Space-Frequency Coded MIMO-OFDM with Variable Multiplexing-Diversity Tradeoff

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Abstract—*Space-frequency coded Orthogonal Frequency Division Multiplexing* **(OFDM) is capable of realizing both spatial and frequency-diversity gains in multipath multiple-input multiple-output (MIMO) fading channels. This naturally leads to the question of variable allocation of the channel's degrees of freedom to multiplexing and diversity transmission modes. In this paper, we provide a systematic method for the design of space-frequency codes with variable multiplexing-diversity tradeoffs. Simulation results illustrate the performance of the proposed codes.**

I. Introduction and Motivation

In this paper, we consider multiple-input multiple-output (MIMO) wireless systems, where both spatial diversity (due to multiple antennas) and frequency-diversity (due to multipath propagation) are present. *Orthogonal frequency division multiplexing* (OFDM) [1], [2] significantly reduces receiver complexity in broadband wireless systems. Space-frequency coded MIMO-OFDM [3]–[5] is a transmission technique which applies coding across transmit antennas and OFDM tones and realizes both spatial and frequency-diversity gains without requiring channel knowledge at the transmitter.

In MIMO systems, two basic signaling modes can be employed, namely spatial multiplexing [6], [7], which aims at increasing the data rate by spatially multiplexing independent data streams, and space-time coding [8], which leverages the spatial degrees of freedom (at the cost of rate) to improve link reliability. In practice, the choice of signaling mode depends on a number of parameters such as the link requirements (high reliability or high data rate) and the channel conditions (correlated vs. uncorrelated spatial fading, Ricean K-factor). The presence of multipath propagation and hence the availability of frequency-diversity generally allows to increase the "spatial signaling rate", i.e., to use the multiple antennas to multiplex independent data streams rather than to provide spatial diversity. It is therefore desirable to be able to allocate the channel's degrees of freedom in space and frequency in a flexible way to multiplexing and diversity transmission modes.

Contributions. In this paper, we introduce a space-frequency signaling scheme which allows variable multiplexing-diversity tradeoffs. Using the design criteria for space-frequency codes established by the authors in [3], we demonstrate that our scheme provably realizes prescribed multiplexing-diversity tradeoffs.

Organization of the paper. In Sec. II, we introduce the channel and signal models, and we briefly review the design criteria for space-frequency codes. In Sec. III, we introduce the new signaling scheme. Sec. IV provides a performance analysis of the new class of space-frequency codes, and Sec. V discusses code design. Finally, Sec. VI contains numerical results, and Sec. VII provides conclusions.

II. Space-Frequency Coded MIMO-OFDM

In this section, we shall first introduce the signal and channel model, and then briefly review space-frequency coded MIMO-OFDM.

A. Channel Model

In the following, M_T and M_R denote the number of transmit and receive antennas, respectively. We assume that the $M_R \times M_T$ matrix-valued channel has order $L-1$ with transfer function given by

$$\mathbf{H}(e^{j2\pi\theta}) = \sum_{l=0}^{L-1} \mathbf{H}_l \, e^{-j2\pi l\theta}, \quad 0 \leq \theta < 1, \qquad (1)$$

where the $M_R \times M_T$ complex-valued random matrix $\mathbf{H}_l$ represents the l-th tap. The channel is assumed to be purely Rayleigh fading, i.e., the elements of the matrices $\mathbf{H}_l$ ($l = 0, 1, \ldots, L-1$) are uncorrelated circularly symmetric complex Gaussian random variables with zero mean and variance σ_l^2, i.e., $[\mathbf{H}_l]_{m,n} \sim \mathcal{CN}(0, \sigma_l^2)$. Different taps are uncorrelated. The path gains σ_l^2 are derived from the power delay profile of the channel.

B. Space-Frequency Coded MIMO-OFDM

MIMO-OFDM. In the following, N denotes the number of subcarriers or tones in the OFDM system. Organizing the transmitted data symbols into vectors[1] $\mathbf{c}_k =$

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[1] The superscripts ^{T,H,*} stand for transpose, conjugate transpose, and element-wise conjugation, respectively.

