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A Necessary and Sufficient Condition for Dual Weyl-Heisenberg Frames to be Compactly Supported

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ABSTRACT. In this note we consider continuous-time Weyl–Heisenberg (Gabor) frame expansions with rational oversampling. We present a necessary and sufficient condition on a compactly supported function $g(t)$ generating a Weyl–Heisenberg frame for $L^2(\mathbb{R})$ for its minimal dual (Wexler–Raz dual) $\gamma^0(t)$ to be compactly supported. We furthermore provide a necessary and sufficient condition for a band-limited function $g(t)$ generating a Weyl–Heisenberg frame for $L^2(\mathbb{R})$ to have a band-limited minimal dual $\gamma^0(t)$. As a consequence of these conditions, we show that in the cases of integer oversampling and critical sampling a compactly supported (band-limited) $g(t)$ has a compactly supported (band-limited) minimal dual $\gamma^0(t)$ if and only if the Weyl–Heisenberg frame operator is a multiplication operator in the time (frequency) domain. Our proofs rely on the Zak transform, on the Zibulski–Zeevi representation of the Weyl–Heisenberg frame operator, and on the theory of polynomial matrices.

1. Introduction and Preparation

Weyl–Heisenberg Frames

In this note we consider signal expansions of the form [1, 9, 4, 8, 12, 5, 6]

$$x(t) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \langle x, \gamma_{k,l} \rangle g_{k,l}(t) , \tag{1.1}$$

with synthesis functions $g_{k,l}(t) = g(t-kT)\, e^{j2\pi lFt}$, analysis functions $\gamma_{k,l}(t) = \gamma(t-kT)\, e^{j2\pi lFt}$, time-shift parameter $T > 0$, and frequency-shift parameter $F > 0$. We say that $g(t)$ generates a

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Weyl–Heisenberg (WH) frame for $L^2(\mathbb{R})$ when there exist constants $A > 0$ and $B < \infty$ such that [4, 8]

$$A\|x\|^2 \leq \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \left|\langle x, g_{k,l}\rangle\right|^2 \leq B\|x\|^2 \qquad \forall \quad x(t) \in L^2(\mathbb{R})\,. \tag{1.2}$$

The constants A and B are called lower and upper frame bound, respectively. It is well known that for $g(t)$ to generate a WH frame for $L^2(\mathbb{R})$, it is necessary that $TF \leq 1$ [4, 11, 15]. Throughout this note we restrict our attention to the case of rational sampling factors $TF = p/q$ with[1] $p \in \mathbb{Z}, q \in \mathbb{Z}$, $p \leq q$ and $\gcd(p, q)$=1. The cases $TF = 1$ and $TF < 1$ are referred to as critical sampling and oversampling, respectively. The frame condition (1.2) can equivalently be written as

$$A\|x\|^2 \leq \langle S_g x, x\rangle \leq B\|x\|^2\,,$$

where $(S_g x)(t) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \langle x, g_{k,l}\rangle\, g_{k,l}(t)$ denotes the WH frame operator [4, 8]. When $g(t)$ generates a WH frame, for $TF < 1$ one possible choice for $\gamma(t)$ satisfying (1.1) for all $x(t) \in L^2(\mathbb{R})$ is $\gamma^0(t) = (S_g^{-1} g)(t)$. The function set $\{\gamma_{k,l}^0(t)\}$ generates the dual WH frame with corresponding frame operator $(S_{\gamma^0} x)(t) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \langle x, \gamma_{k,l}^0\rangle\, \gamma_{k,l}^0(t)$. From the theory of WH frames it is well known that $S_{\gamma^0} = S_g^{-1}$ [4, 8]. In the oversampled case, $\gamma(t)$ in (1.1) is not unique for given $g(t)$. Among all the dual functions[2] $\gamma(t)$ the one with minimum L^2-norm is $\gamma^0(t)$, which will henceforth be called minimal dual or Wexler–Raz dual [18, 12, 5, 16]. For generalities about WH frames the interested reader is referred to [4, 8, 6, 2].

The Zak Transform

An important tool in WH frame theory is the Zak transform (ZT) [10, 3]. The ZT of a signal $x(t)$ is defined as

$$\mathcal{Z}_x(t, f) = \sum_{k=-\infty}^{\infty} x(t + kT)\, e^{-j2\pi kTf}\,. \tag{1.3}$$

$\mathcal{Z}_x(t, f)$ is quasiperiodic in t and periodic in f, i.e.,

$$\begin{aligned} \mathcal{Z}_x(t + T, f) &= e^{j2\pi Tf}\, \mathcal{Z}_x(t, f) \\ \mathcal{Z}_x\left(t, f + \frac{1}{T}\right) &= \mathcal{Z}_x(t, f)\,. \end{aligned}$$

The ZT of $x(t)$ can equivalently be written in terms of the Fourier transform $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$ as

$$\mathcal{Z}_x(t, f) = \frac{1}{T}\, e^{j2\pi ft} \sum_{k=-\infty}^{\infty} X\left(f + \frac{k}{T}\right) e^{j2\pi k\frac{t}{T}}\,.$$

We shall also need the Zak transform on the dual grid [12, 19, 13, 20] defined by

$$\mathcal{Z}_x^{(d)}(t, f) = \sum_{k=-\infty}^{\infty} x\left(t + \frac{k}{F}\right) e^{-j2\pi k\frac{f}{F}}\,.$$

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[1] Here $\gcd(p, q)$ denotes the greatest common divisor of p and q.

