$, fixed $a_1, ..., a_{\varphi(d)}, b_1, ..., b_{\varphi(d)} \in \mathbb{Z}$ not all equal to 0, and fixed $i \in \{1, ..., K\}$,

$$\sum_{j=1}^{\varphi(d)} a_j f_j(\check{\mathbf{h}}) + \sum_{j=1}^{\varphi(d)} b_j h_{ii} f_j(\check{\mathbf{h}}) = 0 \tag{4}$$

is satisfied only on a set of measure 0 with respect to $\mathbf{H}$, as the solutions of (4) are given by the set of zeros of a polynomial in the channel coefficients. Since the set of equations (4) is countable with respect to $d \in \mathbb{N}$, $a_1, ..., a_{\varphi(d)}, b_1, ..., b_{\varphi(d)} \in \mathbb{Z}$, and $i \in \{1, ..., K\}$, the set of channel matrices violating Condition $(*)$ is given by a countable union of sets of measure 0, which again has measure 0. It therefore follows that Condition $(*)$ is satisfied for almost all channel matrices $\mathbf{H}$ and hence Theorem 1 provides conditions on $\mathbf{H}$ that not only guarantee that $K/2$ DoF can be achieved but are also explicit and almost surely satisfied. The operational significance—in terms of achieving full DoF—of linear independence over $\mathbb{Q}$ in Condition $(*)$ will be explained in Section V.

Verifying Condition $(*)$ for a specific channel matrix $\mathbf{H}$ in theory requires checking infinitely many equations of the form (4). It turns out, however, that verifying (4) for finitely many a_j, b_j and up to a finite degree d, already comes with guarantees on the number of DoF achievable. This will be discussed in detail in Remark 5 in Section VI.

We finally note that the prominent example class from [8] with all entries of $\mathbf{H}$ nonzero and rational, shown in [8, Thm. 2] to admit strictly less than $K/2$ DoF, does not satisfy Condition $(*)$, as two rational numbers are always linearly dependent over $\mathbb{Q}$.

IV. Preparatory Material

This section briefly reviews basic material on information dimension, self-similar distributions, iterated function systems, and additive combinatorics needed in the rest of the paper. The reader intimately familiar with these concepts may want to proceed directly to Section V.

A. Information dimension and DoF

Definition 1: Let X be a random variable of arbitrary distribution[6] μ. We define the lower and upper information dimension of X as

$$\underline{d}(X) := \liminf_{k \to \infty} \frac{H(\langle X \rangle_k)}{\log k} \quad \text{and} \quad \overline{d}(X) := \limsup_{k \to \infty} \frac{H(\langle X \rangle_k)}{\log k},$$

where $\langle X \rangle_k := \lfloor kX \rfloor / k$. If $\underline{d}(X) = \overline{d}(X)$, we set $d(X) := \underline{d}(X) = \overline{d}(X)$ and call $d(X)$ the information dimension of X. Since $\underline{d}(X), \overline{d}(X)$, and $d(X)$ depend on μ only, we sometimes also write $\underline{d}(\mu), \overline{d}(\mu)$, and $d(\mu)$, respectively.

The relevance of information dimension in characterizing DoF stems from the following relation [12], [3], [13]

$$\limsup_{\mathsf{snr} \to \infty} \frac{h(\sqrt{\mathsf{snr}} X + Z)}{\frac{1}{2} \log \mathsf{snr}} = \overline{d}(X), \tag{5}$$

which holds for arbitrary independent random variables X and Z (both not depending on snr), with the distribution of Z absolutely continuous and such that $h(Z) > -\infty$ and $H(\lfloor Z \rfloor) < \infty$.

We can apply (5) to ICs as follows. By standard random coding arguments it follows that the sum-rate

$$I(X_1; Y_1) + \ldots + I(X_K; Y_K) \tag{6}$$

is achievable, where $X_1, ..., X_K$ are independent input distributions with $\mathbb{E}[X_i^2] \leqslant 1$, $i = 1, ..., K$. Next note that

$$I(X_i; Y_i) = h(Y_i) - h(Y_i \,|\, X_i) \tag{7}$$

$$= h\left(\sqrt{\mathsf{snr}} \sum_{j=1}^{K} h_{ij} X_j + Z_i\right) - h\left(\sqrt{\mathsf{snr}} \sum_{j \neq i}^{K} h_{ij} X_j + Z_i\right), \tag{8}$$

for $i = 1, ..., K$. Combining (5)-(8), it now follows that [3]

$$\mathsf{dof}(X_1, ..., X_K; \mathbf{H}) := \sum_{i=1}^{K} \left[d\left(\sum_{j=1}^{K} h_{ij} X_j\right) - d\left(\sum_{j \neq i}^{K} h_{ij} X_j\right)\right] \tag{9}$$

$$\leqslant \mathsf{DoF}(\mathbf{H}), \tag{10}$$

for all independent $X_1, ..., X_K$ with[7] i) $\mathbb{E}[X_i^2] < \infty$, $i = 1, ..., K$, and ii) such that all information dimension terms appearing in (9) exist. A striking result in [3] shows that inputs of discrete, continuous, or mixed discrete-continuous distribution can achieve no more than 1 DoF irrespective of K. For $K > 2$, input distributions achieving $K/2$ (i.e., full) DoF therefore necessarily exhibit a singular component.

[6] We consider general distributions which may be discrete, continuous, singular, or mixtures thereof.

[7] We only need the conditions $\mathbb{E}[X_i^2] < \infty$ as scaling of the inputs does not affect $\mathsf{dof}(X_1, ..., X_K; \mathbf{H})$.

Taking the supremum in (10) over all admissible $X_1, \dots, X_K$ yields[8]

$$\mathsf{DoF}(\mathbf{H}) \geqslant \sup_{X_1,\dots,X_K} \sum_{i=1}^{K} \left[d\left(\sum_{j=1}^{K} h_{ij} X_j \right) - d\left(\sum_{j \neq i}^{K} h_{ij} X_j \right) \right]. \quad (11)$$

It was furthermore discovered in [3] that equality in (11) holds for almost all channel matrices $\mathbf{H}$; an explicit characterization of this "almost-all set", however, does not seem to be available. The right-hand side (RHS) of (11) can be difficult to evaluate as explicit expressions for information dimension seem to be available only for mixed discrete-continuous distributions and (singular) self-similar distributions reviewed in the next section.

B. Self-similar distributions and iterated function systems

A class of singular distributions with explicit expressions for their information dimension is that of self-similar distributions [14]. What is more, self-similar distributions can be constructed to retain self-similarity under linear combinations. We can therefore devise self-similar inputs $X_1, \dots, X_K$ such that the information dimension of the corresponding outputs in (9) can be expressed analytically. For an excellent in-depth treatment of the material reviewed in this section, the interested reader is referred to [15].

We proceed to the definition of self-similar distributions. Consider a finite set $\Phi_r := \{\varphi_{i,r} : i = 1, \dots, n\}$ of affine contractions $\varphi_{i,r} \colon \mathbb{R} \to \mathbb{R}$, i.e.,

$$\varphi_{i,r}(x) = rx + w_i, \quad (12)$$

where $r \in I \subseteq (0,1)$ and the w_i are pairwise distinct real numbers. We furthermore set $\mathcal{W} := \{w_1, \dots, w_n\}$. Φ_r is called an iterated function system (IFS) parametrized by the contraction parameter $r \in I$. By classical fractal geometry [15, Ch. 9] every IFS has an associated unique attractor, i.e., a nonempty compact set $\mathcal{A} \subseteq \mathbb{R}$ such that

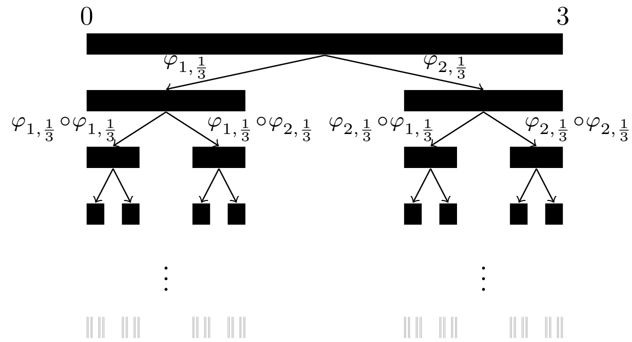
$$\mathcal{A} = \bigcup_{i=1}^{n} \varphi_{i,r}(\mathcal{A}). \quad (13)$$

Moreover, for each probability vector $(p_1, \dots, p_n)$, there is a unique (Borel) probability distribution μ_r on $\mathbb{R}$ such that

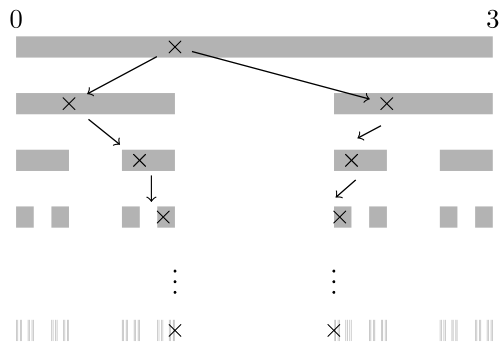
$$\mu_r = \sum_{i=1}^{n} p_i (\varphi_{i,r})_* \mu_r, \quad (14)$$

where $(\varphi_{i,r})_* \mu_r$ is the push-forward of μ_r by $\varphi_{i,r}$ [14]. The distribution μ_r is supported on the attractor set $\mathcal{A}$ in (13) for each probability vector $(p_1, \dots, p_n)$ [14] and is referred to as the self-similar distribution corresponding to the IFS Φ_r with underlying probability vector $(p_1, \dots, p_n)$. We can give the

[8]Note that, throughout the paper, we first take $\mathsf{snr} \to \infty$ and then optimize over all possible input distributions. In particular, the inputs $X_1, \dots, X_K$ do not depend on snr.



