

DISPLACEMENT-COVARIANT TIME-FREQUENCY ENERGY DISTRIBUTIONS*

Franz Hlawatsch and Helmut Bölcskei

INTHFT, Technische Universität Wien, Gusshausstrasse 25/389, A-1040 Vienna, Austria
email address: fhlawats@email.tuwien.ac.at

Abstract—We present a theory of quadratic time-frequency (TF) energy distributions that satisfy a covariance property and generalized marginal properties. The theory coincides with the characteristic function method of Cohen and Baraniuk in the special case of “conjugate operators.”

1 INTRODUCTION AND OUTLINE

Important classes of quadratic time-frequency representations (QTFRs), such as Cohen’s class¹ and the affine, hyperbolic, and power classes [1]–[8], are special cases within a general theory of *displacement-covariant QTFRs* [9]. This theory (briefly reviewed in Section 2) is based on the concept of *time-frequency displacement operators* (DOs).

In Section 3, we shall consider the important *separable case* where a DO can be decomposed into two “partial DOs” (PDOs). Section 4 defines *marginal properties* associated to the PDOs and derives constraints on the QTFR kernels. Section 5 shows that, for “conjugate” PDOs, our theory coincides with the characteristic function method of [10, 11].

2 DISPLACEMENT-COVARIANT QTFRs

Time-Frequency Displacement Operators. A DO is a family of unitary, linear operators \mathbf{D}_θ defined on a linear space $\mathcal{X} \subseteq \mathcal{L}_2(\mathbb{R})$ of finite-energy signals $x(t)$, and indexed by the 2D “displacement parameter” $\theta = (\alpha, \beta) \in \mathcal{D}$ with $\mathcal{D} \subseteq \mathbb{R}^2$. By definition, \mathbf{D}_θ obeys a *composition law*

$$\mathbf{D}_{\theta_2} \mathbf{D}_{\theta_1} = e^{j\sigma(\theta_1, \theta_2)} \mathbf{D}_{\theta_1 \circ \theta_2} \quad (1)$$

where \circ is a binary operation such that \mathcal{D} and \circ form a group² with identity element θ_0 and inverse element θ^{-1} . The TF displacements produced by a DO are described by its *displacement function* (DF) $d(z, \theta)$: if a signal $x(t)$ is localized about a TF point $z = (t, f)$, then $(\mathbf{D}_\theta x)(t)$ is localized about some other TF point $z' = (t', f')$ given by

$$z' = d(z, \theta),$$

which is short for $t' = d_1(t, f; \alpha, \beta)$, $f' = d_2(t, f; \alpha, \beta)$. The DF’s construction is discussed in [9]. The DF is assumed to be an invertible, area-preserving mapping of \mathcal{Z} onto \mathcal{Z} (where $\mathcal{Z} \subseteq \mathbb{R}^2$ denotes the set of TF points $z = (t, f)$), and to obey the composition law (cf. (1))

$$d(d(z, \theta_1), \theta_2) = d(z, \theta_1 \circ \theta_2). \quad (2)$$

The *parameter function* $p(z', z)$ of \mathbf{D}_θ yields the displacement parameter θ that maps z into z' ,

*Funding by FWF grant P10012-ÖPH.

¹Short for *Cohen’s class with signal-independent kernels*.

²The group axioms are (i) $\theta_1 \circ \theta_2 \in \mathcal{D}$ for $\theta_1, \theta_2 \in \mathcal{D}$, (ii) $\theta_1 \circ (\theta_2 \circ \theta_3) = (\theta_1 \circ \theta_2) \circ \theta_3$, (iii) $\theta \circ \theta_0 = \theta_0 \circ \theta = \theta$, and (iv) $\theta^{-1} \circ \theta = \theta \circ \theta^{-1} = \theta_0$.

$$z' = d(z, \theta) \Leftrightarrow \theta = p(z', z),$$

which is short for $\alpha = p_1(t', f'; t, f)$, $\beta = p_2(t', f'; t, f)$.

Two Examples. The *TF-shift operator* $\mathbf{S}_{\tau, \nu}$, defined as $(\mathbf{S}_{\tau, \nu} x)(t) = x(t - \tau) e^{j2\pi\nu t}$, is a DO with composition law (1) $\mathbf{S}_{\tau_2, \nu_2} \mathbf{S}_{\tau_1, \nu_1} = e^{-j2\pi\nu_1\tau_2} \mathbf{S}_{\tau_1 + \tau_2, \nu_1 + \nu_2}$, DF $t' = d_1(t, f; \tau, \nu) = t + \tau$, $f' = d_2(t, f; \tau, \nu) = f + \nu$, and parameter function $\tau = p_1(t', f'; t, f) = t' - t$, $\nu = p_2(t', f'; t, f) = f' - f$. Another DO is the *time-shift/TF-scaling operator* $\mathbf{C}_{a, \tau}$ defined as $(\mathbf{C}_{a, \tau} x)(t) = \sqrt{a} x(a(t - \tau))$ ($a > 0$), with $\mathbf{C}_{a_2, \tau_2} \mathbf{C}_{a_1, \tau_1} = \mathbf{C}_{a_1 a_2, \tau_1/a_2 + \tau_2}$, DF $t' = d_1(t, f; a, \tau) = t/a + \tau$, $f' = d_2(t, f; a, \tau) = af$, and parameter function $a = p_1(t', f'; t, f) = f'/f$, $\tau = p_2(t', f'; t, f) = t' - tf/f'$.

