

WIGNER-TYPE a - b AND TIME-FREQUENCY ANALYSIS BASED ON CONJUGATE OPERATORS*

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Abstract—We extend the Wigner distribution (WD) to conjugate unitary operators \mathbf{A}_α and \mathbf{B}_β . The resulting “AB-WD” is defined both as an a - b representation and as a time-frequency representation. Important properties and relations of the WD are generalized to the AB-WD.

1 INTRODUCTION AND OUTLINE

Recently, general frameworks for *joint a - b representations* and *time-frequency representations* based on pairs of unitary operators \mathbf{A}_α and \mathbf{B}_β have been proposed [1]–[13]. In particular, the time shift operator \mathbf{T}_τ and the frequency shift operator \mathbf{F}_ν underlying Cohen’s class and the Wigner distribution (WD) [5], [14]–[17] have been generalized to the concept of *conjugate* (or *dual*) operators \mathbf{A}_α and \mathbf{B}_β [6, 8, 10, 12, 13]. Classes of a - b or time-frequency (TF) representations based on conjugate operators retain the structure of Cohen’s class. Hence, each such class contains a central “AB-WD.”

This paper introduces and discusses the AB-WD. Section 2 summarizes the theory of conjugate operators. Sections 3 and 4 introduce the AB-WD as an a - b and TF representation, respectively. A special case is considered in Section 5.

Cohen’s class and WD. Let $x(t) \in \mathcal{L}_2(\mathbb{R})$ be a signal with Fourier transform¹ $X(f) = \int_t x(t) e^{-j2\pi ft} dt$. Cohen’s class of quadratic TF representations (QTFRs) [5], [14]–[17] consists of all QTFRs $C_x(t, f)$ that are *covariant* to the time shift operator \mathbf{T}_τ and frequency shift operator \mathbf{F}_ν defined as $(\mathbf{T}_\tau x)(t) = x(t - \tau)$ and $(\mathbf{F}_\nu x)(t) = x(t) e^{j2\pi\nu t}$,

$$C_{\mathbf{F}_\nu \mathbf{T}_\tau x}(t, f) = C_x(t - \tau, f - \nu). \quad (1)$$

Any QTFR of Cohen’s class can be written as

$$C_x(t, f) = \int_{t_1} \int_{t_2} x(t_1) x^*(t_2) h^*(t_1 - t, t_2 - t) e^{-j2\pi f(t_1 - t_2)} dt_1 dt_2 \quad (2)$$

with a 2-D kernel function $h(t_1, t_2)$. The central QTFR of Cohen’s class is the WD [5], [14]–[17]

$$W_x(t, f) = \int_\tau x\left(t + \frac{\tau}{2}\right) x^*\left(t - \frac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau \quad (3)$$

$$= \int_\nu X\left(f + \frac{\nu}{2}\right) X^*\left(f - \frac{\nu}{2}\right) e^{j2\pi\nu t} d\nu, \quad (4)$$

from which any Cohen’s class QTFR can be derived as

$$C_x(t, f) = \int_{t'} \int_{f'} \psi(t - t', f - f') W_x(t', f') dt' df' \quad (5)$$

with the kernel $\psi(t, f) = \int_\tau h^*\left(-t + \frac{\tau}{2}, -t - \frac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau$.

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¹Integrals extend from $-\infty$ to ∞ unless specified otherwise.

2 CONJUGATE OPERATORS

Generalizing the shift operators \mathbf{T}_τ , \mathbf{F}_ν underlying Cohen’s class and the WD, we now consider two linear operators \mathbf{A}_α and \mathbf{B}_β indexed by parameters $\alpha \in \mathcal{G}$ and $\beta \in \mathcal{G}$ with $\mathcal{G} \subseteq \mathbb{R}$. These operators are assumed to be *unitary* on a linear signal space $\mathcal{X} \subseteq \mathcal{L}_2(\mathbb{R})$, and to satisfy identical *composition properties* $\mathbf{A}_{\alpha_2} \mathbf{A}_{\alpha_1} = \mathbf{A}_{\alpha_1 \bullet \alpha_2}$ and $\mathbf{B}_{\beta_2} \mathbf{B}_{\beta_1} = \mathbf{B}_{\beta_1 \bullet \beta_2}$, where (\mathcal{G}, \bullet) is a commutative group [1, 6, 9, 18]. The *eigenvalues* $\lambda_{\alpha, b}^A$ and *eigenfunctions* $u_b^A(t)$ of \mathbf{A}_α , defined by $(\mathbf{A}_\alpha u_b^A)(t) = \lambda_{\alpha, b}^A u_b^A(t)$, are indexed by a “dual” parameter b . The *A-Fourier transform* (A-FT) [1, 5, 13] is defined as

$$X_A(b) \triangleq \langle x, u_b^A \rangle = \int_t x(t) u_b^{A*}(t) dt. \quad (6)$$