(a) The middle-third Cantor set at the bottom is the attractor set generated by the IFS $\{\varphi_{1,\frac{1}{3}}(x) = \frac{1}{3}x + 0, \varphi_{2,\frac{1}{3}}(x) = \frac{1}{3}x + 2\}$.



(b) Illustration of the construction of the self-similar inputs $X = \sum_{k=0}^{\infty} r^k W_k$ for $r = \frac{1}{3}$ and the W_k chosen uniformly at random (i.e., $p_1 = p_2 = \frac{1}{2}$) from the set $\{0, 2\}$. $W_k = 0$ results in branching out to the left and $W_k = 2$ to the right. The resulting realizations of X are depicted in the bottom line of the figure.

Fig. 1: Example of the attractor (a) and the corresponding random variable (b) associated with the self-similar distribution generated by an IFS.

following explicit expression for a random variable X with distribution μ_r as in (14)

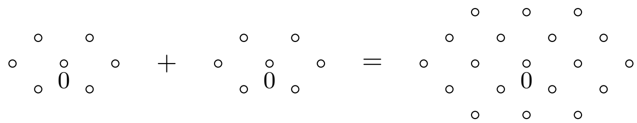
$$X = \sum_{k=0}^{\infty} r^k W_k, \quad (15)$$

where $\{W_k\}_{k \geqslant 0}$ is a set of i.i.d. copies of a random variable W drawn from the set $\mathcal{W}$ according to $(p_1, \dots, p_n)$. Figure 1 illustrates the classical example of the middle-third Cantor set, which constitutes the attractor generated by the IFS $\{\varphi_{1,\frac{1}{3}}(x) = \frac{1}{3}x + 0, \varphi_{2,\frac{1}{3}}(x) = \frac{1}{3}x + 2\}$.

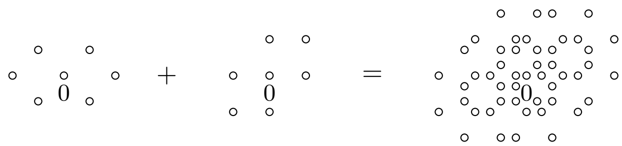
Remark 1: It is not immediately clear how self-similar inputs $X_1, \dots, X_K$ as in (15) would lead to practical signal constellations, which obviously have to be discrete. One natural approach would be to truncate the series in (15) at an snr-dependent index so that the number of constellation points grows with snr. Evaluating the performance of the resulting constellations, albeit interesting, is beyond the scope of this paper.

C. A glimpse of additive combinatorics

The common theme of our two main results is a formal relationship between the study of DoF in constant single-antenna



(a) Sum of two sets with common algebraic structure.



(b) Sum of two sets with different algebraic structures.

Fig. 2: The cardinality of the sum in (a) is 19 and hence small compared to the $7^2 = 49$ pairs summed up, whereas the sum in (b) has cardinality 49.

ICs and the field of additive combinatorics. This connection is enabled by a recent breakthrough result in fractal geometry reported in [2] and summarized in Section V. We next briefly present basic material from additive combinatorics that is relevant for our discussion. Specifically, we will be concerned with sumset theory, which studies, for discrete sets $\mathcal{U}$, $\mathcal{V}$, the cardinality of the sumset $\mathcal{U}+\mathcal{V}=\{u+v\,:\,u\in\mathcal{U}, v\in\mathcal{V}\}$ relative to $|\mathcal{U}|$ and $|\mathcal{V}|$. We begin by noting the trivial bounds

$$\max\{|\mathcal{U}|,|\mathcal{V}|\}\leqslant|\mathcal{U}+\mathcal{V}|\leqslant|\mathcal{U}|\cdot|\mathcal{V}|, \tag{16}$$

for $\mathcal{U}$ and $\mathcal{V}$ finite and nonempty. One of the central ideas in sumset theory says that the left-hand inequality in (16) can be close to equality only if $\mathcal{U}$ and $\mathcal{V}$ have a common algebraic structure (e.g., lattice structures), whereas the right-hand inequality in (16) will be close to equality only if the pairs $\mathcal{U}$ and $\mathcal{V}$ do not have a common algebraic structure, i.e., they are generic relative to each other. Figure 2 illustrates this statement. Algebraic structures relevant in this context are arithmetic progressions, which are sets of the form $\mathcal{S}=\{a,a+d,a+2d,\dots,a+(n-1)d\}$ with $a\in\mathbb{Z}$ and $d\in\mathbb{N}$. If $\mathcal{U}$ and $\mathcal{V}$ are finite nonempty subsets of $\mathbb{Z}$, an improvement of the lower bound in (16) to $|\mathcal{U}|+|\mathcal{V}|-1\leqslant|\mathcal{U}+\mathcal{V}|$ can be obtained. This lower bound is attained if and only if $\mathcal{U}$ and $\mathcal{V}$ are arithmetic progressions of the same step size d [10, Prop. 5.8].

An interesting connection between sumset theory and entropy inequalities was discovered in [16], [17]. This connection revolves around the fact that many sumset inequalities have analogons in terms of entropy inequalities. For example, the entropy version of the trivial bounds (16) is

$$\max\{H(U),H(V)\}\leqslant H(U+V)\leqslant H(U)+H(V),$$

where U and V are independent discrete random variables. Less trivial examples are the sumset inequalities [10], [18]

$$|\mathcal{U}-\mathcal{V}|\cdot|\mathcal{U}|\cdot|\mathcal{V}|\leqslant|\mathcal{U}+\mathcal{V}|^3$$
$$|\mathcal{U}-\mathcal{V}|\leqslant|\mathcal{U}+\mathcal{V}|^{1/2}\cdot(|\mathcal{U}|\cdot|\mathcal{V}|)^{2/3},$$

for finite nonempty sets $\mathcal{U},\mathcal{V}$, with their entropy counterparts [16], [17]

$$H(U-V)+H(U)+H(V)\leqslant 3H(U+V) \tag{17}$$

$$H(U-V)\leqslant\frac{1}{2}H(U+V)+\frac{2}{3}(H(U)+H(V)) \tag{18}$$

for independent discrete random variables U,V. Note that due to the logarithmic scale of entropy, products in sumset inequalities are replaced by sums in their entropy versions.

V. The cornerstones of the proof of Theorem 1

In this section, we discuss the main ideas and conceptual elements underlying the proof of Theorem 1. First, we note that, as already pointed out in Section III, by [11, Prop. 1] we have $\mathsf{DoF}(\mathbf{H})\leqslant K/2$ for all $\mathbf{H}$ satisfying Condition $(*)$. To achieve this upper bound, we construct self-similar input distributions that yield $\mathsf{dof}(X_1,...,X_K;\mathbf{H})=K/2$ for channel matrices satisfying Condition $(*)$. Specifically, we take each input to have a self-similar distribution with contraction parameter r, i.e., $X_i=\sum_{k=0}^{\infty}r^kW_{i,k}$, where, for $i=1,...,K$, $\{W_{i,k}\,:\,k\geqslant 0\}$ are i.i.d. copies of a discrete random variable[9] W_i with value set $\mathcal{W}_i$, possibly different across i. For the random variables $\sum_j h_{ij}X_j$ appearing in (11) we then have[10]

$$\sum_j h_{ij}X_j=\sum_j\sum_{k=0}^{\infty}r^kh_{ij}W_{j,k}=\sum_{k=0}^{\infty}r^k\sum_j h_{ij}W_{j,k}, \tag{19}$$

and thus $\sum_j h_{ij}X_j$ is again self-similar with contraction parameter r. The "output-$\mathcal{W}$" set, i.e., the value set of $\sum_j h_{ij}W_j$ is then given by $\sum_j h_{ij}\mathcal{W}_j$.