Displacement-Covariant QTFRs. A QTFR $T_x(t, f) = T_x(z)$ is called *covariant to a DO* \mathbf{D}_θ if

$$\mathbf{T}_{\mathbf{D}_\theta x}(z) = T_x(\bar{z}) \quad \text{with } \bar{z} = d(z, \theta^{-1}). \quad (3)$$

It can be shown [9] that all QTFRs satisfying the covariance property (3) are given by the 2D inner product³

$$T_x(z) = \int_{t_1} \int_{t_2} x(t_1) x^*(t_2) (\mathbf{D}_{p(z, z_0)}^\otimes h)^*(t_1, t_2) dt_1 dt_2 \quad (4)$$

$$= \int_{f_1} \int_{f_2} X(f_1) X^*(f_2) (\hat{\mathbf{D}}_{p(z, z_0)}^\otimes H)^*(f_1, f_2) df_1 df_2 \quad (5)$$

where $h(t_1, t_2)$ is a 2D “kernel” (independent of $x(t)$), $z_0 \in \mathcal{Z}$ is a fixed reference TF point, $\mathbf{D}_\theta^\otimes$ is the outer product of \mathbf{D}_θ by itself⁴, $X(f) = \mathcal{F}_{t \rightarrow f} x(t)$, $\hat{\mathbf{D}}_\theta = \mathcal{F} \mathbf{D}_\theta \mathcal{F}^{-1}$, and $H(f_1, f_2) = \mathcal{F}_{t_1 \rightarrow f_1} \mathcal{F}_{t_2 \rightarrow -f_2} h(t_1, t_2)$. Conversely, all QTFRs (4),(5) are covariant to \mathbf{D}_θ . We note that (4) can be written as the quadratic form

$$T_x(z) = \langle x, \mathbf{H}_z^D x \rangle \quad \text{with } \mathbf{H}_z^D = \mathbf{D}_{p(z, z_0)} \mathbf{H} \mathbf{D}_{p(z, z_0)}^{-1}, \quad (6)$$

where \mathbf{H} is the linear operator whose kernel is $h(t_1, t_2)$, i.e. $(\mathbf{H}x)(t) = \int_{t'} h(t, t') x(t') dt'$, and $\langle x, y \rangle = \int_t x(t) y^*(t) dt$.

Examples. For $\mathbf{D}_\theta = \mathbf{S}_{\tau, \nu}$ and $z_0 = (0, 0)$, (3) becomes the TF-shift covariance $\mathbf{T}_{\mathbf{S}_{\tau, \nu} x}(t, f) = T_x(t - \tau, f - \nu)$ and (4) becomes *Cohen’s class* [1]–[3]

³Integrals are over the functions’ support.

⁴ $\mathbf{D}_\theta^\otimes$ acts on a 2D function $y(t_1, t_2)$ as $(\mathbf{D}_\theta^\otimes y)(t_1, t_2) = \int_{t'_1} \int_{t'_2} D_\theta(t_1, t'_1) D_\theta^*(t_2, t'_2) y(t'_1, t'_2) dt'_1 dt'_2$, where $D_\theta(t, t')$ is the kernel of \mathbf{D}_θ . For example, $(\mathbf{S}_{\tau, \nu}^\otimes y)(t_1, t_2) = y(t_1 - \tau, t_2 - \tau) e^{j2\pi\nu(t_1 - t_2)}$ and $(\mathbf{C}_{a, \tau}^\otimes y)(t_1, t_2) = a y(a(t_1 - \tau), a(t_2 - \tau))$.

$$T_\tau(t, f) = \int_{t_1}^{t_2} \int_{t_1}^{t_2} x(t_1) x^*(t_2) h^*(t_1 - t, t_2 - t) e^{-j2\pi f(t_1 - t_2)} dt_1 dt_2.$$

For $\mathbf{D}_\theta = \mathbf{C}_{a, \tau}$ and $z_0 = (0, f_0)$ (with fixed $f_0 > 0$), (7) becomes the time-shift/TF-scaling covariance $T_{\mathbf{C}_{a, \tau} x}(t, f) = T_x(a(t - \tau), f/a)$ and (4) becomes the *affine class* [4, 5]

$$T_x(t, f) = \frac{f}{f_0} \int_{t_1}^{t_2} \int_{t_1}^{t_2} x(t_1) x^*(t_2) h^*\left(\frac{f}{f_0}(t_1 - t), \frac{f}{f_0}(t_2 - t)\right) dt_1 dt_2$$

for $f > 0$. Further special cases of (4)-(6) are the *hyperbolic class* and the *power classes* [6]-[9].

3 THE SEPARABLE CASE

The next theorem (obtained from (1), (2)) considers a *separable* DO that can be decomposed into two "partial DOs."

Theorem 1. Let \mathbf{D}_θ with $\theta = (\alpha, \beta)$, $\mathcal{D} = \mathcal{A} \times \mathcal{B}$ be a DO with identity parameter $\theta_0 = (\alpha_0, \beta_0)$, and define $\theta_\alpha = (\alpha, \beta_0)$ and $\theta_\beta = (\alpha_0, \beta)$. If

$$\theta_\alpha \circ \theta_\beta = \theta, \quad \theta_{\alpha_1} \circ \theta_{\alpha_2} = \theta_{\alpha_{12}}, \quad \theta_{\beta_1} \circ \theta_{\beta_2} = \theta_{\beta_{12}} \quad (9)$$

with $\alpha_{12} = \alpha_1 \bullet \alpha_2$ and $\beta_{12} = \beta_1 * \beta_2$, where \bullet and $*$ are commutative operations, then the following results hold:⁵

(i) The DO \mathbf{D}_θ can be decomposed as

$$\mathbf{D}_\theta = e^{-j\sigma(\theta_\alpha, \theta_\beta)} \mathbf{B}_\beta \mathbf{A}_\alpha$$

with the *partial DOs* (PDOs) $\mathbf{A}_\alpha = \mathbf{D}_{\theta_\alpha}$ and $\mathbf{B}_\beta = \mathbf{D}_{\theta_\beta}$.