$[c_k^{(0)} \; c_k^{(1)} \; \ldots \; c_k^{(M_T-1)}]^T$ $(k = 0, 1, \ldots, N-1)$ with $c_k^{(i)}$ denoting the data symbol transmitted from the i-th antenna on the k-th tone, the reconstructed data vector for the k-th tone is given by [9], [10]

$$\mathbf{r}_k = \sqrt{E_s}\, \mathbf{H}(e^{j\frac{2\pi}{N}k})\, \mathbf{c}_k + \mathbf{n}_k, \quad k = 0, 1, \ldots, N-1, \quad (2)$$

where $\mathbf{n}_k$ is complex-valued additive white Gaussian noise satisfying[2] $\mathcal{E}\{\mathbf{n}_k \mathbf{n}_l^H\} = \sigma_n^2 \mathbf{I}_{M_R} \delta[k-l]$. The data symbols $c_k^{(i)}$ are taken from a finite complex alphabet and have average energy 1. The constant E_s is an energy normalization factor.

Space-frequency coding. In a space-frequency coded OFDM system, one data burst consists of N vectors of size $M_T \times 1$, or equivalently one spatial OFDM symbol. Throughout the paper, we assume that the channel is constant over at least one OFDM symbol, the transmitter has no channel knowledge, and the receiver knows the channel perfectly. The maximum likelihood (ML) decoder computes the vector sequence $\hat{\mathbf{c}}_k$ $(k = 0, 1, \ldots, N-1)$ according to

$$\widehat{\mathbf{C}} = \arg\min_{\mathbf{C}} \sum_{k=0}^{N-1} \|\mathbf{r}_k - \mathbf{H}(e^{j\frac{2\pi}{N}k})\, \mathbf{c}_k\|^2,$$

where $\mathbf{C} = [\mathbf{c}_0 \; \mathbf{c}_1 \; \ldots \; \mathbf{c}_{N-1}]$ and $\widehat{\mathbf{C}} = [\hat{\mathbf{c}}_0 \; \hat{\mathbf{c}}_1 \; \ldots \; \hat{\mathbf{c}}_{N-1}]$.

Pairwise error probability. In the following, for the sake of simplicity, we assume a uniform power delay profile, i.e., $\sigma_l^2 = 1$ for $l = 0, 1, \ldots, L-1$. Assume that $\mathbf{C} = [\mathbf{c}_0 \; \mathbf{c}_1 \; \ldots \; \mathbf{c}_{N-1}]$ and $\mathbf{C}' = [\mathbf{c}_0' \; \mathbf{c}_1' \; \ldots \; \mathbf{c}_{N-1}']$ are two different space-frequency codewords of size $M_T \times N$. The average (with respect to the random channel) probability that the receiver decides erroneously in favor of the signal $\mathbf{C}'$ assuming that $\mathbf{C}$ was transmitted can be upper-bounded as[3] [3]

$$P(\mathbf{C} \to \mathbf{C}') \leq \prod_{i=0}^{r(\mathbf{Y})-1} \frac{1}{\left(1 + \frac{E_s}{4\sigma_n^2} \lambda_i(\mathbf{Y})\right)^{M_R}}, \quad (3)$$

where $\mathbf{Y} = \mathbf{G}(\mathbf{E})\mathbf{G}^H(\mathbf{E})$ with $\mathbf{E} = \mathbf{C} - \mathbf{C}'$ and

$$\mathbf{G}(\mathbf{E}) = [\mathbf{E}^T \quad \mathbf{D}\mathbf{E}^T \quad \ldots \quad \mathbf{D}^{L-1}\mathbf{E}^T]$$

with $\mathbf{D} = \mathrm{diag}\{e^{-j\frac{2\pi}{N}k}\}_{k=0}^{N-1}$.