Next, we discuss conditions on X_j and h_{ij} under which analytical expressions for the information dimension of $\sum_j h_{ij}X_j$ can be given. For general self-similar distributions arising from an IFS classical results in fractal geometry impose the so-called open set condition [19, Thm. 2], which requires the existence of a nonempty bounded set $\mathcal{U}\subseteq\mathbb{R}$ such that

$$\bigcup_{i=1}^{n}\varphi_{i,r}(\mathcal{U})\subseteq\mathcal{U} \tag{20}$$

$$\text{and}\qquad \varphi_{i,r}(\mathcal{U})\cap\varphi_{j,r}(\mathcal{U})=\emptyset,\quad\text{for all } i\neq j, \tag{21}$$

for the $\varphi_{i,r}$ defined in (12). Wu et al. [3] ensure that the open set condition is satisfied by imposing an upper bound on the contraction parameter r according to

$$r\leqslant\frac{\mathsf{m}(\mathcal{W})}{\mathsf{m}(\mathcal{W})+\mathsf{M}(\mathcal{W})}, \tag{22}$$

where $\mathsf{m}(\mathcal{W}):=\min_{i\neq j}|w_i-w_j|$ and $\mathsf{M}(\mathcal{W}):=\max_{i,j}|w_i-w_j|$ (see also the example in Figure 1, where this condition is satisfied). The challenge here resides in making (22) hold for the output-$\mathcal{W}$ set. In [3] this is accomplished by building the input sets $\mathcal{W}_i$ from $\mathbb{Z}$-linear combinations (i.e., linear

[9]Henceforth "discrete random variable" refers to a random variable that takes finitely many values.

[10]The change of order in the summations in the second step in (19) is justified by $\sum_j$ being a finite sum and all series $\sum_{k=0}^{\infty}r^kW_{j,k}$ converging for all realizations of the $W_{j,k}$. The latter follows since the realizations of the $W_{j,k}$ are bounded and the geometric series $\sum_{k=0}^{\infty}r^k$ converges for $r\in(0,1)$.

combinations with integer coefficients) of monomials in the off-diagonal channel coefficients and then recognizing that results in Diophantine approximation theory can be used to show that (22) is satisfied for almost all channel matrices. Unfortunately, it does not seem to be possible to obtain an explicit characterization of this "almost-all set". Recent groundbreaking work by Hochman [2] replaces the open set condition by a much weaker condition, which instead of (21) only requires that the images $\varphi_{i,r}(\mathcal{A})$ and $\varphi_{j,r}(\mathcal{A})$, for $i \neq j$, not "overlap exactly". This, in turn, as shown in Theorem 2 below can be accomplished by "wiggling" with r in an arbitrarily small neighborhood of its original value. The resulting improvement turns out to be instrumental in our Theorem 1 as it allows us to abandon the Diophantine approximation approach and thereby opens the doors to an explicit characterization of an "almost-all set" of full-DoF admitting channel matrices. Specifically, we use the following simple consequence of [2, Thm. 1.8].

Theorem 2: If $I \subseteq (0,1)$ is a proper compact interval[11] and μ_r is the self-similar distribution from (14) with contraction parameter $r \in I$ and probability vector $(p_1, ..., p_n)$, then[12]

$$d(\mu_r) = \min\left\{\frac{\sum p_i \log p_i}{\log r}, 1\right\}, \tag{23}$$

for all $r \in I \backslash E$, where E is a set of Hausdorff dimension 0.

Proof: For $\mathbf{i} \in \{1, ..., n\}^k$, let $\varphi_{\mathbf{i},r} := \varphi_{i_1,r} \circ \ldots \circ \varphi_{i_k,r}$ and define

$$\Delta_{\mathbf{i},\mathbf{j}}(r) := \varphi_{\mathbf{i},r}(0) - \varphi_{\mathbf{j},r}(0),$$

for $\mathbf{i}, \mathbf{j} \in \{1, ..., n\}^k$. Extend this definition to infinite sequences $\mathbf{i}, \mathbf{j} \in \{1, ..., n\}^{\mathbb{N}}$ according to

$$\Delta_{\mathbf{i},\mathbf{j}}(r) := \lim_{k \to \infty} \Delta_{(i_1,...,i_k),(j_1,...,j_k)}(r).$$

Using (12) it follows that

$$\Delta_{\mathbf{i},\mathbf{j}}(r) = \sum_{k=1}^{\infty} r^{k-1}(w_{i_k} - w_{j_k}).$$

Since a power series can vanish on a nonempty open set only if it is identically 0, we get that $\Delta_{\mathbf{i},\mathbf{j}} \equiv 0$ on I if and only if $\mathbf{i} = \mathbf{j}$, as a consequence of the w_i being pairwise distinct and I containing a nonempty open set. This is precisely the condition of [2, Thm. 1.8] which asserts that (23) holds for all $r \in I$ with the exception of a set of Hausdorff dimension 0, and thus completes the proof. ∎

Remark 2: Note that (23) can be rewritten in terms of the entropy of the random variable W, defined in (15), which takes value w_i with probability p_i:

$$d(\mu_r) = \min\left\{\frac{H(W)}{\log(1/r)}, 1\right\}. \tag{24}$$

Remark 3: The concept of Hausdorff dimension is rooted in fractal geometry [15]. In the proofs of our main results, we will only need the following aspect: For I as in Theorem 2, we can always find an $\widetilde{r} \in I \backslash E$ for which (23) holds. This can be seen as follows: $I \backslash E = \emptyset$ implies that E contains a nonempty open set and therefore would have Hausdorff dimension 1 [15, Sec. 2.2].

Remark 4: The strength of Theorem 2 stems from (23) holding without any restrictions on the $w_i \in \mathcal{W}$. In particular, the elements in the output-$\mathcal{W}$ set $\sum_j h_{ij}\mathcal{W}_j$ may be arbitrarily close to each other rendering (22), needed to satisfy the open set condition, obsolete. This feature will allow us to prove that we can achieve full DoF for the channel matrices satisfying the conditions of Theorem 1.

We next show how Theorem 2 allows us to derive explicit expressions for the information dimension terms in (9).

Proposition 1: Let $r \in (0,1)$ and let $W_1, ..., W_K$ be independent discrete random variables. Then, we have

$$\sum_{i=1}^{K}\left[\min\left\{\frac{H\left(\sum_{j=1}^{K} h_{ij}W_j\right)}{\log(1/r)}, 1\right\} - \min\left\{\frac{H\left(\sum_{j\neq i}^{K} h_{ij}W_j\right)}{\log(1/r)}, 1\right\}\right] \leqslant \mathsf{DoF}(\mathbf{H}). \tag{25}$$

Proof: For $i = 1, ..., K$, let $\{W_{i,k} : k \geqslant 0\}$ be i.i.d. copies of W_i. We consider the self-similar inputs $X_i = \sum_{k=0}^{\infty} r^k W_{i,k}$, for $i = 1, ..., K$. Then, the signals

$$\sum_{j=1}^{K} h_{ij}X_j = \sum_{k=0}^{\infty} r^k \sum_{j=1}^{K} h_{ij}W_{j,k}$$

and

$$\sum_{j\neq i}^{K} h_{ij}X_j = \sum_{k=0}^{\infty} r^k \sum_{j\neq i}^{K} h_{ij}W_{j,k}$$

also have self-similar distributions with contraction parameter r. Thus, by Theorem 2, for each $\varepsilon > 0$, there exists an $\widetilde{r}$ in the proper compact interval $I_\varepsilon := [r - \varepsilon, r]$ such that

$$d\left(\sum_{j=1}^{K} h_{ij}X_j\right) = \min\left\{\frac{H\left(\sum_{j=1}^{K} h_{ij}W_j\right)}{\log(1/\widetilde{r})}, 1\right\} \tag{26}$$

and

$$d\left(\sum_{j\neq i}^{K} h_{ij}X_j\right) = \min\left\{\frac{H\left(\sum_{j\neq i}^{K} h_{ij}W_j\right)}{\log(1/\widetilde{r})}, 1\right\}. \tag{27}$$

For $\varepsilon \to 0$ we have $\log(1/\widetilde{r}) \to \log(1/r)$ by continuity of $\log(\cdot)$. Thus, inserting (26) and (27) into (10) and letting $\varepsilon \to 0$, we get (25) as desired. ∎

The freedom we exploit in constructing full DoF-achieving X_i lies in the choice of $W_1, ..., W_K$, which thanks to Theorem 2, unlike in [3], is not restricted by distance constraints on the output-$\mathcal{W}$ set. For simplicity of exposition we henceforth choose the same value set $\mathcal{W}$ for each W_i. We want to ensure that the first term inside the sum (9) equals 1 and the second term equals $1/2$, for all i, resulting in a total of $K/2$ DoF. It follows from (26), (27) that this can be accomplished by choosing the W_i such that

$$H\left(h_{ii}W_i + \sum_{j\neq i}^{K} h_{ij}W_j\right) \approx 2H\left(\sum_{j\neq i}^{K} h_{ij}W_j\right) \tag{28}$$

[11] By "proper compact interval" we mean a nonempty interval, which does not consist of a single point only.