Similar definitions apply to $\lambda_{\beta, a}^B$, $u_a^B(t)$ and the B-FT $X_B(a)$.

Two operators \mathbf{A}_α and \mathbf{B}_β as defined above are called *conjugate* [6, 10, 12] if $a \in \mathcal{G}$, $b \in \mathcal{G}$, and if application of one operator to an eigenfunction of the other operator merely produces a shift of the eigenfunction parameter, i.e., $(\mathbf{B}_\beta u_a^A)(t) = u_{b \bullet \beta}^A(t)$ and $(\mathbf{A}_\alpha u_b^B)(t) = u_{a \bullet \alpha}^B(t)$. The eigenvalues of conjugate operators can be written as [6, 10, 12]

$$\lambda_{\alpha, b}^A = e^{\pm j2\pi \mu(\alpha) \mu(b)}, \quad \lambda_{\beta, a}^B = e^{\mp j2\pi \mu(\beta) \mu(a)} = \lambda_{\beta, a}^{A*}, \quad (7)$$

where $\mu(g) \in \mathbb{R}$ maps (\mathcal{G}, \bullet) onto $(\mathbb{R}, +)$ in the sense that $\mu(g_1 \bullet g_2) = \mu(g_1) + \mu(g_2)$, $\mu(g_0) = 0$, and $\mu(g^{-1}) = -\mu(g)$ with g_0 the identity element in \mathcal{G} and g^{-1} the group-inverse of g . Due to (7), we shall write $\lambda_{\alpha, b}^A = \lambda_{\alpha, b}$ and $\lambda_{\beta, a}^B = \lambda_{\beta, a}^*$ in the following. Conjugate operators commute up to a phase factor, $\mathbf{A}_\alpha \mathbf{B}_\beta = \lambda_{\alpha, \beta} \mathbf{B}_\beta \mathbf{A}_\alpha$. Their eigenfunctions are related as $\langle u_a^B, u_b^A \rangle = \lambda_{a, b}$, $\int_{\mathcal{G}} u_a^B(t) \lambda_{b, a}^* d\mu(a) = u_b^A(t)$, and $\int_{\mathcal{G}} u_b^A(t) \lambda_{a, b} d\mu(b) = u_a^B(t)$, where $d\mu(g) = |\mu'(g)| dg$ if $\mu(g)$ is differentiable. The A-FT and B-FT are related as $X_B(a) = \int_{\mathcal{G}} X_A(b) \lambda_{a, b}^* d\mu(b)$ and $X_A(b) = \int_{\mathcal{G}} X_B(a) \lambda_{b, a} d\mu(a)$ (cf. the equivalent concept of “dual operators” in [8, 13]).

Examples. The shift operators \mathbf{T}_τ , \mathbf{F}_ν underlying Cohen’s class and the WD are conjugate with $(\mathcal{G}, \bullet) = (\mathbb{R}, +)$, $\mu(g) = g$, $b = f$, $a = t$, $\lambda_{\tau, f}^T = e^{-j2\pi\tau f}$, $\lambda_{\nu, t}^F = e^{j2\pi\nu t}$, $u_f^T(t) = e^{j2\pi ft}$, $u_t^F(t') = \delta(t' - t)$, $X_T(f) = X(f)$, and $X_F(t) = x(t)$. The operators \mathbf{T}_τ , \mathbf{F}_ν are conjugate since $(\mathbf{F}_\nu u_f^T)(t) = u_{f+\nu}^T(t)$ and $(\mathbf{T}_\tau u_t^F)(t') = u_{t+\tau}^F(t')$. They commute up to a phase factor, $\mathbf{T}_\tau \mathbf{F}_\nu = e^{-j2\pi\tau\nu} \mathbf{F}_\nu \mathbf{T}_\tau$.