[2] A function $\gamma(t)$ is dual to $g(t)$ if it satisfies (1.1) for all $x(t) \in L^2(\mathbb{R})$.

The superscript (d) indicates that the ZT is defined with respect to the dual grid. $\mathcal{Z}_x^{(d)}(t, f)$ satisfies the following periodicity relations:

$$\begin{aligned} \mathcal{Z}_x^{(d)}\left(t + \frac{1}{F}, f\right) &= e^{j2\pi \frac{f}{F}} \mathcal{Z}_x^{(d)}(t, f) \\ \mathcal{Z}_x^{(d)}(t, f + F) &= \mathcal{Z}_x^{(d)}(t, f) \,. \end{aligned}$$

The frequency domain expression of the ZT on the dual grid reads

$$\mathcal{Z}_x^{(d)}(t, f) = F e^{j2\pi f t} \sum_{k=-\infty}^{\infty} X(f + kF)\, e^{j2\pi k F t} \,.$$

Representations of the WH Frame Operator

We shall next provide time, frequency, and time-frequency representations of the WH frame operator S_g. These representations constitute the basis for the results presented in Section 2. The Walnut representation [17] of the WH frame operator is given by[3]

$$(S_g x)(t) = \frac{1}{F} \sum_{l=-\infty}^{\infty} x\left(t - \frac{l}{F}\right) \sum_{k=-\infty}^{\infty} g(t - kT)\, g^*\left(t - kT - \frac{l}{F}\right) \tag{1.4}$$

or in the frequency domain

$$\left(\hat{S}_g X\right)(f) = \frac{1}{T} \sum_{l=-\infty}^{\infty} X\left(f - \frac{l}{T}\right) \sum_{k=-\infty}^{\infty} G(f - kF)\, G^*\left(f - kF - \frac{l}{T}\right), \tag{1.5}$$

where $\hat{S}_g = \boldsymbol{F} S_g \boldsymbol{F}^{-1}$ with $\boldsymbol{F}$ denoting the Fourier transform operator.

With $(S_g x)(t) = y(t)$, the Zibulski–Zeevi representation [19, 20] of the WH frame operator S_g for rational oversampling $(TF = p/q)$ reads

$$\mathcal{Z}_y(t, f) = \frac{T}{p} \sum_{u=0}^{p-1} \mathcal{Z}_x\left(t - u\frac{q}{p}T, f\right) \sum_{i=0}^{q-1} \mathcal{Z}_g\left(t, f - \frac{i}{qT}\right) \mathcal{Z}_g^*\left(t - u\frac{q}{p}T, f - \frac{i}{qT}\right). \tag{1.6}$$

Setting $t \rightarrow t - k\frac{q}{p}T$ in (1.6) we obtain after straightforward manipulations

$$\begin{aligned} \mathcal{Z}_y\left(t - k\frac{q}{p}T, f\right) &= \frac{T}{p} \sum_{u=0}^{p-1} \mathcal{Z}_x\left(t - u\frac{q}{p}T, f\right) \sum_{i=0}^{q-1} \mathcal{Z}_g\left(t - k\frac{q}{p}T, f - \frac{i}{qT}\right) \\ &\qquad \mathcal{Z}_g^*\left(t - u\frac{q}{p}T, f - \frac{i}{qT}\right) \end{aligned}$$

for $k = 0, 1, \ldots, p - 1$. This relation can be rephrased in vector-matrix form as [19, 13, 20]

$$\mathbf{z}_y(t, f) = \mathbf{S}_g(t, f)\, \mathbf{z}_x(t, f) \,, \tag{1.7}$$

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[3] Here * stands for complex conjugation.

where

$$\begin{aligned}\mathbf{z}_y(t,f) &= \left[\mathcal{Z}_y(t,f)\ \mathcal{Z}_y\left(t-\frac{q}{p}T,f\right)\ \ldots\ \mathcal{Z}_y\left(t-(p-1)\frac{q}{p}T,f\right)\right]^T ,\\ \mathbf{z}_x(t,f) &= \left[\mathcal{Z}_x(t,f)\ \mathcal{Z}_x\left(t-\frac{q}{p}T,f\right)\ \ldots\ \mathcal{Z}_x\left(t-(p-1)\frac{q}{p}T,f\right)\right]^T\end{aligned}$$

and $\mathbf{S}_g(t,f)$ is a $p \times p$ matrix with elements ($k=0,1,\ldots,p-1$, $l=0,1,\ldots,p-1$)

$$\left[\mathbf{S}_g(t,f)\right]_{k,l} = \frac{T}{p}\sum_{i=0}^{q-1}\mathcal{Z}_g\left(t-k\frac{q}{p}T, f-\frac{i}{qT}\right)\mathcal{Z}_g^*\left(t-l\frac{q}{p}T, f-\frac{i}{qT}\right). \tag{1.8}$$