[12] The "1" in the minimum simply accounts for the fact that information dimension cannot exceed the dimension of the ambient space.

followed by a suitable choice of the contraction parameter r. Resorting to the analogy of entropy and sumset cardinalities sketched in Section IV-C, the doubling condition (28) becomes

$$\left|h_{ii}\mathcal{W}+\sum_{j\neq i}^{K}h_{ij}\mathcal{W}\right|\approx\left|\sum_{j\neq i}^{K}h_{ij}\mathcal{W}\right|^2, \tag{29}$$

which effectively says that the sum of the desired signal and the interference should be twice as "rich" as the interference alone. Note that by the trivial lower bound in (16)

$$|h_{ii}\mathcal{W}|=|\mathcal{W}|\leqslant\left|\sum_{j\neq i}^{K}h_{ij}\mathcal{W}\right|, \tag{30}$$

and, by the trivial upper bound in (16)

$$\left|h_{ii}\mathcal{W}+\sum_{j\neq i}^{K}h_{ij}\mathcal{W}\right|\leqslant|h_{ii}\mathcal{W}|\cdot\left|\sum_{j\neq i}^{K}h_{ij}\mathcal{W}\right|. \tag{31}$$

The doubling condition (29) can therefore be realized by constructing $\mathcal{W}$ such that the inequalities (30) and (31) are close to equality. In particular, this means that (cf. Section IV-C)

A) the terms in the sum $\sum_{j\neq i}^{K}h_{ij}\mathcal{W}$ must have a common algebraic structure and
B) $h_{ii}\mathcal{W}$ and $\sum_{j\neq i}^{K}h_{ij}\mathcal{W}$ must *not* have a common algebraic structure.

The challenge here is to introduce algebraic structure into $\mathcal{W}$ so that A) is satisfied, but at the same time to keep the algebraic structures of the sets $h_{ii}\mathcal{W}$ and $\sum_{j\neq i}^{K}h_{ij}\mathcal{W}$ different enough so that B) is met. Before describing a construction that accomplishes this, we note that the answer to the question of whether the sets $h_{ij}\mathcal{W}$ have a common algebraic structure or not depends on the channel coefficients h_{ij}. As we want our construction to be universal in the sense of (29) holding independently of the channel coefficients, a channel-independent choice of $\mathcal{W}$ is out of question. Inspired by [7], we therefore build $\mathcal{W}$ as a set of $\mathbb{Z}$-linear combinations of monomials (up to a certain degree $d\in\mathbb{N}$) in the off-diagonal channel coefficients, i.e., the elements of $\mathcal{W}$ are given by $\sum_{j=1}^{\varphi(d)}a_jf_j(\check{\mathbf{h}})$, for $a_j\in\{1,...,N\}$ with $N\in\mathbb{N}$. This construction satisfies A) by inducing the same algebraic structure for $h_{ij}\mathcal{W}$, $j\neq i$, independently of the actual values of the channel coefficients h_{ij}, $j\neq i$. To see this, first note that multiplying the elements $\sum_{j=1}^{\varphi(d)}a_jf_j(\check{\mathbf{h}})$ of $\mathcal{W}$ by an off-diagonal channel coefficient h_{ij}, $j\neq i$, simply increases the degrees of the participating $f_j(\check{\mathbf{h}})$ by 1. For d sufficiently large the number of elements that do not appear both in $h_{ij}\mathcal{W}$ and $\mathcal{W}$ is therefore small, rendering $h_{ij}\mathcal{W}$, $j\neq i$, algebraically "similar" to $\mathcal{W}$, which we denote as $h_{ij}\mathcal{W}\approx\mathcal{W}$. We therefore get $\sum_{j\neq i}h_{ij}\mathcal{W}\approx\mathcal{W}+\ldots+\mathcal{W}$ as the sum of $K-1$ sets with shared algebraic structure and note that the elements of $\mathcal{W}+\ldots+\mathcal{W}$ are given by $\sum_{j=1}^{\varphi(d)}a_jf_j(\check{\mathbf{h}})$ with $a_j\in\{1,...,(K-1)N\}$. Choosing N large relative to K, we finally get $|\sum_{j\neq i}h_{ij}\mathcal{W}|\approx|\mathcal{W}|$. As for Condition B), we begin by noting that h_{ii} does not participate in the monomials $f_j(\check{\mathbf{h}})$ used to construct the elements in $\mathcal{W}$. This means that $\sum_{j\neq i}^{K}h_{ij}\mathcal{W}$ consists of $\mathbb{Z}$-linear combinations of $f_j(\check{\mathbf{h}})$, while $h_{ii}\mathcal{W}$ consists of $\mathbb{Z}$-linear combinations of $h_{ii}f_j(\check{\mathbf{h}})$. By Condition $(*)$ the union of the sets $\{f_j(\check{\mathbf{h}}):j\geqslant1\}$ and $\{h_{ii}f_j(\check{\mathbf{h}}):j\geqslant1\}$ is linearly independent over $\mathbb{Q}$, which ensures that $h_{ii}\mathcal{W}$ and $\sum_{j\neq i}^{K}h_{ij}\mathcal{W}$ do not share an algebraic structure.

VI. Proof of Theorem 1

Since a set containing 0 is always linearly dependent over $\mathbb{Q}$, Condition $(*)$ implies that all entries of $\mathbf{H}$ must be nonzero, i.e., $\mathbf{H}$ must be fully connected. It therefore follows from [11, Prop. 1] that $\mathsf{DoF}(\mathbf{H})\leqslant K/2$.

The remainder of the proof establishes the lower bound $\mathsf{DoF}(\mathbf{H})\geqslant K/2$ under Condition $(*)$. Let N and d be positive integers. We begin by setting

$$\mathcal{W}_N:=\left\{\sum_{i=1}^{\varphi(d)}a_if_i(\check{\mathbf{h}})\,:\,a_1,...,a_{\varphi(d)}\in\{1,...,N\}\right\} \tag{32}$$

and $r:=|\mathcal{W}_N|^{-2}$. Let $W_1,...,W_K$ be i.i.d. uniform random variables on $\mathcal{W}_N$. By Proposition 1 we then have

$$\sum_{i=1}^{K}\left[\min\left\{\frac{H\left(\sum_{j=1}^{K}h_{ij}W_j\right)}{2\log|\mathcal{W}_N|},1\right\}-\min\left\{\frac{H\left(\sum_{j\neq i}^{K}h_{ij}W_j\right)}{2\log|\mathcal{W}_N|},1\right\}\right]\leqslant\mathsf{DoF}(\mathbf{H}). \tag{33}$$

Note that the random variable $\sum_{j\neq i}h_{ij}W_j$ takes value in

$$\left\{\sum_{i=1}^{\varphi(d+1)}a_if_i(\check{\mathbf{h}})\,:\,a_1,...,a_{\varphi(d+1)}\in\{1,...,(K-1)N\}\right\}. \tag{34}$$

By Condition $(*)$ the set $\{f_j(\check{\mathbf{h}}):j\geqslant1\}$ is linearly independent over $\mathbb{Q}$. Therefore, each element in the set (34) has exactly one representation as a $\mathbb{Z}$-linear combination with coefficients $a_1,...,a_{\varphi(d+1)}\in\{1,...,(K-1)N\}$. This allows us to conclude that the cardinality of the set (34) is given by $((K-1)N)^{\varphi(d+1)}$, which implies $H\left(\sum_{j\neq i}h_{ij}W_j\right)\leqslant\varphi(d+1)\log((K-1)N)$. Similarly, we find that $|\mathcal{W}_N|=N^{\varphi(d)}$ and thus get

$$\frac{H\left(\sum_{j\neq i}^{K}h_{ij}W_j\right)}{2\log|\mathcal{W}_N|}\leqslant\frac{\varphi(d+1)\log((K-1)N)}{2\varphi(d)\log N} \tag{35}$$

$$\xrightarrow{d,N\to\infty}\frac{1}{2}, \tag{36}$$

where we used

$$\frac{\varphi(d+1)}{\varphi(d)}=\frac{K(K-1)+d+1}{d+1}\xrightarrow{d\to\infty}1. \tag{37}$$

We next show that Condition $(*)$ implies

$$H\left(h_{ii}W_i+\sum_{j\neq i}h_{ij}W_j\right)=H\left(h_{ii}W_i,\sum_{j\neq i}h_{ij}W_j\right). \tag{38}$$

Applying the chain rule twice we find

$$H\Bigg(h_{ii}W_i, \sum_{j\neq i} h_{ij}W_j\Bigg)$$

$$= H\Bigg(h_{ii}W_i, \sum_{j\neq i} h_{ij}W_j, h_{ii}W_i + \sum_{j\neq i} h_{ij}W_j\Bigg) \tag{39}$$

$$= H\Bigg(h_{ii}W_i + \sum_{j\neq i} h_{ij}W_j\Bigg)$$

$$+ H\Bigg(h_{ii}W_i, \sum_{j\neq i} h_{ij}W_j \,\Bigg|\, h_{ii}W_i + \sum_{j\neq i} h_{ij}W_j\Bigg), \tag{40}$$

and therefore proving (38) amounts to showing that

$$H\Bigg(h_{ii}W_i, \sum_{j\neq i} h_{ij}W_j \,\Bigg|\, h_{ii}W_i + \sum_{j\neq i} h_{ij}W_j\Bigg) = 0. \tag{41}$$