The operators underlying the *hyperbolic* QTFR class [19, 20] are conjugate as well, but the operators underlying the *affine class* and the *power classes* [21]–[24] are *not* conjugate.

3 AB-WD AS a - b REPRESENTATION

We now introduce² the AB-WD as an extension of the WD in (3), (4) to arbitrary conjugate operators \mathbf{A}_α and \mathbf{B}_β :

$$W_x^{AB}(a, b) \triangleq \int_{\mathcal{G}} X_A(b \bullet \beta^{1/2}) X_A^*(b \bullet \beta^{-1/2}) \lambda_{\alpha, \beta}^* d\mu(\beta),$$

where $X_A(b)$ is defined in (6) and $\beta^{1/2}$ is defined by $\beta^{1/2} \bullet \beta^{1/2} = \beta$. The AB-WD is a function of the parameter a of $u_a^B(t)$ and the parameter b of $u_b^A(t)$ (see Section 4 for a time-frequency version of the AB-WD). It can be equivalently expressed in terms of the B-FT,

$$W_x^{AB}(a, b) = \int_{\mathcal{G}} X_B(a \bullet \alpha^{1/2}) X_B^*(a \bullet \alpha^{-1/2}) \lambda_{b, \alpha} d\mu(\alpha).$$

For $\mathbf{A}_\alpha = \mathbf{T}_\tau$ and $\mathbf{B}_\beta = \mathbf{F}_\nu$, these two expressions reduce to (4) and (3), respectively, so that $W_x^{TF}(a, b) = W_x(t, f)$. On the other hand, the AB-WD can be formally obtained from the WD by a unitary signal transformation and a parameter transformation $(t, f) \rightarrow (a, b)$ [10]-[12]: If $\lambda_{\alpha, b} = e^{-j2\pi \mu(\alpha) \mu(b)}$ (minus sign in the exponent), then

$$W_x^{AB}(a, b) = W_{\tilde{x}}\left(t_r \mu(a), \frac{\mu(b)}{t_r}\right), \quad \tilde{x}(t) = \frac{1}{\sqrt{t_r}} X_B\left(\mu^{-1}\left(\frac{t}{t_r}\right)\right)$$

where $t_r > 0$ is a fixed reference time and $\mu^{-1}(\cdot)$ is the function inverse to $\mu(\cdot)$. If $\lambda_{\alpha, b} = e^{j2\pi \mu(\alpha) \mu(b)}$, then

$$W_x^{AB}(a, b) = W_{\tilde{x}}\left(t_r \mu(b), \frac{\mu(a)}{t_r}\right), \quad \tilde{x}(t) = \frac{1}{\sqrt{t_r}} X_A\left(\mu^{-1}\left(\frac{t}{t_r}\right)\right).$$

Properties of the AB-WD. The properties of the AB-WD generalize the properties of the WD [5], [14]-[17]:

Real-valued: $W_x^{AB}(a, b) \in \mathbb{R}$.

Covariance property:

$$W_{\mathbf{B}_\beta \mathbf{A}_\alpha x}^{AB}(a, b) = W_x^{AB}(a \bullet \alpha^{-1}, b \bullet \beta^{-1}). \quad (8)$$

Marginal properties:

$$\int_{\mathcal{G}} W_x^{AB}(a, b) d\mu(b) = |X_B(a)|^2, \quad (9)$$

$$\int_{\mathcal{G}} W_x^{AB}(a, b) d\mu(a) = |X_A(b)|^2. \quad (10)$$

Energy distribution property:

$$\iint_{\mathcal{G}^2} W_x^{AB}(a, b) d\mu(a) d\mu(b) = \|x\|^2 = \int_t |x(t)|^2 dt.$$

Moyal's formula/unitarity:

$$\iint_{\mathcal{G}^2} W_x^{AB}(a, b) W_y^{AB}(a, b) d\mu(a) d\mu(b) = |\langle x, y \rangle|^2.$$

Eigenfunction localization properties:

$$W_{u_{b_0}^A}^{AB}(a, b) = \delta(\mu(b \bullet b_0^{-1})) = \delta(\mu(b) - \mu(b_0)),$$

$$W_{u_{a_0}^B}^{AB}(a, b) = \delta(\mu(a \bullet a_0^{-1})) = \delta(\mu(a) - \mu(a_0)).$$

²While only the auto AB-WD will be considered for simplicity, we note that extension to the cross AB-WD is straightforward.