The matrix $\mathbf{S}_g(t,f)$ is T-periodic in t, i.e., $\mathbf{S}_g(t+T,f)=\mathbf{S}_g(t,f)$ and $\frac{1}{qT}$-periodic in f, i.e., $\mathbf{S}_g\left(t,f+\frac{1}{qT}\right)=\mathbf{S}_g(t,f)$ [13]. Furthermore, it is easily seen from (1.8) that $\mathbf{S}_g(t,f)$ is hermitian, i.e., $\mathbf{S}_g(t,f)=\mathbf{S}_g^H(t,f)$, where the superscript H stands for the conjugate transpose.

We shall also need the Zibulski–Zeevi representation of the WH frame operator in terms of the ZT on the dual grid given by [19, 13, 20]

$$\mathcal{Z}_y^{(d)}(t,f) = \frac{1}{pF}\sum_{u=0}^{p-1}\mathcal{Z}_x^{(d)}\left(t,f-u\frac{F}{p}\right)\sum_{i=0}^{q-1}\mathcal{Z}_g^{(d)}\left(t-i\frac{p}{qF},f\right)\mathcal{Z}_g^{(d)*}\left(t-i\frac{p}{qF},f-u\frac{F}{p}\right).$$

We furthermore have $\mathcal{Z}_y^{(d)}\left(t,f-k\frac{F}{p}\right)=$

$$\frac{1}{pF}\sum_{u=0}^{p-1}\mathcal{Z}_x^{(d)}\left(t,f-u\frac{F}{p}\right)\sum_{i=0}^{q-1}\mathcal{Z}_g^{(d)}\left(t-i\frac{p}{qF},f-k\frac{F}{p}\right)\mathcal{Z}_g^{(d)*}\left(t-i\frac{p}{qF},f-u\frac{F}{p}\right) \tag{1.9}$$

for $k=0,1,\ldots,p-1$. Equation (1.9) can be rewritten in vector-matrix form as [19, 13, 20]

$$\mathbf{z}_y^{(d)}(t,f)=\mathbf{S}_g^{(d)}(t,f)\,\mathbf{z}_x^{(d)}(t,f), \tag{1.10}$$

where

$$\begin{aligned}\mathbf{z}_y^{(d)}(t,f) &= \left[\mathcal{Z}_y^{(d)}(t,f)\ \mathcal{Z}_y^{(d)}\left(t,f-\frac{F}{p}\right)\ \ldots\ \mathcal{Z}_y^{(d)}\left(t,f-(p-1)\frac{F}{p}\right)\right]^T ,\\ \mathbf{z}_x^{(d)}(t,f) &= \left[\mathcal{Z}_x^{(d)}(t,f)\ \mathcal{Z}_x^{(d)}\left(t,f-\frac{F}{p}\right)\ \ldots\ \mathcal{Z}_x^{(d)}\left(t,f-(p-1)\frac{F}{p}\right)\right]^T\end{aligned}$$

and $\mathbf{S}_g^{(d)}(t,f)$ is a $p \times p$ matrix with elements

$$\left[\mathbf{S}_g^{(d)}(t,f)\right]_{k,l} = \frac{1}{pF}\sum_{i=0}^{q-1}\mathcal{Z}_g^{(d)}\left(t-i\frac{p}{qF},f-k\frac{F}{p}\right)\mathcal{Z}_g^{(d)*}\left(t-i\frac{p}{qF},f-l\frac{F}{p}\right). \tag{1.11}$$

The matrix $\mathbf{S}_g^{(d)}(t,f)$ is $\frac{p}{qF}$-periodic in t, i.e., $\mathbf{S}_g^{(d)}(t,f)=\mathbf{S}_g^{(d)}(t+\frac{p}{qF},f)$, and F-periodic in f, i.e., $\mathbf{S}_g^{(d)}(t,f)=\mathbf{S}_g^{(d)}(t,f+F)$ [13]. Furthermore, it is easily seen from (1.11) that $\mathbf{S}_g^{(d)}(t,f)$ is hermitian, i.e., $\mathbf{S}_g^{(d)}(t,f)=\mathbf{S}_g^{(d)^H}(t,f)$.

2. Results

In practical applications one is often interested in WH frames with compactly supported synthesis functions $g_{k,l}(t)$ and compactly supported analysis functions $\gamma_{k,l}^{0}(t)$. Therefore, a question of great practical relevance is whether a given compactly supported function $g(t)$ generating a WH frame for $L^2(\mathbb{R})$ has a compactly supported minimal dual $\gamma^0(t)$. It is well known [2] that a compactly supported $g(t)$ having a frame operator S_g which is a multiplication operator in the time domain has a compactly supported minimal dual $\gamma^0(t)$. However, it appears that no general condition on a compactly supported $g(t)$ to have a compactly supported minimal dual $\gamma^0(t)$ has been reported in the literature.