In order to establish (41), suppose that $w_1, ..., w_K$ and $\widetilde{w}_1, ..., \widetilde{w}_K$ are realizations of $W_1, ..., W_K$ such that

$$h_{ii}w_i + \sum_{j\neq i} h_{ij}w_j = h_{ii}\widetilde{w}_i + \sum_{j\neq i} h_{ij}\widetilde{w}_j, \tag{42}$$

or equivalently

$$h_{ii}(w_i - \widetilde{w}_i) + \sum_{j\neq i} h_{ij}(w_j - \widetilde{w}_j) = 0. \tag{43}$$

The first term on the left-hand side (LHS) of (43) is a $\mathbb{Z}$-linear combination of elements in $\{h_{ii}f_j(\check{\mathbf{h}}) : j \geqslant 1\}$, whereas the second term is a $\mathbb{Z}$-linear combination of elements in $\{f_j(\check{\mathbf{h}}) : j \geqslant 1\}$. Thanks to the linear independence of the union in Condition $(*)$, it follows that the two terms in (43) have to equal 0 individually and hence $w_i = \widetilde{w}_i$ and $\sum_{j\neq i} h_{ij}w_j = \sum_{j\neq i} h_{ij}\widetilde{w}_j$. This shows that the sum $h_{ii}W_i + \sum_{j\neq i} h_{ij}W_j$ uniquely determines the terms $h_{ii}W_i$ and $\sum_{j\neq i} h_{ij}W_j$ and therefore proves (41). Next, we note that

$$H\Bigg(\sum_{j=1}^{K} h_{ij}W_j\Bigg) = H\Bigg(h_{ii}W_i + \sum_{j\neq i}^{K} h_{ij}W_j\Bigg) \tag{44}$$

$$= H\Bigg(h_{ii}W_i, \sum_{j\neq i}^{K} h_{ij}W_j\Bigg) \tag{45}$$

$$= H(h_{ii}W_i) + H\Bigg(\sum_{j\neq i}^{K} h_{ij}W_j\Bigg), \tag{46}$$

where the last equality is thanks to the independence of the W_j, $1 \leqslant j \leqslant K$. Putting the pieces together, we finally obtain

$$\frac{H\Big(\sum_{j=1}^{K} h_{ij}W_j\Big) - H\Big(\sum_{j\neq i}^{K} h_{ij}W_j\Big)}{2\log|\mathcal{W}_N|} \tag{47}$$

$$= \frac{H(h_{ii}W_i)}{2\varphi(d)\log N} = \frac{\varphi(d)\log N}{2\varphi(d)\log N} = \frac{1}{2}, \tag{48}$$

where we used the scaling invariance of entropy, the fact that W_i is uniform on $\mathcal{W}$, and $|\mathcal{W}| = N^{\varphi(d)}$. This allows us to conclude that, for all d and N, we have

$$\min\Bigg\{\frac{H\Big(\sum_{j=1}^{K} h_{ij}W_j\Big)}{2\log|\mathcal{W}_N|}, 1\Bigg\} - \min\Bigg\{\frac{H\Big(\sum_{j\neq i}^{K} h_{ij}W_j\Big)}{2\log|\mathcal{W}_N|}, 1\Bigg\}$$

$$\geqslant 1 - \frac{\varphi(d+1)\log((K-1)N)}{2\varphi(d)\log N}, \tag{49}$$

as either the first minimum on the LHS of (49) coincides with the nontrivial term in which case by (46) the second minimum coincides with the nontrivial term as well, and therefore by (48) the LHS of (49) equals $1/2 \geqslant 1 - \frac{\varphi(d+1)\log((K-1)N)}{2\varphi(d)\log N}$, or the first minimum coincides with 1 in which case we apply $\min\Big\{\frac{H(\sum_{j\neq i}^{K} h_{ij}W_j)}{2\log|\mathcal{W}_N|}, 1\Big\} \leqslant \frac{H(\sum_{j\neq i}^{K} h_{ij}W_j)}{2\log|\mathcal{W}_N|} \leqslant \frac{\varphi(d+1)\log((K-1)N)}{2\varphi(d)\log N}$, where we used (35) for the second inequality. As, by (36), the RHS of (49) converges to $1/2$ for $d, N \to \infty$, it follows that the LHS of (33) is asymptotically lower-bounded by $K/2$. This completes the proof. ■

Remark 5: Verifying Condition $(*)$ for a given channel matrix $\mathbf{H}$ in theory requires checking infinitely many equations of the form (4). It is therefore natural to ask whether we can say anything about the DoF achievable for a given $\mathbf{H}$ when (4) is known to hold only for finitely many coefficients a_j, b_j and up to a finite degree d. To address this question we consider the same input distributions as in the proof of Theorem 1 and carefully analyze the steps in the proof that employ Condition $(*)$. Specifically, there are only two such steps, namely the argument on the uniqueness of the representation of elements in the set (34) and the argument leading to (46). First, as for uniqueness in (34) we need to verify that

$$\sum_{j=1}^{\varphi(d+1)} a_j f_j(\check{\mathbf{h}}) \neq \sum_{j=1}^{\varphi(d+1)} \widetilde{a}_j f_j(\check{\mathbf{h}}) \tag{50}$$

for all $a_j, \widetilde{a}_j \in \{1, ..., (K-1)N\}$ with $(a_1, ..., a_{\varphi(d+1)}) \neq (\widetilde{a}_1, ..., \widetilde{a}_{\varphi(d+1)})$. Note that we have to consider monomials up to degree $d+1$ as the multiplication of W_j by an off-diagonal channel coefficient h_{ij} increases the degrees of the involved monomials by 1, as already formalized in (34). Second, to get (46), we need to ensure that $h_{ii}W_i + \sum_{j\neq i} h_{ij}W_j$ uniquely determines $h_{ii}W_i$ and $\sum_{j\neq i} h_{ij}W_j$, for $i = 1, ..., K$, which amounts to requiring $h_{ii}w_i + \sum_{j\neq i} h_{ij}w_j \neq h_{ii}\widetilde{w}_i + \sum_{j\neq i} h_{ij}\widetilde{w}_j$ whenever $(h_{ii}w_i, \sum_{j\neq i} h_{ij}w_j) \neq (h_{ii}\widetilde{w}_i, \sum_{j\neq i} h_{ij}\widetilde{w}_j)$. Inserting the elements in (32) for $w_i, \widetilde{w}_i$ this condition reads

$$\sum_{j=1}^{\varphi(d+1)} a_j f_j(\check{\mathbf{h}}) + \sum_{j=1}^{\varphi(d)} b_j h_{ii} f_j(\check{\mathbf{h}})$$

$$\neq \sum_{j=1}^{\varphi(d+1)} \widetilde{a}_j f_j(\check{\mathbf{h}}) + \sum_{j=1}^{\varphi(d)} \widetilde{b}_j h_{ii} f_j(\check{\mathbf{h}}), \tag{51}$$

for all $a_j, \widetilde{a}_j \in \{1, ..., (K-1)N\}$ and $b_j, \widetilde{b}_j \in \{1, ..., N\}$ with $(a_1, ..., a_{\varphi(d+1)}, b_1, ..., b_{\varphi(d)}) \neq (\widetilde{a}_1, ..., \widetilde{a}_{\varphi(d+1)}, \widetilde{b}_1, ..., \widetilde{b}_{\varphi(d)})$.

Next, we note that (50) is a special case of (51) obtained by setting $b_j = \widetilde{b}_j$, for all j, in (51). Finally, subtracting the RHS of (51) from both sides in (51) we obtain the condition that nontrivial $\mathbb{Z}$-linear combinations of the elements participating in Condition $(*)$ do not equal 0, which in turn is equivalent to (4) restricted to a finite number of coefficients a_j, b_j and a finite degree d.

Now, assuming that, for a given $\mathbf{H}$, (51) is verified for all $a_j, \widetilde{a}_j \in \{1, ..., (K-1)N\}$ and $b_j, \widetilde{b}_j \in \{1, ..., N\}$ with

$$(a_1, ..., a_{\varphi(d+1)}, b_1, ..., b_{\varphi(d)}) \neq (\widetilde{a}_1, ..., \widetilde{a}_{\varphi(d+1)}, \widetilde{b}_1, ..., \widetilde{b}_{\varphi(d)})$$

and fixed d and N, we can proceed as in the proof of Theorem 1 to get the following from (49):

$$\begin{aligned}
&\min\left\{\frac{H\left(\sum_{j=1}^{K} h_{ij} W_j\right)}{\log(1/r)}, 1\right\} - \min\left\{\frac{H\left(\sum_{j\neq i}^{K} h_{ij} W_j\right)}{\log(1/r)}, 1\right\} \\
&\geqslant 1 - \frac{\varphi(d+1)\log((K-1)N)}{2\varphi(d)\log N} \\
&= 1 - \frac{(K(K-1)+d+1)\log((K-1)N)}{2(d+1)\log N}.
\end{aligned}$$

Upon insertion into (33) this yields the DoF lower bound

$$\frac{K}{2}\left[2 - \frac{(K(K-1)+d+1)\log((K-1)N)}{(d+1)\log N}\right].$$

VII. Condition $(*)$ is not necessary

While Condition $(*)$ is sufficient for $\mathsf{DoF}(\mathbf{H}) = K/2$, we next show that it is not necessary. This will be accomplished by constructing a class of example channel matrices that fail to satisfy Condition $(*)$ but still admit $K/2$ DoF. As, however, almost all channel matrices satisfy Condition $(*)$ this example class is necessarily of Lebesgue measure 0. Specifically, we consider channel matrices that have $h_{ii} \in \mathbb{R} \setminus \mathbb{Q}$, $i = 1, ..., K$, and $h_{ij} \in \mathbb{Q} \setminus \{0\}$, for $i, j = 1, ..., K$ with $i \neq j$. This assumption implies that all entries of $\mathbf{H}$ are nonzero, i.e., $\mathbf{H}$ is fully connected, which, again by [11, Prop. 1], yields $\mathsf{DoF}(\mathbf{H}) \leqslant K/2$. Moreover, as two rational numbers are linearly dependent over $\mathbb{Q}$, these channel matrices violate Condition $(*)$. We next show that nevertheless $\mathsf{DoF}(\mathbf{H}) \geqslant K/2$ and hence $\mathsf{DoF}(\mathbf{H}) = K/2$. This will be accomplished by constructing corresponding full DoF-achieving input distributions.