(ii) The PDO \mathbf{A}_α is a family of linear operators indexed by the 1D displacement parameter $\alpha \in \mathcal{A}$ with $\mathcal{A} \subseteq \mathbb{R}$. \mathbf{A}_α is unitary on \mathcal{X} and satisfies the composition law

$$\mathbf{A}_{\alpha_2} \mathbf{A}_{\alpha_1} = e^{j\sigma(\theta_{\alpha_1}, \theta_{\alpha_2})} \mathbf{A}_{\alpha_1 \bullet \alpha_2},$$

where \mathcal{A} and \bullet form a commutative group with identity element α_0 . Analogous results hold for the PDO \mathbf{B}_β .

(iii) The DF of \mathbf{D}_θ can be decomposed as $d(z, \theta) = d^B(d^A(z, \alpha), \beta)$ with the *partial DFs* $d^A(z, \alpha) = d(z, \theta_\alpha)$ and $d^B(z, \beta) = d(z, \theta_\beta)$.

In the following, we assume $\sigma(\theta_{\alpha_1}, \theta_{\alpha_2}) = \sigma(\theta_{\beta_1}, \theta_{\beta_2}) \equiv 0$ so that $\mathbf{A}_{\alpha_2} \mathbf{A}_{\alpha_1} = \mathbf{A}_{\alpha_1 \bullet \alpha_2}$ and $\mathbf{B}_{\beta_2} \mathbf{B}_{\beta_1} = \mathbf{B}_{\beta_1 * \beta_2}$.

Eigenvalues and Eigenfunctions [10, 12]. The *eigenvalues* $\lambda_{\alpha, \tilde{\alpha}}^A$ and *eigenfunctions* $u_\alpha^A(t)$ of \mathbf{A}_α are defined by

$$(\mathbf{A}_\alpha u_\alpha^A)(t) = \lambda_{\alpha, \tilde{\alpha}}^A u_\alpha^A(t); \quad (10)$$

they are indexed by a "dual parameter" $\tilde{\alpha} \in \tilde{\mathcal{A}}$ with $\tilde{\mathcal{A}} \subseteq \mathbb{R}$. The composition law $\mathbf{A}_{\alpha_2} \mathbf{A}_{\alpha_1} = \mathbf{A}_{\alpha_1 \bullet \alpha_2}$ implies $\lambda_{\alpha_1 \bullet \alpha_2, \tilde{\alpha}}^A = \lambda_{\alpha_1, \tilde{\alpha}}^A \lambda_{\alpha_2, \tilde{\alpha}}^A$, and the unitarity of \mathbf{A}_α implies $|\lambda_{\alpha, \tilde{\alpha}}^A| \equiv 1$. It follows [13] that $\tilde{\alpha}$ belongs to a commutative "dual" group $(\tilde{\mathcal{A}}, \tilde{\circ})$ and that there is $\lambda_{\alpha, \tilde{\alpha}_1 \tilde{\circ} \tilde{\alpha}_2}^A = \lambda_{\alpha, \tilde{\alpha}_1}^A \lambda_{\alpha, \tilde{\alpha}_2}^A$. These relations show that the eigenvalues must be of the form

$$\lambda_{\alpha, \tilde{\alpha}}^A = e^{j2\pi \mu_A(\alpha) \tilde{\mu}_A(\tilde{\alpha})}, \quad (11)$$

where $\mu_A(\alpha_1 \bullet \alpha_2) = \mu_A(\alpha_1) + \mu_A(\alpha_2)$, $\mu_A(\alpha_0) = 0$, $\mu_A(\alpha^{-1}) = -\mu_A(\alpha)$, and $\tilde{\mu}_A(\tilde{\alpha}_1 \tilde{\circ} \tilde{\alpha}_2) = \tilde{\mu}_A(\tilde{\alpha}_1) + \tilde{\mu}_A(\tilde{\alpha}_2)$, $\tilde{\mu}_A(\tilde{\alpha}_0) = 0$, $\tilde{\mu}_A(\tilde{\alpha}^{-1}) = -\tilde{\mu}_A(\tilde{\alpha})$. This implies $\lambda_{\alpha_0, \tilde{\alpha}}^A = \lambda_{\tilde{\alpha}, \alpha_0}^A = 1$ and $\lambda_{\alpha^{-1}, \tilde{\alpha}}^A = \lambda_{\tilde{\alpha}, \alpha}^A = \lambda_{\alpha, \tilde{\alpha}}^{A*}$. Analogous results hold for \mathbf{B}_β .

⁵Analogous results hold if $\theta_\beta \circ \theta_\alpha = \theta$.