In this note we present a necessary and sufficient condition on a compactly supported $g(t)$ generating a WH frame for $L^2(\mathbb{R})$ for its minimal dual $\gamma^0(t)$ to be compactly supported. We furthermore present a necessary and sufficient condition on a band-limited[4] $g(t)$ generating a WH frame for $L^2(\mathbb{R})$ for its minimal dual $\gamma^0(t)$ to be band-limited. In addition to that, we show that in the cases of integer oversampling and critical sampling ($TF = 1/q$ with $q \in \mathbb{N}$) a compactly supported (band-limited) $g(t)$ has a compactly supported (band-limited) minimal dual $\gamma^0(t)$ if and only if the WH frame operator S_g is a multiplication operator in the time (frequency) domain. Proofs of the following results will be provided in Section 3.

Theorem 1.

A compactly supported $g(t)$ generating a WH frame for $L^2(\mathbb{R})$ has a compactly supported minimal dual $\gamma^0(t)$ if and only if the matrix $\mathbf{S}_g(t,f)$ is unimodular[5] for all t, i.e., the determinant $\det[\mathbf{S}_g(t,f)] = c(t)$ *is a function of t only.*

Note that $g(t)$ and $\gamma^0(t)$ need not necessarily have the same support.

Corollary 1.

In the cases of integer oversampling and critical sampling, i.e., $TF = \frac{1}{q}$ with $q \in \mathbb{N}$, a compactly supported $g(t)$ generating a WH frame for $L^2(\mathbb{R})$ has a minimal dual $\gamma^0(t)$ with compact support if and only if the WH frame operator S_g is a multiplication operator in the time domain, i.e.,

$$\left(S_g x\right)(t) = qT\, x(t) \sum_{k=-\infty}^{\infty} |g(t-kT)|^2 \,. \tag{2.1}$$

We note that in general a multiplication operator acting on a signal $x(t)$ performs a multiplication of the signal $x(t)$ with some function $h(t)$. Here, it follows from the time-domain Walnut representation (1.4) that if the WH frame operator is a multiplication operator it necessarily has the form (2.1).

Corollary 2.

A band-limited $g(t)$ generating a WH frame for $L^2(\mathbb{R})$ has a band-limited minimal dual $\gamma^0(t)$ if and only if the matrix $\mathbf{S}_g^{(d)}(t,f)$ is unimodular for all f, i.e., $\det[\mathbf{S}_g^{(d)}(t,f)] = d(f)$ *is a function of f only.*

Corollary 3.

In the cases of integer oversampling and critical sampling, i.e., $TF = \frac{1}{q}$ with $q \in \mathbb{N}$, a band-limited $g(t)$ generating a WH frame for $L^2(\mathbb{R})$ has a band-limited minimal dual $\gamma^0(t)$ if and only if

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[4] By band-limited we mean that the Fourier transform is compactly supported.

[5] A square polynomial matrix $\mathbf{A}(z)$ is said to be unimodular if it has a constant nonzero determinant.

the WH frame operator $\mathbf{S}_g$ *is a multiplication operator in the frequency domain, i.e.,*

$$\left(\hat{S}_g X\right)(f) = qF\, X(f) \sum_{k=-\infty}^{\infty} |G(f-kF)|^2 . \tag{2.2}$$

From the frequency domain Walnut representation (1.5) it follows that if $\hat{S}_g$ is a multiplication operator, it necessarily has the form (2.2). We conclude this section by noting that a compactly supported (band-limited) $g(t)$ in general is very unlikely to have a compactly supported (band-limited) minimal dual $\gamma^0(t)$.

3. Derivations and Proofs

For the proof of Theorem 1 we need the following lemma.

Lemma 1.

For a compactly supported $g(t)$ *the ZT* $\mathcal{Z}_g(t,f)$ *is a polynomial in* $e^{j2\pi Tf}$ *for all* t. *If* $g(t)$ *is supported in* $t \in [0, T_0)$ *the maximum degree of* $\mathcal{Z}_g(t,f)$ *for* $t \in [0,T)$ *is given by* $\left\lfloor \frac{T_0}{T} \right\rfloor$. *Conversely, if* $\mathcal{Z}_g(t,f)$ *has a finite maximum degree for* $t \in [0,T)$ *the function* $g(t)$ *is compactly supported.*

Proof. It follows from (1.3) that the ZT of a compactly supported $g(t)$ is a polynomial in $e^{j2\pi Tf}$ for all t. For $g(t)$ supported in $t \in [0,T_0)$ the maximum degree of $\mathcal{Z}_g(t,f)$ for $t \in [0,T)$ follows from the definition of the ZT (1.3). Conversely, if $\mathcal{Z}_g(t,f)$ has a finite maximum degree for $t \in [0,T)$ the sequences $g(t+kT)$ ($t \in \mathbb{R}$) are nonzero on intervals of finite length which implies that $g(t)$ is compactly supported. □