We begin by arguing that we may assume $h_{ij} \in \mathbb{Z}$, for $i \neq j$. Indeed, since $\mathsf{DoF}(\mathbf{H})$ is invariant to scaling of rows or columns of $\mathbf{H}$ by a nonzero constant [13, Lem. 3], we can, without affecting $\mathsf{DoF}(\mathbf{H})$, multiply the channel matrix by a common denominator of the h_{ij}, $i \neq j$, thus rendering the off-diagonal entries integer-valued while retaining irrationality of the diagonal entries h_{ii}.

Let

$$\mathcal{W} := \{0, ..., N-1\}, \tag{52}$$

for some $N > 0$, and take $W_1, ..., W_K$ to be i.i.d. uniformly distributed on $\mathcal{W}$. We set the contraction parameter to

$$r = 2^{-2\log(2h_{\max}KN)}, \tag{53}$$

where $h_{\max} := \max\{|h_{ij}| \,:\, i \neq j\}$. Writing $\sum_{j=1}^{K} h_{ij} W_j = h_{ii} \cdot W_i + 1 \cdot \sum_{j\neq i} h_{ij} W_j$, where $W_i, \sum_{j\neq i} h_{ij} W_j \in \mathbb{Z}$, and realizing that $\{h_{ii}, 1\}$ is linearly independent over $\mathbb{Q}$, we can mimic the arguments leading to (46) to conclude that

$$H\left(\sum_{j=1}^{K} h_{ij} W_j\right) = H(h_{ii} W_i) + H\left(\sum_{j\neq i} h_{ij} W_j\right), \tag{54}$$

for $i = 1, ..., K$. In fact, it is precisely the linear independence of $\{h_{ii}, 1\}$ over $\mathbb{Q}$ that makes this example class work. Next, we note that

$$\sum_{j\neq i}^{K} h_{ij} W_j \in \{-h_{\max}(K-1)N, ..., 0, ..., h_{\max}(K-1)N\}$$

and hence $H\left(\sum_{j\neq i} h_{ij} W_j\right) \leqslant \log(2h_{\max}KN)$. Since the W_j, $1 \leqslant j \leqslant K$, are identically distributed, we have $H(h_{ii} W_i) = H(h_{ij} W_j)$, for all i, j, and therefore $H(h_{ii} W_i) \leqslant H(\sum_{j\neq i} h_{ij} W_j)$ as a consequence of the fact that the entropy of a sum of independent random variables is greater than the entropy of each participating random variable [20, Ex. 2.14]. Thus (54) implies that

$$H\left(\sum_{j=1}^{K} h_{ij} W_j\right) \leqslant 2H\left(\sum_{j\neq i}^{K} h_{ij} W_j\right) \leqslant 2\log(2h_{\max}KN).$$

With (53) we therefore obtain

$$\min\left\{\frac{H\left(\sum_{j=1}^{K} h_{ij} W_j\right)}{\log(1/r)}, 1\right\} = \frac{H\left(\sum_{j=1}^{K} h_{ij} W_j\right)}{\log(1/r)},$$

and since

$$H\left(\sum_{j\neq i}^{K} h_{ij} W_j\right) \leqslant H\left(\sum_{j=1}^{K} h_{ij} W_j\right), \tag{55}$$

again by [20, Ex. 2.14], we also have

$$\min\left\{\frac{H\left(\sum_{j\neq i}^{K} h_{ij} W_j\right)}{\log(1/r)}, 1\right\} = \frac{H\left(\sum_{j\neq i}^{K} h_{ij} W_j\right)}{\log(1/r)}.$$

Applying Proposition 1 with (54) and using $H(h_{ii} W_i) = \log N$, we finally obtain

$$\mathsf{DoF}(\mathbf{H}) \geqslant \frac{\sum_{i=1}^{K} H(h_{ii} W_i)}{\log(1/r)} = \frac{K\log N}{\log(1/r)} = \frac{K\log N}{2\log(2h_{\max}KN)}. \tag{56}$$

Since (56) holds for all N, in particular for $N \to \infty$, this establishes that $\mathsf{DoF}(\mathbf{H}) \geqslant K/2$ and thereby completes our argument.

Recall that in the case of channel matrices satisfying Condition $(*)$ the value set $\mathcal{W}$ in (32) is channel-dependent. Specifically, this channel dependence induces an algebraic structure that is shared by all channel matrices satisfying Condition $(*)$. Here, however, the assumption of the diagonal entries of $\mathbf{H}$ being irrational and the off-diagonal entries rational already induces enough algebraic structure to allow the construction of full-DoF achieving input distributions based on the channel-independent set $\mathcal{W}$ in (52).

We conclude by noting that the example class studied here was investigated before in [8, Thm. 1] and [3, Thm. 6]. In contrast to [3], [8] our proof of DoF-optimality is, however, not based on arguments from Diophantine approximation theory.

VIII. DoF-characterization in terms of Shannon entropy

To put our second main result, reported in this section, into context, we first note that the DoF-characterization [3, Thm. 4], see also (11) and the statement thereafter, is in terms of information dimension. As already noted, information dimension is, in general, difficult to evaluate. Now, it turns out that the DoF lower bound in Proposition 1 can be developed into a DoF-characterization in the spirit of [3, Thm. 4], which, however, will be entirely in terms of Shannon entropy.

Theorem 3: (*Achievability*): For all channel matrices $\mathbf{H} \neq \mathbf{0}$, we have

$$\sup_{W_1,\dots,W_K} \frac{\sum_{i=1}^K \left[H\left(\sum_{j=1}^K h_{ij}W_j\right) - H\left(\sum_{j\neq i}^K h_{ij}W_j\right)\right]}{\max_{i=1,\dots,K} H\left(\sum_{j=1}^K h_{ij}W_j\right)} \leqslant \mathsf{DoF}(\mathbf{H}), \tag{57}$$

where the supremum in (57) is over all independent discrete $W_1, \dots, W_K$ such that the denominator $\max_{i=1,\dots,K} H\left(\sum_{j=1}^K h_{ij}W_j\right)$ in (57) is nonzero.[13]

Converse: We have equality in (57) for almost all $\mathbf{H}$ including channel matrices with arbitrary diagonal entries and all off-diagonal entries algebraic numbers.

Proof: We first prove the achievability statement. The idea of the proof is to apply Proposition 1 with a suitably chosen contraction parameter r. Specifically, let $W_1, \dots, W_K$ be independent discrete random variables such that the denominator in (57) is nonzero, and apply Proposition 1 with

$$r := 2^{-\max_{i=1,\dots,K} H\left(\sum_{j=1}^K h_{ij}W_j\right)},$$

which ensures that all minima in (25) coincide with the respective nontrivial terms. Specifically, for $i = 1, \dots, K$, we have

$$\min\left\{\frac{H\left(\sum_{j=1}^K h_{ij}W_j\right)}{\log(1/r)}, 1\right\} = \frac{H\left(\sum_{j=1}^K h_{ij}W_j\right)}{\max\limits_{i=1,\dots,K} H\left(\sum_{j=1}^K h_{ij}W_j\right)}$$

and

$$\min\left\{\frac{H\left(\sum_{j\neq i}^K h_{ij}W_j\right)}{\log(1/r)}, 1\right\} = \frac{H\left(\sum_{j\neq i}^K h_{ij}W_j\right)}{\max\limits_{i=1,\dots,K} H\left(\sum_{j=1}^K h_{ij}W_j\right)},$$

where the latter follows from $H\left(\sum_{j=1}^K h_{ij}W_j\right) \geqslant H\left(\sum_{j\neq i}^K h_{ij}W_j\right)$ (cf. (55)). Proposition 1 now yields

$$\frac{\sum_{i=1}^K \left[H\left(\sum_{j=1}^K h_{ij}W_j\right) - H\left(\sum_{j\neq i}^K h_{ij}W_j\right)\right]}{\max_{i=1,\dots,K} H\left(\sum_{j=1}^K h_{ij}W_j\right)} \leqslant \mathsf{DoF}(\mathbf{H}). \tag{58}$$

Finally, (57) is obtained by supremization of the LHS of (58) over all admissible $W_1, \dots, W_K$.