Diversity and coding gains. Following [8] we define the diversity gain d as the minimum rank of $\mathbf{G}(\mathbf{E})$ over all possible code difference matrices $\mathbf{E} = \mathbf{C} - \mathbf{C}'$. The corresponding coding gain is given by the minimum of $\left(\prod_{i=0}^{d-1} \lambda_i(\mathbf{Y})\right)^{1/d}$ over all matrices $\mathbf{Y}$ with $r(\mathbf{Y}) = d$. Space-frequency code design in general aims at maximizing the diversity and coding gains.

III. The New Signaling Scheme

In this section, we shall introduce the new signaling scheme. We start by providing a time-domain formulation of the design criteria given in the previous section. This will allow us a better

understanding of the basic idea behind the new modulation scheme.

Time-domain formulation of design criteria. Set[4] $\mathbf{C}^T = \mathbf{F}\mathbf{C}_t^T$ with $[\mathbf{F}]_{m,n} = \frac{1}{\sqrt{N}} e^{-j2\pi\frac{mn}{N}}$ denoting the $N \times N$ FFT matrix. Next, we note that $\mathbf{F}^H \mathbf{D}^l \mathbf{C}^T = \mathbf{F}^H \mathbf{D}^l \mathbf{F}\mathbf{C}_t^T = \mathbf{C}_{t-l}^T$, where $\mathbf{C}_{t-l}^T$ denotes the matrix obtained by cyclically shifting the columns of $\mathbf{C}_t^T$ by l positions downwards. Now it is straightforward to derive the equivalent design criteria in the time domain. First, we define

$$\mathbf{G}_t(\mathbf{E}_t) = \mathbf{F}^H \mathbf{G}(\mathbf{E}) = [\mathbf{E}_t^T \quad \mathbf{E}_{t-1}^T \quad \ldots \quad \mathbf{E}_{t-L+1}^T], \quad (4)$$

where $\mathbf{E}_t = \mathbf{C}_t - \mathbf{C}_t'$. Now, since $\mathbf{Y}$ and $\mathbf{Y}_t = \mathbf{F}^H \mathbf{Y}\mathbf{F} = \mathbf{G}_t(\mathbf{E}_t)\mathbf{G}_t^H(\mathbf{E}_t)$ are unitarily equivalent, and hence $\lambda_i(\mathbf{Y}) = \lambda_i(\mathbf{Y}_t)$, we immediately obtain

$$P(\mathbf{C} \to \mathbf{C}') \leq \prod_{i=0}^{r(\mathbf{Y}_t)-1} \frac{1}{\left(1 + \frac{E_s}{4\sigma_n^2} \lambda_i(\mathbf{Y}_t)\right)^{M_R}}. \quad (5)$$

This criterion basically says that in order to realize frequency-diversity gain, the cyclically shifted versions of $\mathbf{E}_t^T$ should be as orthogonal as possible to each other, so that the receiver can separate the delayed replica of the transmitted code-word matrix. We emphasize that the design criteria (5) and (4) employ the *cyclically shifted* versions of $\mathbf{E}_t$ rather than the linearly shifted versions as would be the case in a single-carrier based system [11].

The new class of space-frequency codes. We construct the $N \times M_T$ space-frequency codewords through a linear transformation according to

$$\mathbf{C}^T = \mathbf{B}\widetilde{\mathbf{C}}_t^T, \quad (6)$$

where $\mathbf{B}$ is an $N \times K$ matrix given by

$$\mathbf{B} = [\mathbf{b}_0 \quad \mathbf{b}_1 \quad \ldots \quad \mathbf{b}_{K-1}],$$

and $\widetilde{\mathbf{C}}_t^T$ is a $K \times M_T$ matrix of data symbols. Next, we write the columns of $\mathbf{B}$ as linear combinations of the columns of the FFT matrix $\mathbf{F} = [\mathbf{f}_0 \; \mathbf{f}_1 \; \ldots \; \mathbf{f}_{N-1}]$, i.e.,

$$\mathbf{b}_i = \sum_{l=0}^{N-1} \alpha_{i,l}\mathbf{f}_l.$$

Defining the $N \times K$ matrix $\mathbf{A}_t$ as

$$\mathbf{A}_t = \begin{bmatrix} \alpha_{0,0} & \alpha_{1,0} & \cdots & \alpha_{K-1,0} \\ \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{K-1,1} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,N-1} & \alpha_{1,N-1} & \cdots & \alpha_{K-1,N-1} \end{bmatrix}$$