Interference formula:

$$\begin{aligned} [W_x^{AB}(a, b)]^2 &= \iint_{\mathcal{G}^2} W_x^{AB}(a \bullet \alpha^{1/2}, b \bullet \beta^{1/2}) \\ &\quad \cdot W_x^{AB}(a \bullet \alpha^{-1/2}, b \bullet \beta^{-1/2}) d\mu(\alpha) d\mu(\beta). \end{aligned}$$

Relation with AB-AF: We next introduce the AB-ambiguity function (AB-AF) as

$$\begin{aligned} A_x^{AB}(\alpha, \beta) &\triangleq \langle \mathbf{B}_{\beta^{-1/2}} \mathbf{A}_{\alpha^{-1/2}} x, \mathbf{B}_{\beta^{1/2}} \mathbf{A}_{\alpha^{1/2}} x \rangle \\ &= \int_{\mathcal{G}} X_A(b \bullet \beta^{1/2}) X_A^*(b \bullet \beta^{-1/2}) \lambda_{\alpha, b}^* d\mu(b) \\ &= \int_{\mathcal{G}} X_B(a \bullet \alpha^{1/2}) X_B^*(a \bullet \alpha^{-1/2}) \lambda_{\beta, a} d\mu(a). \end{aligned}$$

For $\mathbf{A}_\alpha = \mathbf{T}_\tau$ and $\mathbf{B}_\beta = \mathbf{F}_\nu$, the AB-AF reduces to the conventional AF [5], [14]-[17]: $A_x^{TF}(\alpha, \beta) = A_x(\tau, \nu) = \int_t x(t + \frac{\tau}{2}) x^*(t - \frac{\tau}{2}) e^{-j2\pi \nu t} dt = \int_f X(f + \frac{\nu}{2}) X^*(f - \frac{\nu}{2}) e^{j2\pi \tau f} df$. For $\lambda_{\alpha, b} = e^{-j2\pi \mu(\alpha) \mu(b)}$ there is $A_x^{AB}(\alpha, \beta) = A_{\tilde{x}}(t_r \mu(\alpha), \mu(\beta)/t_r)$ with $\tilde{x}(t) = \frac{1}{\sqrt{t_r}} X_B(\mu^{-1}(t/t_r))$, and for $\lambda_{\alpha, b} = e^{j2\pi \mu(\alpha) \mu(b)}$ there is $A_x^{AB}(\alpha, \beta) = A_{\tilde{x}}(t_r \mu(\beta), \mu(\alpha)/t_r)$ with $\tilde{x}(t) = \frac{1}{\sqrt{t_r}} X_A(\mu^{-1}(t/t_r))$.

The AB-AF is related to the AB-WD as

$$A_x^{AB}(\alpha, \beta) = \iint_{\mathcal{G}^2} W_x^{AB}(a, b) \lambda_{\beta, a} \lambda_{\alpha, b}^* d\mu(a) d\mu(b)$$

and

$$\begin{aligned} |A_x^{AB}(\alpha, \beta)|^2 &= \iint_{\mathcal{G}^2} W_x^{AB}(a \bullet \alpha^{1/2}, b \bullet \beta^{1/2}) \\ &\quad \cdot W_x^{AB}(a \bullet \alpha^{-1/2}, b \bullet \beta^{-1/2}) d\mu(a) d\mu(b). \end{aligned}$$

Uncertainty relations: We define the A -spread σ_x^A , B -spread σ_x^B , and AB -radius $\rho_x^{AB}(b_0)$ as $\sigma_x^{A2} \triangleq \frac{\int_{\mathcal{G}} \mu^2(b) |X_A(b)|^2 d\mu(b)}{\int_{\mathcal{G}} |X_A(b)|^2 d\mu(b)}$, $\sigma_x^{B2} \triangleq \frac{\int_{\mathcal{G}} \mu^2(a) |X_B(a)|^2 d\mu(a)}{\int_{\mathcal{G}} |X_B(a)|^2 d\mu(a)}$, and $\rho_x^{AB2}(b_0) \triangleq \left(\frac{\sigma_x^A}{b_0}\right)^2 + (b_0 \sigma_x^B)^2$ with $b_0 \neq 0$. These quantities are related to the AB-WD as