A-Fourier Transform. Assuming suitable normalization of the eigenfunctions $u_\alpha^A(t)$, it can be shown [10, 12] that any $x(t) \in \mathcal{X}$ can be expanded into the $u_\alpha^A(t)$ as

$$x(t) = \int_{\tilde{\mathcal{A}}} X_A(\tilde{\alpha}) u_\alpha^A(t) |\tilde{\mu}'_A(\tilde{\alpha})| d\tilde{\alpha} = (\mathcal{F}_A^{-1} X_A)(t), \quad (12)$$

with the *A-Fourier transform* (A-FT) [10, 12]

$$X_A(\tilde{\alpha}) = \langle x, u_\alpha^A \rangle = \int_t x(t) u_\alpha^{A*}(t) dt = (\mathcal{F}_A x)(\tilde{\alpha}). \quad (13)$$

$|X_A(\tilde{\alpha})|^2$ is an *energy density* since $\int_{\tilde{\mathcal{A}}} |X_A(\tilde{\alpha})|^2 |\tilde{\mu}'_A(\tilde{\alpha})| d\tilde{\alpha} = \int_t |x(t)|^2 dt = \|x\|^2$. With (10), (12), and (13) we easily show

$$(\mathbf{A}_\alpha x)(t) = \int_{\tilde{\mathcal{A}}} \lambda_{\alpha, \tilde{\alpha}}^A \langle x, u_\alpha^A \rangle u_\alpha^A(t) |\tilde{\mu}'_A(\tilde{\alpha})| d\tilde{\alpha}. \quad (14)$$

Displacement Curves. The TF displacements produced by a PDO \mathbf{A}_α are described by the partial DF $z' = d^A(z, \alpha)$ (see Theorem 1), which is short for $t' = d^A(t, f; \alpha)$, $f' = d^A(f, t; \alpha)$. For given z , the set of all $z' = d^A(z, \alpha)$ obtained by varying α is a curve $C_z^A \in \mathcal{Z}$ that passes through z . This curve will be called a *displacement curve* (DC) of the PDO \mathbf{A}_α . The eigenequation (10) implies that \mathbf{A}_α does not cause a TF displacement of $u_\alpha^A(t)$. Hence, $u_\alpha^A(t)$ must be *TF-localized along a DC* C_z^A , where z is related to the eigenfunction index $\tilde{\alpha}$. Two cases will be considered:

Case 1. The eigenfunction can be written as

$$u_\alpha^A(t) = r_\alpha^A(t) e^{j2\pi [b_A(\tilde{\alpha}) \phi_A(t) + \psi_A(t)]}, \quad (15)$$

where $b_A(\tilde{\alpha})$ and $\phi_A(t)$ are one-to-one functions and $r_\alpha^A(t) = \sqrt{|b'_A(\tilde{\alpha}) \phi'_A(t) / \tilde{\mu}'_A(\tilde{\alpha})|}$ in order to be consistent with (12), (13). Here, the DC C_z^A is postulated to coincide with the *instantaneous frequency*

$$\nu_\alpha^A(t) = b_A(\tilde{\alpha}) \phi'_A(t) + \psi'_A(t) \quad (16)$$

of $u_\alpha^A(t)$, where $z = (t, f)$ in C_z^A is related to $\tilde{\alpha}$ in that z lies on the instantaneous-frequency curve, i.e. $f = \nu_\alpha^A(t)$.

Case 2. The Fourier transform of $u_\alpha^A(t)$ can be written as

$$U_\alpha^A(f) = R_\alpha^A(f) e^{-j2\pi [b_A(\tilde{\alpha}) \Phi_A(f) + \Psi_A(f)]}, \quad (17)$$

where $b_A(\tilde{\alpha})$ and $\Phi_A(f)$ are one-to-one functions and $R_\alpha^A(f) = \sqrt{|b'_A(\tilde{\alpha}) \Phi'_A(f) / \tilde{\mu}'_A(\tilde{\alpha})|}$. Here, C_z^A is postulated to coincide with the *group delay*

$$\tau_\alpha^A(f) = b_A(\tilde{\alpha}) \Phi'_A(f) + \Psi'_A(f) \quad (18)$$

of $u_\alpha^A(t)$, where $z = (t, f)$ in C_z^A is related to $\tilde{\alpha}$ as $t = \tau_\alpha^A(f)$.

Since in both cases the DC C_z^A is really parameterized by $\tilde{\alpha}$, we shall henceforth write C_α^A .

Examples. The DOs $\mathbf{S}_{\tau, \nu}$ and $\mathbf{C}_{a, \tau}$ are both separable. We have $\mathbf{S}_{\tau, \nu} = \mathbf{F}_\nu \mathbf{T}_\tau$ and $\mathbf{C}_{a, \tau} = \mathbf{T}_\tau \mathbf{L}_a$ with the time-shift operator \mathbf{T}_τ , frequency-shift operator \mathbf{F}_ν , and TF-scaling operator \mathbf{L}_a defined by $(\mathbf{T}_\tau x)(t) = x(t - \tau)$, $(\mathbf{F}_\nu x)(t) = x(t) e^{j2\pi \nu t}$, and $(\mathbf{L}_a x)(t) = \sqrt{a} x(at)$ ($a > 0$).

\mathbf{T}_τ is a "case-1 PDO" with $(\mathcal{A}, \bullet) = (\mathbb{R}, +)$, $\lambda_{\tau, f}^T = e^{-j2\pi \tau f}$, $u_f^T(t) = e^{j2\pi f t}$, $\tilde{\tau} = f$, $\mu_T(\tau) = -\tau$, $\tilde{\mu}_T(f) = f$, $b_T(f) = f$, $\phi_T(t) = t$, and $\psi_T(t) \equiv 0$. The DC $C_{t, f}^T: (t', f') = (t + \tau, f)$ coincides with the instantaneous frequency $\nu_f^T(t) = f$, and the T-FT is the Fourier transform, $X_T(f) = \int_t x(t) e^{-j2\pi f t} dt = X(f)$.

F_ν is a “case-2 PDO” with $(A, \bullet) = (\tilde{A}, \tilde{\bullet}) = (\mathbb{R}, +)$, $\lambda_{\nu,t}^F = e^{j2\pi\nu t}$, $U_t^F(f) = e^{-j2\pi t f}$, $\tilde{\nu} = t$, $\mu_F(\nu) = \nu$, $\tilde{\mu}_F(t) = t$, $b_F(t) = t$, $\Phi_F(f) = f$, and $\Psi_F(f) \equiv 0$. The DC $C_{t,f}^F: (t', f') = (t, f + \nu)$ coincides with the group delay $\tau_t^F(f) = t$, and the F-FT is the identity transform, $X_F(t) = x(t)$.