it follows that $\mathbf{B} = \mathbf{F}\mathbf{A}_t$ and hence $\mathbf{C}_t^T = \mathbf{A}_t\widetilde{\mathbf{C}}_t^T$, which implies

$$\mathbf{G}_t(\mathbf{E}_t) = \left[\mathbf{A}_t\widetilde{\mathbf{E}}_t^T \quad \mathbf{A}_{t-1}\widetilde{\mathbf{E}}_t^T \quad \ldots \quad \mathbf{A}_{t-L+1}\widetilde{\mathbf{E}}_t^T\right],$$

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[2]$\mathcal{E}\{\cdot\}$ denotes the expectation operator.

[3]$r(\mathbf{A})$ denotes the rank of the matrix $\mathbf{A}$.

[4]The subscript t indicates that $\mathbf{C}_t$ denotes the time-domain version of $\mathbf{C}$.

where $\mathbf{E}_t^T = \mathbf{A}_t \widetilde{\mathbf{E}}_t^T$ and $\mathbf{A}_{t-l}$ denotes the matrix obtained by cyclically shifting the columns of $\mathbf{A}_t$ by l positions downwards.

Interpretation of the new signaling scheme. Starting from the representation $\mathbf{C}^T = \mathbf{F} \mathbf{A}_t \widetilde{\mathbf{C}}_t^T$ and noting that an OFDM modulator basically applies an IFFT, i.e., transmits $\mathbf{F}^H \mathbf{C}^T = \mathbf{A}_t \widetilde{\mathbf{C}}_t^T$, it follows that the block of N data symbols (neglecting the cyclic prefix) transmitted from the i-th antenna during an OFDM symbol interval is given by the i-th column of $\mathbf{A}_t \widetilde{\mathbf{C}}_t^T$. For each antenna, the N data symbols transmitted in a block of length $N > K$ are hence obtained by left-multiplying the corresponding $K \times 1$ data vector by the matrix $\mathbf{A}_t$. Multiplication by $\mathbf{A}_t$ can thus be interpreted as applying an outer code over the complex field whereas the matrices $\widetilde{\mathbf{C}}_t^T$ are chosen from an (arbitrary) inner space-time code. Neglecting the loss due to the cyclic prefix, the transmission rate of the proposed scheme is given by $r = \frac{K}{N} r_I \, r_c \, \log_2 |\mathcal{S}|$ bits per tone, where $|\mathcal{S}|$ denotes the size of the scalar constellation employed, r_I is the spatial rate of the inner space-time code, and $r_c \leq 1$ is the rate of a potential forward error correction code used on top of the inner space-time and the outer space-frequency code. For example, in the case of spatial multiplexing, we have $r_I = M_T$. We emphasize that our modulation scheme introduces redundancy if $K < N$. The basic idea of our approach is to choose the matrix $\mathbf{A}_t$ (and hence the outer code) such that a certain amount of frequency-diversity is guaranteed irrespectively of the inner space-time code and the scalar constellation used.

We conclude this section by noting that our coding scheme can be reformulated as a linear dispersion code [12]. The designs proposed in [12] are for the frequency-flat fading case and aim at maximizing mutual information. Hence they do not provide any guarantees on the achievable spatial and/or frequency-diversity gains.

IV. PERFORMANCE ANALYSIS

In this section, we elaborate on how proper design of the modulation matrix $\mathbf{A}_t$ can yield guaranteed frequency-diversity gains which consequently allows to increase the spatial rate of the inner space-time code and hence realizes a flexible multiplexing-diversity tradeoff.