$$\sigma_x^{A2} = \frac{\iint_{\mathcal{G}^2} \mu^2(b) W_x^{AB}(a, b) d\mu(a) d\mu(b)}{\iint_{\mathcal{G}^2} W_x^{AB}(a, b) d\mu(a) d\mu(b)}$$

$$\sigma_x^{B2} = \frac{\iint_{\mathcal{G}^2} \mu^2(a) W_x^{AB}(a, b) d\mu(a) d\mu(b)}{\iint_{\mathcal{G}^2} W_x^{AB}(a, b) d\mu(a) d\mu(b)}$$

$$\rho_x^{AB2}(b_0) = \frac{\iint_{\mathcal{G}^2} \left[\left(\frac{\mu(b)}{b_0}\right)^2 + (b_0 \mu(a))^2 \right] W_x^{AB}(a, b) d\mu(a) d\mu(b)}{\iint_{\mathcal{G}^2} W_x^{AB}(a, b) d\mu(a) d\mu(b)}$$

and there exist the bounds (uncertainty relations)

$$\sigma_x^A \sigma_x^B \geq \frac{1}{4\pi}, \quad \rho_x^{AB}(b_0) \geq \frac{1}{\sqrt{2\pi}} \quad \forall b_0 \neq 0.$$

For the next properties, we assume $\lambda_{\alpha, b} = e^{-j2\pi \mu(\alpha) \mu(b)}$. Analogous properties hold for $\lambda_{\alpha, b} = e^{j2\pi \mu(\alpha) \mu(b)}$.

Generalized chirp localization property:

$$W_x^{AB}(a, b) = \delta(\mu(b) - c\mu(a))$$

for $X_B(a) = e^{j\pi c \mu^2(a)}$, i.e., $x(t) = \int_{\mathcal{G}} e^{j\pi c \mu^2(a)} u_a^B(t) d\mu(a)$.

Generalized instantaneous frequency property:

$$\frac{\int_{\mathcal{G}} \mu(b) W_x^{AB}(a, b) d\mu(b)}{\int_{\mathcal{G}} W_x^{AB}(a, b) d\mu(b)} = \frac{1}{2\pi\mu'(a)} \frac{d}{da} \arg\{X_B(a)\}.$$

Generalized group delay property:

$$\frac{\int_{\mathcal{G}} \mu(a) W_x^{AB}(a, b) d\mu(a)}{\int_{\mathcal{G}} W_x^{AB}(a, b) d\mu(a)} = -\frac{1}{2\pi\mu'(b)} \frac{d}{db} \arg\{X_A(b)\}.$$

Multiplication property: Let the B-FTs of $x(t)$, $y(t)$, and $z(t)$ be related as $Z_B(a) = X_B(a)Y_B(a)$, which can be shown to entail $Z_A(b) = \int_{\mathcal{G}} X_A(b \bullet \beta^{-1}) Y_A(\beta) d\mu(\beta)$ and $z(t) = \int_{\mathcal{G}} Y_A(\beta) (\mathbf{B}_\beta x)(t) d\mu(\beta) = \int_{\mathcal{G}} X_A(\beta) (\mathbf{B}_\beta y)(t) d\mu(\beta)$. Then

$$W_z^{AB}(a, b) = \int_{\mathcal{G}} W_x^{AB}(a, b \bullet \beta^{-1}) W_y^{AB}(a, \beta) d\mu(\beta).$$

Convolution property: Similarly, if $Z_B(a) = \int_{\mathcal{G}} X_B(a \bullet \alpha^{-1}) Y_B(\alpha) d\mu(\alpha)$ such that $Z_A(b) = X_A(b)Y_A(b)$ and $z(t) = \int_{\mathcal{G}} Y_B(\alpha) (\mathbf{A}_\alpha x)(t) d\mu(\alpha) = \int_{\mathcal{G}} X_B(\alpha) (\mathbf{A}_\alpha y)(t) d\mu(\alpha)$, then

$$W_z^{AB}(a, b) = \int_{\mathcal{G}} W_x^{AB}(a \bullet \alpha^{-1}, b) W_y^{AB}(\alpha, b) d\mu(\alpha).$$