To prove the converse, we begin by referring to the proof of [3, Thm. 4], where the following is shown to hold for almost all $\mathbf{H}$ including channel matrices $\mathbf{H}$ with arbitrary diagonal entries and all off-diagonal entries algebraic numbers: For every $\delta > 0$, there exist independent discrete random variables $W_1, \dots, W_K$ and an $r \in (0,1)$ satisfying[14]

$$\log(1/r) \geqslant \max_{i=1,\dots,K} H\left(\sum_{j=1}^K h_{ij}W_j\right) \tag{59}$$

such that

$$\mathsf{DoF}(\mathbf{H}) \leqslant \delta + \frac{\sum_{i=1}^K \left[H\left(\sum_{j=1}^K h_{ij}W_j\right) - H\left(\sum_{j\neq i}^K h_{ij}W_j\right)\right]}{\log(1/r)}. \tag{60}$$

By (59) it follows that

$$\frac{\sum_{i=1}^K \left[H\left(\sum_{j=1}^K h_{ij}W_j\right) - H\left(\sum_{j\neq i}^K h_{ij}W_j\right)\right]}{\log(1/r)} \leqslant \frac{\sum_{i=1}^K \left[H\left(\sum_{j=1}^K h_{ij}W_j\right) - H\left(\sum_{j\neq i}^K h_{ij}W_j\right)\right]}{\max_{i=1,\dots,K} H\left(\sum_{j=1}^K h_{ij}W_j\right)}.$$

Finally, letting $\delta \to 0$ and taking the supremum over all admissible $W_1, \dots, W_K$, we get

$$\mathsf{DoF}(\mathbf{H}) \leqslant \sup_{W_1,\dots,W_K} \frac{\sum_{i=1}^K \left[H\left(\sum_{j=1}^K h_{ij}W_j\right) - H\left(\sum_{j\neq i}^K h_{ij}W_j\right)\right]}{\max_{i=1,\dots,K} H\left(\sum_{j=1}^K h_{ij}W_j\right)}$$

for almost all $\mathbf{H}$ including channel matrices $\mathbf{H}$ with arbitrary diagonal entries and all off-diagonal entries algebraic numbers. This completes the proof. ■

Remark 6: A natural question to ask is whether the DoF-characterization in Theorem 3 is equivalent to that in [3, Thm. 4], or, more specifically, for which $\mathbf{H}$ the LHS of (57) equals the RHS of (11). In the achievability part of the proof of Theorem 3, we have actually shown that for all $\mathbf{H}$

$$\sup_{W_1,\dots,W_K} \frac{\sum_{i=1}^K \left[H\left(\sum_{j=1}^K h_{ij}W_j\right) - H\left(\sum_{j\neq i}^K h_{ij}W_j\right)\right]}{\max_{i=1,\dots,K} H\left(\sum_{j=1}^K h_{ij}W_j\right)} \leqslant \sup_{X_1,\dots,X_K} \sum_{i=1}^K \left[d\left(\sum_{j=1}^K h_{ij}X_j\right) - d\left(\sum_{j\neq i}^K h_{ij}X_j\right)\right], \tag{61}$$

which yields the final result (57) when combined with (11). The LHS of (61) is obtained by reasoning along the same lines as in the proof of Proposition 1, namely by applying the RHS of (61) to self-similar $X_1, \dots, X_K$ with suitable contraction parameter r, invoking Theorem 2, and noting that the supremization is then carried out over a smaller set of distributions.

[13] This condition only excludes the cases where all W_i that appear with nonzero channel coefficients are chosen as deterministic. In fact, such choices yield $\mathsf{dof}(X_1, \dots, X_K; \mathbf{H}) = 0$ (irrespective of the choice of the contraction parameter r) and are thus not of interest. The assumption $\mathbf{H} \neq \mathbf{0}$ guarantees that there do exist $W_1, \dots, W_K$ such that the denominator $\max_{i=1,\dots,K} H\left(\sum_{j=1}^K h_{ij}W_j\right)$ in (57) is nonzero.

[14] This statement is obtained from the proof of [3, Thm. 4] as follows. The W_i and r here correspond to the W_i and r^n defined in [3, Eq. (146)] and [3, Eq. (147)], respectively. The relation in (59) is then simply a consequence of [3, Eq. (153)] and the cardinality bound for entropy.

By the converse statements in Theorem 3 and [3, Thm. 4] we know that our alternative DoF-characterization is equivalent to the original DoF-characterization in [3, Thm. 4] for almost all $\mathbf{H}$ including $\mathbf{H}$-matrices with arbitrary diagonal entries and all off-diagonal entries algebraic numbers, since in all these cases we have a converse for both DoF-characterizations. As shown in the next section, this includes cases where $\mathsf{DoF}(\mathbf{H}) < K/2$. Moreover, the two DoF-characterizations are trivially equivalent on the "almost-all set" characterized by Condition $(*)$, as in this case the LHS of (61) equals $K/2$ and therefore by (11) and $\mathsf{DoF}(\mathbf{H}) \leqslant K/2$ [11, Prop. 1], the RHS of (61) equals $K/2$ as well. What we do not know is whether (61) is always satisfied with equality, but certainly the set of channel matrices where this is not the case is of Lebesgue measure 0.

Remark 7: Compared to the original DoF-characterization [3, Thm. 4] the alternative expression in Theorem 3 exhibits two advantages. First, the supremization has to be carried out over discrete random variables only, whereas in [3, Thm. 4] the supremum is taken over general distributions. Second, Shannon entropy is typically much easier to evaluate than information dimension. Our alternative characterization is therefore more amenable to both analytical statements and numerical evaluations. This is demonstrated in the next section.

IX. DoF-characterization and additive combinatorics

In this section, we apply our alternative DoF-characterization in Theorem 3 to establish a formal connection between the characterization of DoF for arbitrary channel matrices and sumset problems in additive combinatorics. We also show how Theorem 3 can be used to improve the best known bounds on the DoF of a particular channel matrix studied in [3].

We begin by noting that according to [8, Thm. 2] channel matrices with all entries rational admit strictly less than $K/2$ DoF, i.e.,

$$\mathsf{DoF}(\mathbf{H}) < \frac{K}{2}.$$

However, finding the exact number of DoF for $\mathbf{H}$ with all entries rational, even for simple examples, turns out to be a very difficult problem. Based on our alternative DoF-characterization (57) in Theorem 3, which holds with equality when all entries of $\mathbf{H}$ are rational, we will be able to explain why this problem is so difficult. Specifically, we establish that characterizing the DoF for $\mathbf{H}$ with all entries rational is equivalent to solving very hard problems in sumset theory. As noted before, however, finding the exact number of DoF is difficult only on a set of channel matrices of Lebesgue measure 0, since $\mathsf{DoF}(\mathbf{H}) = K/2$ for almost all $\mathbf{H}$.

The simplest nontrivial example is the 3-user case with

$$\mathbf{H} = \begin{pmatrix} h_1 & 0 & 0 \\ h_2 & h_3 & 0 \\ h_4 & h_5 & h_6 \end{pmatrix},$$

where $h_1, \dots, h_6 \in \mathbb{Q} \setminus \{0\}$. Since $\mathsf{DoF}(\mathbf{H})$ is invariant to scaling of rows or columns of $\mathbf{H}$ by a nonzero constant [13, Lem. 3], we can transform this channel matrix as follows:

$$\begin{pmatrix} h_1 & 0 & 0 \\ h_2 & h_3 & 0 \\ h_4 & h_5 & h_6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ h_2 & h_3 & 0 \\ 1 & \frac{h_5}{h_4} & \frac{h_6}{h_4} \end{pmatrix} \longrightarrow$$
$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ h_2 & h_3 & 0 \\ 1 & \frac{h_5}{h_4} & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{h_3 h_4}{h_2 h_5} & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

We can therefore restrict ourselves to the analysis of channel matrices of the form

$$\mathbf{H}_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \lambda & 0 \\ 1 & 1 & 1 \end{pmatrix}, \tag{62}$$

where $\lambda \in \mathbb{Q} \setminus \{0\}$. This example class was studied before in [3], [8]. In particular, using the DoF-characterization in terms of information dimension (11), Wu et al. showed that [3, Thm. 11]

$$\mathsf{DoF}(\mathbf{H}_\lambda) = 1 + \sup_{X_1, X_2} \left[d(X_1 + \lambda X_2) - d(X_1 + X_2) \right], \tag{63}$$

where the supremum is taken over all independent X_1, X_2 such that $\mathbb{E}[X_1^2], \mathbb{E}[X_2^2] < \infty$ and the information dimension terms appearing in (63) exist. Based on (63) $\mathsf{DoF}(\mathbf{H}_\lambda)$ can be lower-bounded through concrete choices of the input distributions X_1 and X_2. Upper bounds on $\mathsf{DoF}(\mathbf{H}_\lambda)$ can be established by employing general upper and lower bounds on information dimension. However, there is not much one can get beyond what basic inequalities deliver.