Setting $z = e^{j2\pi Tf}$, we can write $\mathcal{Z}_g(t,z) = \sum_{k=-\infty}^{\infty} g_t[k]\, z^{-k}$ with the sequence $g_t[k] = g(t+kT)$ obtained by sampling $g(t)$. For each t the ZT $\mathcal{Z}_g(t,z)$ is the discrete-time Fourier transform of the sequence $g_t[k]$. Obviously $\mathcal{Z}_g\left(t - l\frac{q}{p}T, z\right) = \sum_{k=-\infty}^{\infty} g_{t-l\frac{q}{p}T}[k]\, z^{-k}$ ($l = 0, 1, ..., p-1$) is also a polynomial in z for all t. Straightforward manipulations reveal that[6]

$$[\mathbf{S}_g(t,z)]_{k,l} = \frac{Tq}{p} \sum_{u=-\infty}^{\infty} z^{-uq} \sum_{s=-\infty}^{\infty} g\left(t - k\frac{q}{p}T - sT\right) g^*\left(t - l\frac{q}{p}T - sT - uqT\right) . \tag{3.1}$$

Thus, for a compactly supported $g(t)$ the matrix $\mathbf{S}_g(t,z)$ is a polynomial matrix in z for all t. We are now ready to give the proof of Theorem 1.

Proof of Theorem 1. We shall first show the sufficiency. Let us assume that $g(t)$ is compactly supported and that $\det[\mathbf{S}_g(t,z)] = c(t)$, i.e., the matrix $\mathbf{S}_g(t,z)$ is unimodular for all t. Setting $x(t) = \gamma^0(t)$ in (1.7) and using $y(t) = (S_g\gamma^0)(t) = g(t)$ we get[7] $\mathbf{z}_{\gamma^0}(t,z) = \mathbf{S}_g^{-1}(t,z)\,\mathbf{z}_g(t,z) = \frac{\text{adj}[\mathbf{S}_g(t,z)]}{\det[\mathbf{S}_g(t,z)]}\mathbf{z}_g(t,z)$. Now, using $\det[\mathbf{S}_g(t,z)] = c(t)$ it follows that $\mathbf{z}_{\gamma^0}(t,z) = \frac{\text{adj}[\mathbf{S}_g(t,z)]}{c(t)}\mathbf{z}_g(t,z)$. The frame property of $g(t)$ (1.2) guarantees that $c(t) > 0$. Since for a compactly supported $g(t)$ $\text{adj}[\mathbf{S}_g(t,z)]$ is a polynomial matrix in z for all t and $\mathbf{z}_g(t,z)$ is a polynomial vector in z for all t, it follows that $\mathbf{z}_{\gamma^0}(t,z)$ is a polynomial vector in z for all t. It remains to show that the vector $\mathbf{z}_{\gamma^0}(t,z)$ has a finite maximum degree for $t \in [0,T)$. From Lemma 1 and (1.8) it follows that for a

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[6] In the following we shall always write $\mathbf{S}_g(t,z)$ instead of $\mathbf{S}_g(t,f)$.

[7] Here $\text{adj}[\mathbf{S}_g(t,z)]$ denotes the adjoint (adjugate) of the matrix $\mathbf{S}_g(t,z)$.

compactly supported $g(t)$ the matrix $\mathbf{S}_g(t,z)$ has a finite maximum degree for $t \in [0,T)$. Denoting this maximum degree by N it furthermore follows that $\mathrm{adj}[\mathbf{S}_g(t,z)]$ has a finite maximum degree for $t \in [0,T)$ which can roughly be upper bounded by N^p. This finally implies that $\mathbf{z}_{\gamma^0}(t,z)$ has a finite maximum degree for $t \in [0,T)$ which using Lemma 1 proves that $\gamma^0(t)$ is compactly supported.