L_a (defined for analytic signals) is a “case-2 PDO” with $(A, \bullet) = (\mathbb{R}_+, \cdot)$, $(\tilde{A}, \tilde{\bullet}) = (\mathbb{R}, +)$, $\lambda_{a,c}^L = e^{j2\pi c \ln a}$, $U_c^L(f) = e^{-j2\pi c \ln(f/f_r)}/\sqrt{f}$ for $f > 0$ (with fixed $f_r > 0$), $\tilde{a} = c$, $\mu_L(a) = \ln a$, $\tilde{\mu}_L(c) = c$, $b_L(c) = c$, $\Phi_L(f) = \ln(f/f_r)$, and $\Psi_L(f) \equiv 0$. The DC $C_{t,f}^L: (t', f') = (at, f/a)$ coincides with the group delay $\tau_c^L(f) = c/f$, and the L-FT is the Mellin transform [6, 14, 11] $X_L(c) = \int_0^\infty X(f) e^{j2\pi c \ln(f/f_r)} df/\sqrt{f}$.

Furthermore, also the DOs underlying the hyperbolic and power classes [6]-[9] are separable.

4 MARGINAL PROPERTIES

We now consider a separable DO $D_\theta = e^{-j\sigma(\theta_\alpha, \theta_\beta)} \mathbf{B}_\beta \mathbf{A}_\alpha$ where \mathbf{A}_α is a case-1 PDO and \mathbf{B}_β is a case-2 PDO (analogous results hold if \mathbf{A}_α is case 2 and \mathbf{B}_β is case 1).

Marginal Properties and Kernel Constraints. The *marginal property* associated to the PDO \mathbf{A}_α states that integration of a QTFR $T_x(t, f)$ over the DC C_α^A (the TF locus of $u_\alpha^A(t)$) yields the energy density $|X_A(\tilde{\alpha})|^2 = |\langle x, u_\alpha^A \rangle|^2$:

$$\int_t T_x(t, \nu_\alpha^A(t)) [r_\alpha^A(t)]^2 dt = |X_A(\tilde{\alpha})|^2. \quad (19)$$

Similarly, the marginal property associated to \mathbf{B}_β reads

$$\int_f T_x(\tau_\beta^B(f), f) [R_\beta^B(f)]^2 df = |X_B(\tilde{\beta})|^2. \quad (20)$$

It can be shown that a QTFR $T_x(t, f)$ covariant to the DO D_θ satisfies the marginal property (19) if and only if its kernel $h(t_1, t_2)$ (cf. (4)) satisfies the constraint

$$\int_t (\mathbf{D}_{p(z(t), z_0)}^\otimes h)(t_1, t_2) [r_\alpha^A(t)]^2 dt = u_\alpha^A(t_1) u_\alpha^{A*}(t_2) \quad (21)$$

with $z(t) = (t, \nu_\alpha^A(t))$. Similarly, (20) holds if and only if

$$\int_f (\tilde{\mathbf{D}}_{p(z(f), z_0)}^\otimes H)(f_1, f_2) [R_\beta^B(f)]^2 df = U_\beta^B(f_1) U_\beta^{B*}(f_2) \quad (22)$$

with $z(f) = (\tau_\beta^B(f), f)$, where $H(f_1, f_2)$ is the kernel in (5).

Examples. From (19), (20), the marginal properties associated to \mathbf{T}_τ , F_ν , and L_a follow as $\int_t T_x(t, f) dt = |X(f)|^2$, $\int_f T_x(t, f) df = |x(t)|^2$, and $\int_f T_x(c/f, f) df/f = |X_L(c)|^2$, respectively. For Cohen’s class (7), the constraints for the \mathbf{T}_τ and F_ν marginal properties follow from (21), (22), after simplification, as $\int_t h(t_1 - t, t_2 - t) dt = 1 \forall t_1, t_2$ and $\int_f H(f_1 - f, f_2 - f) df = 1 \forall f_1, f_2$, respectively. For the affine class (8), the constraints for the L_a and \mathbf{T}_τ marginal properties follow as $\int_0^\infty H(f_0 f_1/f, f_0 f_2/f) e^{-j2\pi(f_1 - f_2)c/f} df/f^2 = e^{-j2\pi c \ln(f_1/f_2)}/\sqrt{f_1 f_2}$ and $(f/f_0) \int_t h(f(t_1 - t)/f_0, f(t_2 - t)/f_0) dt = e^{j2\pi f(t_1 - t_2)}$, respectively.

Localization Function. We now assume that the DCs C_α^A, C_β^B corresponding to a dual parameter pair $\tilde{\theta} = (\tilde{\alpha}, \tilde{\beta})$ intersect in a unique TF point

$$z = l(\tilde{\theta}),$$

which is short for $t = l_1(\tilde{\alpha}, \tilde{\beta})$, $f = l_2(\tilde{\alpha}, \tilde{\beta})$. We shall call