By applying Theorem 3 to the channel matrix (62), we next develop an alternative characterization to (63), which will in turn allow us to obtain tighter bounds on $\mathsf{DoF}(\mathbf{H}_\lambda)$. The resulting expression for $\mathsf{DoF}(\mathbf{H}_\lambda)$ involves the minimization of the ratio of entropies of linear combinations of discrete random variables and is analytically and numerically more tractable than (63).

Theorem 4: For

$$\mathbf{H}_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \lambda & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

we have

$$\mathsf{DoF}(\mathbf{H}_\lambda) = 2 - \inf_{U,V} \frac{H(U+V)}{H(U+\lambda V)}, \tag{64}$$

where the infimum is taken over all independent discrete random variables U, V such that[15] $H(U + \lambda V) > 0$.

Proof: As the off-diagonal entries of $\mathbf{H}_\lambda$ are all rational and therefore algebraic numbers, we have equality in (57), which upon insertion of $\mathbf{H}_\lambda$ yields

$$\mathsf{DoF}(\mathbf{H}_\lambda) = \sup_{U,V,W} \frac{H(U+\lambda V) + H(U+V+W) - H(U+V)}{\max\{H(U), H(U+\lambda V), H(U+V+W)\}}, \tag{65}$$

[15] Again, this condition simply prevents the denominator in (64) from being 0. The case $H(U + \lambda V) = 0$ is equivalent to U and V deterministic. This choice would, however, yield $\mathsf{dof}(X_1, \dots, X_K; \mathbf{H}) \leqslant 1$ and is thus not of interest.

where the supremum is taken over all independent discrete random variables U, V, W such that the denominator in (65) is nonzero. Now, again using [20, Ex. 2.14], we have $H(U) \leqslant H(U + \lambda V)$, which when inserted into (65) yields

$$\mathsf{DoF}(\mathbf{H}_\lambda) = \tag{66}$$

$$\sup_{U,V,W} \frac{H(U+\lambda V) + H(U+V+W) - H(U+V)}{\max\{H(U+\lambda V), H(U+V+W)\}} \tag{67}$$

$$\leqslant 1 + \sup_{U,V,W} \frac{H(U+\lambda V) - H(U+V)}{\max\{H(U+\lambda V), H(U+V+W)\}} \tag{68}$$

$$\leqslant 1 + \sup_{U,V} \frac{H(U+\lambda V) - H(U+V)}{H(U+\lambda V)} \tag{69}$$

$$= 2 - \inf_{U,V} \frac{H(U+V)}{H(U+\lambda V)}, \tag{70}$$

where we used the fact that the supremum in (68) is nonnegative (as seen, e.g., by choosing U to be nondeterministic and V deterministic); invoking $\max\{H(U+\lambda V), H(U+V+W)\} \geqslant H(U+\lambda V)$ in the denominator of (68) then yields the upper bound (69).

For the converse part, let U, V be independent discrete random variables such that $H(U+\lambda V) > 0$. We take W to be discrete, independent of U and V, and to satisfy

$$H(W) \geqslant H(U+\lambda V), \tag{71}$$

e.g., we may simply choose W to be uniformly distributed on a sufficiently large finite set. Applying Proposition 1 with $W_1 = U$, $W_2 = V$, $W_3 = W$, and $r := 2^{-H(U+\lambda V)}$, we obtain

$$\begin{aligned}&\min\left\{\frac{H(U)}{H(U+\lambda V)}, 1\right\} + \min\left\{\frac{H(U+\lambda V)}{H(U+\lambda V)}, 1\right\} \\ &- \min\left\{\frac{H(U)}{H(U+\lambda V)}, 1\right\} + \min\left\{\frac{H(U+V+W)}{H(U+\lambda V)}, 1\right\} \\ &- \min\left\{\frac{H(U+V)}{H(U+\lambda V)}, 1\right\} \leqslant \mathsf{DoF}(\mathbf{H}_\lambda).\end{aligned} \tag{72}$$

Since $H(U+V+W) \geqslant H(W) \geqslant H(U+\lambda V)$, where the first inequality is by [20, Ex. 2.14] and the second by the assumption (71), we get from (72) that

$$2 - \min\left\{\frac{H(U+V)}{H(U+\lambda V)}, 1\right\} \leqslant \mathsf{DoF}(\mathbf{H}_\lambda). \tag{73}$$

We treat the cases $H(U+V) > H(U+\lambda V)$ and $H(U+V) \leqslant H(U+\lambda V)$ separately. If $H(U+V) > H(U+\lambda V)$, then

$$2 - \frac{H(U+V)}{H(U+\lambda V)} \tag{74}$$

$$< 1 \tag{75}$$

$$= 2 - \min\left\{\frac{H(U+V)}{H(U+\lambda V)}, 1\right\} \tag{76}$$

$$\leqslant \mathsf{DoF}(\mathbf{H}_\lambda). \tag{77}$$

On the other hand, if $H(U+V) \leqslant H(U+\lambda V)$, (73) becomes

$$2 - \frac{H(U+V)}{H(U+\lambda V)} \leqslant \mathsf{DoF}(\mathbf{H}_\lambda). \tag{78}$$

Combining (74)–(77) and (78), we finally get

$$2 - \frac{H(U+V)}{H(U+\lambda V)} \leqslant \mathsf{DoF}(\mathbf{H}_\lambda), \tag{79}$$

for all independent U, V such that $H(U+\lambda V) > 0$. Taking the supremum in (79) over all admissible U and V completes the proof. ■

Through Theorem 4 we reduced the DoF-characterization of $\mathbf{H}_\lambda$ to the optimization of the ratio of entropies of linear combinations of discrete random variables. This optimization problem has a counterpart in additive combinatorics, namely the following sumset problem: find finite sets $\mathcal{U}, \mathcal{V} \subseteq \mathbb{R}$ such that the relative size

$$\frac{|\mathcal{U}+\mathcal{V}|}{|\mathcal{U}+\lambda\mathcal{V}|} \tag{80}$$

of the sumsets $\mathcal{U}+\mathcal{V}$ and $\mathcal{U}+\lambda\mathcal{V}$ is minimal. The additive combinatorics literature provides a considerable body of useful bounds—as a function of $|\mathcal{U}|$ and $|\mathcal{V}|$—on (80) [18]. A complete answer to this minimization problem does, however, not seem to be available. Generally, finding the minimal value of sumset quantities as in (80) or corresponding entropic quantities, i.e., $H(U+V)/H(U+\lambda V)$ in this case, appears to be a very hard problem, which indicates why finding the exact number of DoF of channel matrices with rational entries is so difficult.

The formal relationship between DoF-characterization and sumset theory, by virtue of Theorem 3, goes beyond $\mathbf{H}$ with rational entries and applies to general $\mathbf{H}$. The resulting linear combinations one has to deal with, however, quickly lead to very hard optimization problems.

We finally show how our alternative DoF-characterization can be put to use to improve the best known bounds on $\mathsf{DoF}(\mathbf{H}_\lambda)$ for $\lambda = -1$. Similar improvements are possible for other values of λ. For brevity we restrict ourselves, however, to the case $\lambda = -1$.

Proposition 2: We have

$$1.13258 \leqslant \mathsf{DoF}(\mathbf{H}_{-1}) \leqslant \frac{4}{3}.$$

Proof: For the lower bound, we choose U and V to be independent and distributed according to

$$\begin{aligned}&\mathbb{P}[U=0] = \mathbb{P}[V=0] = (0.08)^3 \\ &\mathbb{P}[U=1] = \mathbb{P}[V=1] = (0.08)^2 \\ &\mathbb{P}[U=2] = \mathbb{P}[V=2] = 0.08 \\ &\mathbb{P}[U=3] = \mathbb{P}[V=3] = 1 - 0.08 - (0.08)^2 - (0.08)^3.\end{aligned}$$

This choice is motivated by numerical investigations not reported here. It then follows from (64) that

$$\mathsf{DoF}(\mathbf{H}_{-1}) \geqslant 2 - \frac{H(U+V)}{H(U-V)} = 1.13258. \tag{81}$$

A more careful construction of U and V should allow improvements of this lower bound.

For the upper bound, let U and V be independent discrete random variables such that $H(U-V) > 0$ as required in the infimum in (64). Recall the entropy inequalities (17) and (18). Multiplying (17) by $2/3$ and adding the result to (18) yields

$$\frac{5}{3}H(U-V) \leqslant \frac{5}{2}H(U+V),$$

and hence

$$\frac{H(U+V)}{H(U-V)} \geqslant \frac{2}{3}. \tag{82}$$

Using (82) in (64), we then obtain

$$\mathsf{DoF}(\mathbf{H}_{-1}) = 2 - \inf_{U,V} \frac{H(U+V)}{H(U-V)} \leqslant \frac{4}{3},$$

which completes the proof. ■

The bounds in Proposition 2 improve on the best known bounds obtained in [3, Thm. 11][16] as $1.0681 \leqslant \mathsf{DoF}(\mathbf{H}_{-1}) \leqslant \frac{7}{5}$.

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