We shall next prove necessity. Assume that both $g(t)$ and its minimal dual $\gamma^0(t)$ are compactly supported. Then, both $\mathbf{S}_g(t,z)$ and $\mathbf{S}_{\gamma^0}(t,z)$ are polynomial matrices in z for all t. Janssen showed in [13] that $\mathbf{S}_{\gamma^0}(t,z) = \mathbf{S}_g^{-1}(t,z)$, which here implies that both $\mathbf{S}_g(t,z)$ and its inverse $\mathbf{S}_g^{-1}(t,z)$ have to be polynomial matrices in z for all t. Since a square polynomial matrix $\mathbf{A}(z)$ has a polynomial inverse $\mathbf{A}^{-1}(z)$ if and only if $\det[\mathbf{A}(z)] = cz^{-K}$ with some $K \in \mathbb{Z}$ [7] and a constant $c \in \mathbb{C}$ independent of z, it follows that $\det[\mathbf{S}_g(t,z)] = c(t)\,z^{-K(t)}$ with an integer-valued function $K(t)$ and $c(t) \neq 0$. Furthermore, $\mathbf{S}_g(t,z)$ is hermitian for all t which implies $\det[\mathbf{S}_g(t,z)] = \left(\det[\mathbf{S}_g(t,z)]\right)^*$ and consequently $c(t)\,z^{-K(t)} = c^*(t)\,z^{K(t)}$ for all z with $|z| = 1$. In particular, for $z = 1$ we have $c(t) = c^*(t)$ which proves that $c(t)$ is real-valued. This furthermore implies $z^{-K(t)} = z^{K(t)}$ which is satisfied only for $K(t) = 0$. Thus, $\det[\mathbf{S}_g(t,z)] = c(t)$, which concludes the proof. □

It can be shown that $\det[\mathbf{S}_g(t,z)]$ is $\frac{T}{p}$-periodic in t. Therefore, it suffices to check the unimodularity of $\mathbf{S}_g(t,z)$ for $t \in \left[0, \frac{T}{p}\right)$.

When the WH frame operator S_g is a multiplication operator in the time domain, i.e., $(S_g x)(t) = \frac{Tq}{p}\,x(t)\,\sum_{k=-\infty}^{\infty} |g(t-kT)|^2$, and $g(t)$ is compactly supported, then obviously $\gamma^0(t)$ will be compactly supported. In fact, $\gamma^0(t)$ has the same support as $g(t)$. The following lemma will be used to show that such a $g(t)$ trivially satisfies the conditions of Theorem 1.

Lemma 2.

Let $g(t)$ be compactly supported such that S_g is a multiplication operator in the time domain. In this case, the matrix $\mathbf{S}_g(t,z)$ is diagonal with elements $[\mathbf{S}_g(t,z)]_{k,k} = [\mathbf{S}_g(t)]_{k,k} = \frac{Tq}{p}\sum_{l=-\infty}^{\infty} |g(t - k\frac{q}{p}T - lT)|^2$.

Proof. For compactly supported $g(t)$ such that S_g is a multiplication operator in the time domain we have

$$\left(S_g x\right)(t) = \frac{Tq}{p} x(t) \sum_{l=-\infty}^{\infty} |g(t-lT)|^2 .$$

The ZT of $y(t) = (S_g x)(t)$ is given by

$$\mathcal{Z}_y(t,f) = \mathcal{Z}_x(t,f)\frac{Tq}{p}\sum_{l=-\infty}^{\infty} |g(t-lT)|^2 ,$$

which implies

$$\mathcal{Z}_y\left(t - k\frac{q}{p}T, f\right) = \mathcal{Z}_x\left(t - k\frac{q}{p}T, f\right)\frac{Tq}{p}\sum_{k=-\infty}^{\infty}\left|g\left(t - k\frac{q}{p}T - lT\right)\right|^2 .$$

This establishes the result. □

From Lemma 2 it follows that the matrix $\mathbf{S}_g(t,z)$ is a function of t only. Hence, $\det[\mathbf{S}_g(t,z)]$ is a function of t only, which shows that a compactly supported $g(t)$ with S_g a multiplication operator in the time domain trivially satisfies the conditions of Theorem 1. Note finally that in the special case of critical sampling $TF = 1$, where $\mathcal{Z}_{\gamma^0}(t,z) = \frac{1}{T\mathcal{Z}_g^*(t,z)}$ [4, 8], the minimal dual is compactly supported if and only if $\mathcal{Z}_g(t,z)$ is a monomial in z for all t.

Proof of Corollary 1. The sufficiency is obvious. For the proof of the necessity assume that $g(t)$ is compactly supported and note that in the cases of integer oversampling and critical sampling

($p = 1$) the $p \times p$ matrix $\mathbf{S}_g(t, z)$ reduces to a scalar given by $S_g(t, z) = qT \sum_{u=-\infty}^{\infty} z^{-uq} \sum_{s=-\infty}^{\infty} g(t - sT)g^*(t - sT - uqT)$. Since here $\det[\mathbf{S}_g(t, z)] = S_g(t, z)$ it follows from Theorem 1 that $\gamma^0(t)$ is compactly supported if and only if $S_g(t, z)$ is a function of t only or equivalently $\sum_{s=-\infty}^{\infty} g(t - sT)g^*(t - sT - uqT) = \sum_{s=-\infty}^{\infty} |g(t - sT)|^2 \delta[u]$, where $\delta[u] = 1$ for $u = 0$ and $\delta[u] = 0$ for $u \neq 0$. Inserting this into (1.4), which in the case of integer oversampling reads

$$(S_g x) = qT \sum_{l=-\infty}^{\infty} x(t - lqT) \sum_{k=-\infty}^{\infty} g(t - kT)\, g^*(t - kT - lqT)\,,$$

it follows that $(S_g x)(t) = qT\, x(t) \sum_{k=-\infty}^{\infty} |g(t - kT)|^2$, which concludes the proof. □

For the proof of Corollary 2 we need the following lemma.