Covariant a - b representations. The class of all quadratic a - b representations $Q_x(a, b)$ that are covariant to conjugate operators \mathbf{A}_α , \mathbf{B}_β as (cf. (8),(1)) $Q_{\mathbf{B}_\beta \mathbf{A}_\alpha x}(a, b) = Q_x(a \bullet \alpha^{-1}, b \bullet \beta^{-1})$ can be formulated as [6, 10, 12]

$$Q_x(a, b) = \langle x, \mathbf{H}_{a,b}^{AB} x \rangle = \quad (11)$$

$$\int_{t_1} \int_{t_2} (\mathbf{A}_{\alpha^{-1} \mathbf{B}_{\beta^{-1}}} x)(t_1) (\mathbf{A}_{\alpha^{-1} \mathbf{B}_{\beta^{-1}}} x)^*(t_2) h^*(t_1, t_2) dt_1 dt_2$$

with $\mathbf{H}_{a,b}^{AB} = \mathbf{B}_b \mathbf{A}_a \mathbf{H} \mathbf{A}_{a^{-1}} \mathbf{B}_{b^{-1}}$, where \mathbf{H} is an arbitrary linear operator with kernel $h(t_1, t_2)$. Equivalently,

$$Q_x(a, b) = \iint_{\mathcal{G}^2} \Psi(\alpha, \beta) A_x^{AB}(\alpha, \beta) \lambda_{\beta, a}^* \lambda_{\alpha, b} d\mu(\alpha) d\mu(\beta) \quad (12)$$

where the kernel $\Psi(\alpha, \beta)$ is related to $h(t_1, t_2)$ [6, 10, 12]. The covariant class $\{Q_x(a, b)\}$ is the extension of Cohen's class in (2) to arbitrary conjugate operators \mathbf{A}_α , \mathbf{B}_β . In particular, the AB-WD is obtained with $\Psi(\alpha, \beta) \equiv 1$.

It can be shown that the covariant class $\{Q_x(a, b)\}$ can be derived from the AB-WD $W_x^{AB}(a, b)$ as (cf. (5))

$$Q_x(a, b) = \iint_{\mathcal{G}^2} \psi(a \bullet \alpha^{-1}, b \bullet \beta^{-1}) W_x^{AB}(\alpha, \beta) d\mu(\alpha) d\mu(\beta), \quad (13)$$

where the kernel $\psi(a, b)$ is related to the kernel $\Psi(\alpha, \beta)$ in (12) as $\psi(a, b) = \iint_{\mathcal{G}^2} \Psi(\alpha, \beta) \lambda_{\beta, a}^* \lambda_{\alpha, b} d\mu(\alpha) d\mu(\beta)$.

AB-spectrogram. Setting $h(t_1, t_2) = g(t_1)g^*(t_2)$ in (11) yields the AB-spectrogram

$$S_x^{AB}(a, b) = |L_x^{AB}(a, b)|^2 \quad \text{with } L_x^{AB}(a, b) = \langle x, \mathbf{B}_b \mathbf{A}_a g \rangle.$$

Here, $\psi(a, b) = W_g^{AB}(a^{-1}, b^{-1})$ so that $S_x^{AB}(a, b) = \iint_{\mathcal{G}^2} W_g^{AB}(\alpha \bullet a^{-1}, \beta \bullet b^{-1}) W_x^{AB}(\alpha, \beta) d\mu(\alpha) d\mu(\beta)$.

4 AB-WD AS TF REPRESENTATION

The AB-WD can be re-formulated as a quadratic time-frequency (TF) representation. Let $\nu_b^A(t)$ denote the instantaneous frequency of the eigenfunction $u_b^A(t)$, and let $\tau_a^B(f)$ denote the group delay of the eigenfunction $u_a^B(t)$. For any $(a, b) \in \mathcal{G}^2$, the corresponding functions $\nu_b^A(t)$ and $\tau_a^B(f)$ are assumed³ to intersect in a unique TF point (t, f) . Hence, there is a one-to-one correspondence $(t, f) = l(a, b)$, $(a, b) = l^{-1}(t, f)$ where $l(\cdot, \cdot)$ will be called the *localization function* of the operator pair \mathbf{A}_α , \mathbf{B}_β . The localization function is constructed by solving the system of equations $\nu_b^A(t) = f$, $\tau_a^B(f) = t$ for (t, f) given (a, b) [11, 3, 6].