In order to simplify the following discussion, we consider a channel with $L = 4$ taps and uniform power delay profile, i.e., $\sigma_l^2 = 1$ for $l = 0, 1, 2, 3$. This example will suffice to explain the basic features of the new modulation scheme. A more detailed and systematic discussion can be found in [13]. We shall furthermore assume that the space-time (inner) code with $K \times M_T$ codeword matrices was designed to achieve spatial diversity order sM_R in a flat-fading environment, i.e., the minimum rank of $\widetilde{\mathbf{E}}_t = \widetilde{\mathbf{C}}_t - \widetilde{\mathbf{C}}_t'$ over the set of all possible codeword matrix pairs $\{\widetilde{\mathbf{C}}_t, \widetilde{\mathbf{C}}_t'\}$ equals s. Moreover, we let $K > M_T$ and $N > M_T L$.

Exploiting[5] $\sigma(\mathbf{A}^H \mathbf{A}) = \{\sigma(\mathbf{A}\mathbf{A}^H), 0, \ldots, 0\}$ with $\mathbf{A}$ a $P \times Q$ matrix where $Q > P$ and the number of additional eigenvalues equal to zero is $Q - P$, it follows

[5] $\sigma(\mathbf{A})$ denotes the eigenvalue spectrum of the matrix $\mathbf{A}$.

that $\sigma(\mathbf{G}_t(\mathbf{E}_t)\mathbf{G}_t^H(\mathbf{E}_t)) = \{\sigma(\mathbf{G}_t^H(\mathbf{E}_t)\mathbf{G}_t(\mathbf{E}_t)), 0, \ldots, 0\}$, where the number of additional eigenvalues equal to zero is $N - M_T L$. The design of the matrix $\mathbf{A}_t$ is more conveniently studied by investigating the eigenvalue behavior of $\mathbf{G}_t^H(\mathbf{E}_t)\mathbf{G}_t(\mathbf{E}_t)$, which is obtained from

$$\mathbf{G}_t^H(\mathbf{E}_t)\mathbf{G}_t(\mathbf{E}_t) = \mathrm{diag}\{\widetilde{\mathbf{E}}_t^*\}_{i=0}^3 \, \mathbf{A}^H \mathbf{A} \, \mathrm{diag}\{\widetilde{\mathbf{E}}_t^T\}_{i=0}^3,$$

where $\mathbf{A} = [\mathbf{A}_t \ \mathbf{A}_{t-1} \ \ldots \ \mathbf{A}_{t-L+1}]$. Our goal is now to control the rank of $\mathbf{G}_t^H(\mathbf{E}_t)\mathbf{G}_t(\mathbf{E}_t)$ and hence the diversity gain through proper design of $\mathbf{A}_t$. The maximum diversity order achievable in the 4-tap case is given by $d_{max} = 4sM_R$ [3]. Now, it is easily seen that choosing $\mathbf{A}_t$ such that

$$\begin{aligned} \mathbf{A}_t^H \mathbf{A}_t &= \mathbf{I}_K \\ \mathbf{A}_{t-l}^H \mathbf{A}_t &= \mathbf{0}_K \end{aligned} \tag{7}$$

for $l = 1, 2, 3$ results in $\mathbf{A}^H \mathbf{A} = \mathbf{I}_{KL}$ and hence

$$\mathbf{G}_t^H(\mathbf{E}_t)\mathbf{G}_t(\mathbf{E}_t) = \mathrm{diag}\{\widetilde{\mathbf{E}}_t^* \widetilde{\mathbf{E}}_t^T\}_{i=0}^3.$$