$l(\tilde{\theta})$ the *localization function* (LF) of the separable DO D_θ . The LF is constructed by solving the system of equations $\nu_\alpha^A(t) = f$, $\tau_\beta^B(f) = t$ for $(t, f) = z$ [12]. We assume that to any $z \in \mathcal{Z}$, there exists a unique $\tilde{\theta} = (\tilde{\alpha}, \tilde{\beta})$ such that $z = l(\tilde{\theta})$. Hence, $\tilde{\theta} = l^{-1}(z)$ with the inverse LF $l^{-1}(z)$. The marginal properties (19), (20) can now be written as

$$\int_{\tilde{\beta}} T_x(l(\tilde{\theta})) n_1(\tilde{\theta}) d\tilde{\beta} = |X_A(\tilde{\alpha})|^2 \quad (23)$$

$$\int_{\tilde{\alpha}} T_x(l(\tilde{\theta})) n_2(\tilde{\theta}) d\tilde{\alpha} = |X_B(\tilde{\beta})|^2 \quad (24)$$

with $n_1(\tilde{\theta}) = [r_\alpha^A(l_1(\tilde{\theta}))]^2 |\frac{\partial}{\partial \tilde{\beta}} l_1(\tilde{\theta})|$, $n_2(\tilde{\theta}) = [R_\beta^B(l_2(\tilde{\theta}))]^2 |\frac{\partial}{\partial \tilde{\alpha}} l_2(\tilde{\theta})|$. With (15)-(18), it can be shown that

$$n_1(\tilde{\theta}) = |J(\tilde{\theta})/\tilde{\mu}'_A(\tilde{\alpha})|, \quad n_2(\tilde{\theta}) = |J(\tilde{\theta})/\tilde{\mu}'_B(\tilde{\beta})| \quad (25)$$

where $J(\tilde{\theta}) = \frac{\partial l_1}{\partial \tilde{\alpha}} \frac{\partial l_2}{\partial \tilde{\beta}} - \frac{\partial l_2}{\partial \tilde{\alpha}} \frac{\partial l_1}{\partial \tilde{\beta}}$ is the Jacobian of $l(\tilde{\theta})$.

Characteristic Function Method. Following [10, 11], a class of QTFRs can be constructed as

$$\tilde{T}_x(z) = \int_D g(\theta) \langle x, \mathbf{D}_\theta x \rangle \Lambda(l^{-1}(z), \theta) d\theta \quad (26)$$

with

$$\Lambda(\tilde{\theta}, \theta) = \lambda_{\alpha, \tilde{\alpha}}^A \lambda_{\beta, \tilde{\beta}}^B |\mu'_A(\alpha) \mu'_B(\beta)|, \quad (27)$$

where $g(\theta) = g(\alpha, \beta)$ is a kernel independent of $x(t)$ and $\langle x, \mathbf{D}_\theta x \rangle$ is the “characteristic function.” If

$$g(\theta_\alpha) = g(\alpha, \beta_0) = 1 \quad \text{and} \quad g(\theta_\beta) = g(\alpha_0, \beta) = 1, \quad (28)$$

then $\tilde{T}_x(z)$ can be shown [10] to satisfy the marginal properties (generally different from (23), (24))

$$\int_{\tilde{\beta}} \tilde{T}_x(l(\tilde{\theta})) |\tilde{\mu}'_B(\tilde{\beta})| d\tilde{\beta} = |X_A(\tilde{\alpha})|^2 \quad (29)$$

$$\int_{\tilde{\alpha}} \tilde{T}_x(l(\tilde{\theta})) |\tilde{\mu}'_A(\tilde{\alpha})| d\tilde{\alpha} = |X_B(\tilde{\beta})|^2. \quad (30)$$

5 THE CONJUGATE CASE

Two PDOs \mathbf{A}_α and \mathbf{B}_β with composition laws $\mathbf{A}_{\alpha_2} \mathbf{A}_{\alpha_1} = \mathbf{A}_{\alpha_1 \bullet \alpha_2}$ and $\mathbf{B}_{\beta_2} \mathbf{B}_{\beta_1} = \mathbf{B}_{\beta_1 \bullet \beta_2}$ are called *conjugate* [15] if⁶

$$(\mathbf{B}_\beta u_\alpha^A)(t) = u_{\tilde{\alpha} \bullet \beta}^A(t), \quad (\mathbf{A}_\alpha u_\beta^B)(t) = u_{\tilde{\beta} \bullet \alpha}^B(t). \quad (31)$$

This implies $(\mathcal{F}_A \mathbf{B}_\beta x)(\tilde{\alpha}) = (\mathcal{F}_A x)(\tilde{\alpha} \bullet \beta^{-1})$ and $(\mathcal{F}_B \mathbf{A}_\alpha x)(\tilde{\beta}) = (\mathcal{F}_B x)(\tilde{\beta} \bullet \alpha^{-1})$. Furthermore, using (14) we can show

Theorem 2. Conjugate PDOs \mathbf{A}_α and \mathbf{B}_β commute up to a phase factor,

$$\mathbf{A}_\alpha \mathbf{B}_\beta = \lambda_{\alpha, \beta}^A \mathbf{B}_\beta \mathbf{A}_\alpha, \quad (32)$$

and their eigenvalues and eigenfunctions are related as $\lambda_{\alpha, \beta}^A = \lambda_{\tilde{\alpha}, \tilde{\beta}}^{B*}$ and $\langle u_\alpha^A, u_\beta^B \rangle = \lambda_{\tilde{\alpha}, \tilde{\beta}}^B$.

With (11), it follows that

$$\lambda_{\alpha, \tilde{\alpha}}^A = e^{\pm j2\pi \mu(\alpha) \mu(\tilde{\alpha})} \quad \text{and} \quad \lambda_{\beta, \tilde{\beta}}^B = e^{\mp j2\pi \mu(\beta) \mu(\tilde{\beta})}.$$

⁶Note that the groups and dual groups underlying $\mathbf{A}_\alpha, \mathbf{B}_\beta$ have to be identical: $(A, \bullet) = (B, *) = (\tilde{A}, \tilde{\bullet}) = (\tilde{B}, \tilde{*})$. Furthermore, the functions $\mu_A(\cdot), \mu_B(\cdot), \tilde{\mu}_A(\cdot)$, and $\tilde{\mu}_B(\cdot)$ are all equal up to sign factors, so that we will simply write $\mu(\cdot)$ in the following.