The TF version of the AB-WD is now defined as

$$\widetilde{W}_x^{AB}(t, f) \triangleq W_x^{AB}(a, b)|_{(a,b)=l^{-1}(t,f)}.$$

All properties of the a - b version of the AB-WD (see Section 3) can be re-formulated for the TF version of the AB-WD. For example, the TF version of the covariance (8) reads

$$\widetilde{W}_{\mathbf{B}_\beta \mathbf{A}_\alpha x}^{AB}(t, f) = \widetilde{W}_x^{AB}\left(l\left(l^{-1}(t, f) \circ (\alpha^{-1}, \beta^{-1})\right)\right), \quad (14)$$

where $(a, b) \circ (\alpha, \beta) \triangleq (a \bullet \alpha, b \bullet \beta)$. The marginal properties (9), (10) become

$$\int_f \widetilde{W}_x^{AB}(\tau_a^B(f), f) d\bar{\mu}_1(f; a) = |X_B(a)|^2, \quad (15)$$

$$\int_t \widetilde{W}_x^{AB}(t, \nu_b^A(t)) d\bar{\mu}_2(t; b) = |X_A(b)|^2, \quad (16)$$

where $d\bar{\mu}_1(f; a)$ and $d\bar{\mu}_2(t; b)$ follow from $\tau_a^B(\cdot)$, $\nu_b^A(\cdot)$, and $d\mu(\cdot)$. All other properties and relations listed in Section 3 can be transferred to the TF domain as well.

5 AN EXAMPLE

We shall finally consider an example. Let the operators \mathbf{A}_α and \mathbf{B}_β be defined on the space $\mathcal{X} = \mathcal{L}_2(\mathbb{R}_+)$ as

$$(\mathbf{A}_\alpha x)(t) = \frac{1}{\sqrt{\alpha}} x\left(\frac{t}{\alpha}\right), \quad (\mathbf{B}_\beta x)(t) = e^{j2\pi \ln \beta \ln(t/t_r)} x(t)$$

where $t, \alpha, \beta > 0$ with $t_r > 0$ fixed. Since $\mathbf{A}_{\alpha_2} \mathbf{A}_{\alpha_1} = \mathbf{A}_{\alpha_1 \alpha_2}$ and $\mathbf{B}_{\beta_2} \mathbf{B}_{\beta_1} = \mathbf{B}_{\beta_1 \beta_2}$, the underlying group is the multiplicative group, $(\mathcal{G}, \bullet) = (\mathbb{R}_+, \cdot)$, with identity element $g_0 = 1$ and inverse elements $g^{-1} = 1/g$. The eigenvalues/functions of \mathbf{A}_α and \mathbf{B}_β are $\lambda_{\alpha, b}^A = e^{-j2\pi \ln \alpha \ln b}$, $u_b^A(t) = \frac{1}{\sqrt{t}} e^{j2\pi \ln b \ln(t/t_r)}$ and $\lambda_{\beta, a}^B = e^{j2\pi \ln \beta \ln a}$, $u_a^B(t) = \frac{1}{\sqrt{t_r \alpha}} \delta(\ln \frac{t}{t_r} - \ln a)$. Note that $\mu(g) = \ln g$ and $d\mu(g) = \frac{dg}{g}$. The A-FT is the Mellin-type transform $X_A(b) = \int_0^\infty x(t) e^{-j2\pi \ln b \ln(t/t_r)} \frac{dt}{\sqrt{t}}$ and the B-FT is $X_B(a) = \sqrt{t_r a} x(t_r a)$. The operators \mathbf{A}_α and \mathbf{B}_β are conjugate, $(\mathbf{B}_\beta u_b^A)(t) = u_b^A(t)$ and $(\mathbf{A}_\alpha u_a^B)(t) = u_a^B(t)$. They commute up to a phase factor, $\mathbf{A}_\alpha \mathbf{B}_\beta = e^{-j2\pi \ln \alpha \ln \beta} \mathbf{B}_\beta \mathbf{A}_\alpha$.