Lemma 3.

For a band-limited $g(t)$ the function $\bar{Z}_g^{(d)}(t, f) = e^{-j2\pi ft} Z_g^{(d)}(t, f) = F \sum_{k=-\infty}^{\infty} G(f + kF)\, e^{j2\pi kFt}$ is a polynomial in $e^{j2\pi Ft}$ for all f. If $G(f)$ is supported in $f \in [0, F_0)$ the maximum degree of $\bar{Z}_g^{(d)}(t, f)$ for $f \in [0, F)$ is given by $\left\lfloor \frac{F_0}{F} \right\rfloor$. Conversely, if $\bar{Z}_g^{(d)}(t, f)$ has a finite maximum degree for $f \in [0, F_0)$, the function $g(t)$ is band-limited.

Proof. The proof of the lemma is similar to that of Lemma 1 and will therefore be omitted. □

Proof of Corollary 2. Let us define the $p \times 1$ vectors

$$\begin{aligned} \bar{\mathbf{z}}_y^{(d)}(t, f) &= \left[\bar{Z}_y^{(d)}(t, f)\ \bar{Z}_y^{(d)}\left(t, f - \frac{F}{p}\right) \ldots \bar{Z}_y^{(d)}\left(t, f - (p-1)\frac{F}{p}\right)\right]^T \\ \bar{\mathbf{z}}_x^{(d)}(t, f) &= \left[\bar{Z}_x^{(d)}(t, f)\ \bar{Z}_x^{(d)}\left(t, f - \frac{F}{p}\right) \ldots \bar{Z}_x^{(d)}\left(t, f - (p-1)\frac{F}{p}\right)\right]^T . \end{aligned}$$

Straightforward manipulations reveal that

$$\bar{\mathbf{z}}_y^{(d)}(t, f) = \bar{\mathbf{S}}_g^{(d)}(t, f)\, \bar{\mathbf{z}}_x^{(d)}(t, f)\,, \tag{3.2}$$

with

$$\bar{\mathbf{S}}_g^{(d)}(t, f) = \mathbf{D}(t, f)\mathbf{S}_g^{(d)}(t, f)\mathbf{D}^{-1}(t, f)\,,$$

where $\mathbf{D}(t, f) = \mathrm{diag}\{e^{-j2\pi(f - i\frac{F}{p})t}\}_{i=0}^{p-1}$. For a band-limited $g(t)$, $\mathbf{S}_g^{(d)}(t, f)$ and $\bar{\mathbf{S}}_g^{(d)}(t, f)$ are polynomial matrices in $e^{j2\pi\frac{F}{p}t}$ for all f. Setting $z = e^{j2\pi\frac{F}{p}t}$ we shall henceforth write $\mathbf{S}_g^{(d)}(z, f)$ and $\bar{\mathbf{S}}_g^{(d)}(z, f)$ instead of $\mathbf{S}_g^{(d)}(t, f)$ and $\bar{\mathbf{S}}_g^{(d)}(t, f)$. Note that for band-limited signals $x(t)$ and $y(t)$ the vectors $\bar{\mathbf{z}}_x^{(d)}(z, f)$ and $\bar{\mathbf{z}}_y^{(d)}(z, f)$ are polynomial vectors in z for all f. Noting that $\det[\mathbf{S}_g^{(d)}(z, f)] = \det[\bar{\mathbf{S}}_g^{(d)}(z, f)]$ the rest of the proof is straightforward using the arguments developed in the proof of Theorem 1. □

Since $\det[\mathbf{S}_g^{(d)}(z, f)]$ is $\frac{F}{p}$-periodic in f [13, 14], it suffices to check the unimodularity of $\mathbf{S}_g^{(d)}(z, f)$ for $f \in \left[0, \frac{F}{p}\right)$.

When the WH frame operator S_g is a multiplication operator in the frequency-domain, i.e., $(\hat{S}_g X)(f) = \frac{Fq}{p} X(f) \sum_{k=-\infty}^{\infty} |G(f - kF)|^2$, and $g(t)$ is band-limited, then obviously the minimal dual $\gamma^0(t)$ is band-limited. In fact, the Fourier transform of $\gamma^0(t)$ has the same support as the Fourier transform of $g(t)$. The following lemma will be used to show that such a $g(t)$ trivially satisfies the conditions of Corollary 2.

Lemma 4.

Let $g(t)$ be band-limited such that $\hat{S}_g$ is a multiplication operator in the frequency domain. In this case the matrix $\mathbf{S}_g^{(d)}(z, f)$ is diagonal with elements $[\mathbf{S}_g^{(d)}(z, f)]_{k,k} = [\mathbf{S}_g^{(d)}(f)]_{k,k} = \frac{Fq}{p} \sum_{l=-\infty}^{\infty} |G(f - k\frac{F}{p} - lF)|^2$.