We now consider the composite operator $\mathbf{D}_\theta = \mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta \mathbf{A}_\alpha$. With (32), it is easily shown that \mathbf{D}_θ satisfies the central DO composition property (1),

$$\mathbf{D}_{\theta_2} \mathbf{D}_{\theta_1} = \lambda_{\alpha_2, \beta_1}^A \mathbf{D}_{\alpha_1 \bullet \alpha_2, \beta_1 \bullet \beta_2}, \quad (33)$$

as well as the relation

$$\mathbf{D}_{\theta'}^{-1} \mathbf{D}_\theta \mathbf{D}_{\theta'} = \lambda_{\alpha, \beta'}^A \lambda_{\beta, \alpha'}^B \mathbf{D}_\theta. \quad (34)$$

Eq. (33) implies that the separability condition (9) is met and that the group (\mathcal{D}, \circ) is commutative, $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$.

We conjecture that, in the conjugate case, the DF and LF of \mathbf{D}_θ are related as $d(l(\tilde{\alpha}, \tilde{\beta}); \alpha, \beta) = l(\tilde{\alpha} \bullet \beta, \tilde{\beta} \bullet \alpha)$ or briefly

$$d(l(\tilde{\theta}), \theta) = l(\tilde{\theta} \circ \theta^T) \quad \text{with } \theta^T = (\alpha, \beta)^T \triangleq (\beta, \alpha). \quad (35)$$

To motivate (35), recall that $z = l(\tilde{\alpha}, \tilde{\beta})$ is the intersection of $\nu_{\tilde{\alpha}}^A(t)$ and $\tau_{\tilde{\beta}}^B(f)$. With (10) and (31), $(\mathbf{D}_\theta u_\alpha^A)(t) = \lambda_{\alpha, \tilde{\alpha}}^A u_{\tilde{\alpha} \bullet \beta}^A(t)$ and $(\mathbf{D}_\theta u_\beta^B)(f) = \lambda_{\beta, \tilde{\beta}}^B u_{\tilde{\beta} \bullet \alpha}^B(f)$. These signals are located along the curves $\nu_{\tilde{\alpha} \bullet \beta}^A(t)$ and $\tau_{\tilde{\beta} \bullet \alpha}^B(f)$, respectively, whose intersection is $z' = l(\tilde{\alpha} \bullet \beta, \tilde{\beta} \bullet \alpha)$. On the other hand, since z' has been derived from z through a displacement by θ , there should be $z' = d(z, \theta)$. This finally gives $d(l(\tilde{\alpha}, \tilde{\beta}); \alpha, \beta) = l(\tilde{\alpha} \bullet \beta, \tilde{\beta} \bullet \alpha)$. Note that the covariance (3) can now be rewritten as

$$T_{\mathbf{D}_\theta z}(l(\tilde{\theta})) = T_x(l(\tilde{\theta} \circ \theta^{-T})) \quad \text{with } \theta^{-T} = (\beta^{-1}, \alpha^{-1}).$$

Choosing, for simplicity, the reference TF point z_0 in (4)-(6) as $z_0 = l(\tilde{\theta}_0)$, (35) implies

$$l(\tilde{\theta}) = d(z_0, \tilde{\theta}^T) \quad \text{and} \quad p(l(\tilde{\theta}), z_0) = \tilde{\theta}^T. \quad (36)$$

Theorem 3. If $\mathbf{D}_\theta = \mathbf{B}_\beta \mathbf{A}_\alpha$ is a separable DO with conjugate PDOs \mathbf{A}_α and \mathbf{B}_β , and if (36) holds, then the \mathbf{D}_θ -covariant QTFR class (6) equals the QTFR class (26). The kernels $h(t_1, t_2)$ in (6) and $g(\theta)$ in (26) are related as

$$h(t_1, t_2) = \int_{\mathcal{D}} g^*(\theta) D_\theta(t_1, t_2) |\mu'(\alpha) \mu'(\beta)| d\theta, \quad (37)$$

where $D_\theta(t_1, t_2)$ is the kernel of the DO \mathbf{D}_θ .

Proof. The QTFR $\bar{T}_x(z)$ in (26) can be written as $\bar{T}_x(z) = \langle x, \tilde{\mathbf{H}}_z^D x \rangle$ with $\tilde{\mathbf{H}}_z^D = \int_{\mathcal{D}} g^*(\theta) \Lambda^*(l^{-1}(z), \theta) \mathbf{D}_\theta d\theta$. Comparing with (6), it remains to show that

$$\mathbf{D}_{p(z, z_0)} \mathbf{H} \mathbf{D}_{p(z, z_0)}^{-1} = \int_{\mathcal{D}} g^*(\theta) \Lambda^*(l^{-1}(z), \theta) \mathbf{D}_\theta d\theta$$

for all z . Setting $z = l(\tilde{\theta})$, using (36), and multiplying by $\mathbf{D}_{\tilde{\theta}^T}^{-1}$ and $\mathbf{D}_{\tilde{\theta}^T}$ from left and right, respectively, this becomes