The a - b version of the AB-WD is the a - b , time-domain version of the Q -distribution [19, 25, 26]

$$W_x^{AB}(a, b) = t_r a \int_{-\infty}^{\infty} x(t_r a e^{u/2}) x^*(t_r a e^{-u/2}) e^{-j2\pi(\ln b)u} du$$

³In certain cases, this assumption holds if one uses the group delay of $u_b^A(t)$ and the instantaneous frequency of $u_a^B(t)$; here, an analogous theory can be formulated.

with $a, b > 0$. It satisfies the covariance property

$$W_{\mathbf{B}\beta\mathbf{A}\alpha x}^{AB}(a, b) = W_x^{AB}\left(\frac{a}{\alpha}, \frac{b}{\beta}\right),$$

the marginal properties

$$\int_0^\infty W_x^{AB}(a, b) \frac{db}{b} = t_r a |x(t_r a)|^2,$$

$$\int_0^\infty W_x^{AB}(a, b) \frac{da}{a} = \left| \int_0^\infty x(t) e^{-j2\pi \ln b \ln(t/t_r)} \frac{dt}{\sqrt{t}} \right|^2,$$

and other properties (cf. Section 3). The covariant class (11) is the a - b , time-domain version of the *hyperbolic class* [19, 20]

$$Q_x(a, b) = \frac{1}{a} \int_0^\infty \int_0^\infty x(t_1) x^*(t_2) h^*\left(\frac{t_1}{a}, \frac{t_2}{a}\right) e^{-j2\pi(\ln b) \ln(t_1/t_2)} dt_1 dt_2;$$

it can be derived from $W_x^{AB}(a, b)$ as (see (13))

$$Q_x(a, b) = \int_0^\infty \int_0^\infty \psi\left(\frac{a}{\alpha}, \frac{b}{\beta}\right) W_x^{AB}(\alpha, \beta) \frac{d\alpha}{\alpha} \frac{d\beta}{\beta}.$$

With $\nu_b^A(t) = (\ln b)/t$ and $\tau_a^B(f) \equiv t_r a$, the localization function is obtained as $(t, f) = l(a, b) = (t_r a, \frac{\ln b}{t_r a})$ with inverse $(a, b) = l^{-1}(t, f) = (\frac{t}{t_r}, e^{tf})$. The TF version of the AB-WD is then [19, 25, 26]

$$\widetilde{W}_x^{AB}(t, f) = W_x^{AB}\left(\frac{t}{t_r}, e^{tf}\right)$$

$$= t \int_{-\infty}^\infty x(t e^{u/2}) x^*(t e^{-u/2}) e^{-j2\pi t f u} du$$

for $t > 0$; it satisfies the covariance property (cf. (14))

$$\widetilde{W}_{\mathbf{B}\beta\mathbf{A}\alpha x}^{AB}(t, f) = \widetilde{W}_x^{AB}\left(\frac{t}{\alpha}, \alpha\left(f - \frac{\ln \beta}{t}\right)\right) \quad (17)$$

and the marginal properties (cf. (15), (16))

$$\int_{-\infty}^\infty \widetilde{W}_x^{AB}(t, f) df = |x(t)|^2,$$

$$\int_0^\infty \widetilde{W}_x^{AB}\left(t, \frac{\ln b}{t}\right) \frac{dt}{t} = \left| \int_0^\infty x(t) e^{-j2\pi \ln b \ln(t/t_r)} \frac{dt}{\sqrt{t}} \right|^2.$$

The class of QTFRs satisfying the covariance (17) is [19, 20]

$$\tilde{Q}_x(t, f) = Q_x\left(\frac{t}{t_r}, e^{tf}\right) = \frac{t_r}{t} \int_0^\infty \int_0^\infty x(t_1) x^*(t_2) h^*\left(t_r \frac{t_1}{t}, t_r \frac{t_2}{t}\right) e^{-j2\pi t f \ln(t_1/t_2)} dt_1 dt_2;$$

it can be derived from $\widetilde{W}_x^{AB}(t, f)$ as

$$\tilde{Q}_x(t, f) = \int_{t'=0}^\infty \int_{f'=-\infty}^\infty \psi\left(\frac{t}{t'}, e^{tf-t'f'}\right) \widetilde{W}_x^{AB}(t', f') dt' df'.$$

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