Proof. For $g(t)$ band-limited such that $\hat{S}_g$ is a multiplication operator in the frequency domain we have

$$\left(\hat{S}_g X\right)(f) = \frac{Fq}{p} X(f) \sum_{l=-\infty}^{\infty} |G(f - lF)|^2 .$$

With $y(t) = (S_g x)(t)$ we have

$$\mathcal{Z}_y^{(d)}(t, f) = \mathcal{Z}_x^{(d)}(t, f) \frac{Fq}{p} \sum_{l=-\infty}^{\infty} |G(f - lF)|^2 ,$$

which implies

$$\mathcal{Z}_y^{(d)}\left(t, f - k\frac{F}{p}\right) = \mathcal{Z}_x^{(d)}\left(t, f - k\frac{F}{p}\right) \frac{Fq}{p} \sum_{l=-\infty}^{\infty} \left|G\left(f - k\frac{F}{p} - lF\right)\right|^2 .$$

This establishes the result. ☐

From Lemma 4 it follows that the matrix $\mathbf{S}_g^{(d)}(z, f)$ is a function of f only and hence its determinant satisfies $\det[\mathbf{S}_g^{(d)}(z, f)] = d(f)$, which establishes that a band-limited $g(t)$ having a frame operator that is a multiplication operator in the frequency domain trivially satisfies the conditions of Corollary 2. For critical sampling, i.e., $TF = 1$, where $\bar{\mathcal{Z}}_{\gamma^0}^{(d)}(z, f) = \frac{1}{T \bar{\mathcal{Z}}_g^{(d)*}(z, f)}$, the minimal dual is band-limited if and only if $\bar{\mathcal{Z}}_g^{(d)}(z, f) = F \sum_{k=-\infty}^{\infty} G(f + kF)\, z^{kp}$ is a monomial in z for all f.

Proof of Corollary 3. The sufficiency is obvious. For the proof of necessity note that in the cases of integer oversampling and critical sampling ($p = 1$) the matrix $\mathbf{S}_g^{(d)}(z, f)$ reduces to a scalar given by $S_g^{(d)}(z, f) = qF \sum_{u=-\infty}^{\infty} z^{uq} \sum_{s=-\infty}^{\infty} G(f - sF) G^*(f - sF - uqF)$. It therefore follows from Corollary 2 that $\gamma^0(t)$ is band-limited if and only if $\sum_{s=-\infty}^{\infty} G(f - sF) G^*(f - sF - uqF) = \sum_{s=-\infty}^{\infty} |G(f - sF)|^2 \,\delta[u]$. Inserting this into the frequency domain version of the Walnut representation (1.5) of the WH frame operator, which in the case of integer oversampling reads

$$\left(\hat{S}_g X\right)(f) = qF \sum_{l=-\infty}^{\infty} X(f - lqF) \sum_{s=-\infty}^{\infty} G(f - sF)\, G^*(f - sF - lqF) ,$$

it follows that $(\hat{S}_g X)(f) = qF\, X(f) \sum_{k=-\infty}^{\infty} |G(f - kF)|^2$ which concludes the proof. ☐

4. Example

In this section we provide a simple example that demonstrates how to use the conditions derived in this note in practice. Consider

$$g(t) = \begin{cases} \frac{1}{\sqrt{2T}}, & 0 \le t \le 2T \\ 0, & \text{else} \end{cases}$$

and take $TF = \frac{2}{3}$, i.e., $p = 2$ and $q = 3$. Using (3.1) it follows that the Zibulski–Zeevi representation of the corresponding WH frame operator is given by

$$\mathbf{S}_g(t,z) = \begin{pmatrix} \frac{3}{2} & \frac{3}{4}\mathrm{rect}_{[T/2,T]}(t) + \frac{3}{4}z^{3}\mathrm{rect}_{[0,T/2]}(t) \\ \frac{3}{4}\mathrm{rect}_{[T/2,T]}(t) + \frac{3}{4}z^{-3}\mathrm{rect}_{[0,T/2]}(t) & \frac{3}{2} \end{pmatrix}, \quad (4.1)$$

where $\mathrm{rect}_{[a,b]}(t)$ denotes the T-periodic extension of a function that is equal to 1 for $t \in [a, b]$ and zero else. From the time domain Walnut representation (1.4) it follows that here the frame operator S_g is not a multiplication operator in the time domain. Furthermore, (4.1) shows that $g(t)$ does not generate a tight WH frame, since for a tight WH frame $\mathbf{S}_g(t, z) = \mathbf{I}_p$. Thus, it is not clear *a priori* that the minimal dual will be compactly supported. Nevertheless, using Theorem 1 with $\det[\mathbf{S}_g(t, z)] = \frac{27}{16}$ it follows that the minimal dual $\gamma^0(t)$ will be compactly supported. The calculation of $\gamma^0(t)$ can be accomplished using techniques described in [19, 20].

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