$$\begin{aligned} \mathbf{H} &= \int_{\mathcal{D}} g^*(\theta) \Lambda^*(\tilde{\theta}, \theta) \mathbf{D}_{\tilde{\theta}^T}^{-1} \mathbf{D}_\theta \mathbf{D}_{\tilde{\theta}^T} d\theta \\ &= \int_{\mathcal{D}} g^*(\theta) |\lambda_{\alpha, \tilde{\alpha}}^A|^2 |\lambda_{\beta, \tilde{\beta}}^B|^2 |\mu'(\alpha) \mu'(\beta)| \mathbf{D}_\theta d\theta \end{aligned}$$

where (27) and (34) have been used. With $|\lambda_{\alpha, \tilde{\alpha}}^A|^2 = |\lambda_{\beta, \tilde{\beta}}^B|^2 = 1$, we obtain $\mathbf{H} = \int_{\mathcal{D}} g^*(\theta) |\mu'(\alpha) \mu'(\beta)| \mathbf{D}_\theta d\theta$, which is (37), and relates the kernels $h(t_1, t_2)$ and $g(\alpha, \beta)$ independently of the external parameter $\tilde{\theta}$. ■

Theorem 3 states that the covariance approach and the characteristic function method are equivalent in the conjugate case. Two important conclusions can now be drawn:

- The \mathbf{D}_θ -covariant QTFR class in (4)-(6) satisfies the marginal properties⁷ (29), (30) if the simple kernel constraint (28) is met.
- The QTFR class (26) obtained with the characteristic function method satisfies the \mathbf{D}_θ -covariance (3).

Examples. The PDOs \mathbf{T}_τ and \mathbf{F}_ν underlying Cohen's class (7) are conjugate. Hence, Cohen's class can be constructed using either the covariance method or the characteristic function method. It is $\mathbf{S}_{\tau, \nu}$ -covariant and (assuming that (28) is met) it satisfies also the marginal properties. An analogous result holds for the hyperbolic class [6].

The PDOs \mathbf{L}_a and \mathbf{T}_τ underlying the affine class (8) are not conjugate. Hence, the characteristic function method yields a class [11] that is different from the affine class and that is not $\mathbf{C}_{a, \tau}$ -covariant. Similarly, the power classes [7, 8] are also based on non-conjugate operators.

Acknowledgment

We thank H.G. Feichtinger, A. Papandreou, and R. Baraniuk for interesting discussions.

References

- [1] P. Flandrin, *Temps-fréquence*. Paris: Hermès, 1993.
- [2] F. Hlawatsch and G.F. Boudreaux-Bartels, "Linear and quadratic time-frequency signal representations," *IEEE Signal Proc. Mag.*, vol. 9, no. 2, pp. 21-67, April 1992.
- [3] L. Cohen, "Generalized phase-space distribution functions," *J. Math. Phys.*, vol. 7, pp. 781-786, 1966.
- [4] J. Bertrand and P. Bertrand, "Affine time-frequency distributions," Chapter 5 in *Time-Frequency Signal Analysis—Methods and Applications*, ed. B. Boashash, Longman-Cheshire, Melbourne, Australia, 1992, pp. 118-140.
- [5] O. Rioul and P. Flandrin, "Time-scale energy distributions: A general class extending wavelet transforms," *IEEE Trans. Sig. Proc.*, vol. 40, no. 7, pp. 1746-1757, July 1992.
- [6] A. Papandreou, F. Hlawatsch, and G.F. Boudreaux-Bartels, "The hyperbolic class of quadratic time-frequency representations, Part I," *IEEE Trans. Signal Processing*, vol. 41, no. 12, pp. 3425-3444, Dec. 1993.
- [7] F. Hlawatsch, A. Papandreou, and G.F. Boudreaux-Bartels, "The power classes of quadratic time-frequency representations: A generalization of the affine and hyperbolic classes," *Proc. 27th Asilomar Conf.*, Pacific Grove, CA, pp. 1265-1270, Nov. 1993.
- [8] A. Papandreou, F. Hlawatsch, and G.F. Boudreaux-Bartels, "A unified framework for the scale covariant affine, hyperbolic, and power class quadratic time-frequency representations using generalized time shifts," *Proc. IEEE ICASSP-95*, Detroit, MI, May 1995.
- [9] F. Hlawatsch and H. Bölcskei, "Unified theory of displacement-covariant time-frequency analysis," *Proc. IEEE-SP Int. Sympos. Time-Frequency Time-Scale Analysis*, Philadelphia, PA, Oct. 1994, pp. 524-527.
- [10] R.G. Baraniuk, "Beyond time-frequency analysis: Energy densities in one and many dimensions," *Proc. IEEE ICASSP-94*, Adelaide, Australia, Apr. 1994, vol. 3, pp. 357-360.
- [11] L. Cohen, "The scale representation," *IEEE Trans. Signal Processing*, vol. 41, no. 12, pp. 3275-3292, Dec. 1993.
- [12] R.G. Baraniuk and D.L. Jones, "Unitary equivalence: A new twist on signal processing," to appear in *IEEE Trans. Signal Processing*.
- [13] W. Rudin, *Fourier Analysis on Groups*. Wiley, 1967.
- [14] J.P. Ovarlez, "La transformation de Mellin: un outil pour l'analyse des signaux à large bande," Thèse Univ. Paris 6, 1992.
- [15] F. Hlawatsch and H. Bölcskei, "Quadratic time-frequency distributions based on conjugate operators," in preparation.

⁷Due to (25), the marginal properties (29), (30) will be identical to the marginal properties (23), (24) and, in turn, (19), (20) if and only if the LF's Jacobian is $J(\tilde{\theta}) = \pm \mu'_A(\tilde{\alpha}) \mu'_B(\tilde{\beta})$. We conjecture that, in the conjugate case, this relation is always satisfied and the two sets of marginal properties are thus equivalent.