Consequently, $\min_{\mathbf{E}_t} r(\mathbf{G}_t^H(\mathbf{E}_t)\mathbf{G}_t(\mathbf{E}_t)) = 4s$, where again the minimum is taken over all possible code difference matrices $\mathbf{E}_t$. This implies that the resulting diversity order is given by $d = 4sM_R$ which equals the maximum achievable diversity order. It furthermore follows that the coding gain in the frequency-selective case satisfies $c_s = c_f^4$ with c_f denoting the coding gain achieved by the inner code in the frequency-flat case. If the inner space-time code achieves full spatial diversity gain, i.e., $s = M_T$, the resulting overall space-frequency code achieves full space-frequency diversity gain $d = 4M_T M_R$. From (7) it is obvious that in order to achieve full frequency-diversity we need $K \leq \frac{N}{4}$, which imposes a limit on the rate r according to $r \leq \frac{1}{4} r_I \, r_c \, \log_2 |\mathcal{S}|$. Maximum rate $r = \frac{1}{4} r_I \, r_c \, \log_2 |\mathcal{S}|$ is achieved for $K = N/4$. If we want to increase the transmission rate, say by a factor of 2, we can do so by relaxing the constraints (7) such that $\mathbf{A}_t^H \mathbf{A}_t = \mathbf{I}_K$ and $\mathbf{A}_{t-l}^H \mathbf{A}_t = \mathbf{0}_K$ for $l = 1$ only, for example. This means that we impose orthogonality constraints on $\mathbf{A}_t$ and $\mathbf{A}_{t-1}$ only. Clearly, this condition can be met for $K \leq N/2$, which implies that we can double our transmission rate to $r = \frac{1}{2} r_I \, r_c \, \log_2 |\mathcal{S}|$. Doubling the transmission rate, however, comes at the cost of reduced guaranteed frequency-diversity gain. More specifically, we can show that the guaranteed diversity gain is reduced by a factor of 2. This is easily seen by evaluating the matrix $\mathbf{G}_t^H(\mathbf{E}_t)\mathbf{G}_t(\mathbf{E}_t)$, and noting that it has the following structure

$$\mathbf{G}_t^H(\mathbf{E}_t)\mathbf{G}_t(\mathbf{E}_t) = \begin{bmatrix} \widetilde{\mathbf{E}}_t^* \widetilde{\mathbf{E}}_t^T & \mathbf{0} & \times & \times \\ \mathbf{0} & \widetilde{\mathbf{E}}_t^* \widetilde{\mathbf{E}}_t^T & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}, \tag{8}$$

where the matrices marked with $\times$ are irrelevant. From (8) we can conclude that

$$r(\mathbf{G}_t^H(\mathbf{E}_t)\mathbf{G}_t(\mathbf{E}_t)) \geq 2r(\widetilde{\mathbf{E}}_t^* \widetilde{\mathbf{E}}_t^T) \geq 2s,$$

which implies that the guaranteed diversity-order is $d = 2sM_R$. It must be stressed that the construction presented

above does not necessarily yield the optimal multiplexing-diversity tradeoff, however its value lies in the fact that a certain frequency-diversity order can be guaranteed *irrespectively* of the inner space-time code and the scalar symbol constellation used.

The propesed scheme results in a very rough multiplexing-diversity tradeoff. A finer tradeoff behavior can be obtained by imposing orthogonality constraints on the individual columns of $\mathbf{A}_{t-l}$ and $\mathbf{A}_t$ rather than on the entire matrices as was done in (7). This approach is described in detail in [13].

We conclude this section by noting that the flexible multiplexing-diversity tradeoff is obtained through varying the spatial rate of the inner space-time code and noting that the overall diversity gain achieved by our code construction is given by the product of the spatial diversity order and the frequency-diversity order. We note that the spatial code rate is maximized for spatial multiplexing, i.e., transmitting independent data streams from the individual antennas.

V. Design of the Modulation Matrix

We shall now turn to the design of the modulation matrix $\mathbf{A}_t$. Recall that we are interested in finding $N \times K$ matrices $\mathbf{A}_t$ such that $\mathbf{A}_t^H \mathbf{A}_t = \mathbf{I}_K$ and $\mathbf{A}_{t-l}^H \mathbf{A}_t = \mathbf{0}_K$ for prescribed values of $l \in \{1, 2, \ldots, L-1\}$. In order to keep the following discussion simple, we focus on the case where $N = PK$ and orthogonality of $\mathbf{A}_t$ with respect to shifts by integer multiples of K is desired, i.e., $\mathbf{A}_{t-rK}^H \mathbf{A}_t = \mathbf{I}_K \delta[r]$. The general case is described in [13]. The following theorem provides a simple algorithm for designing such matrices.

Theorem 1: Let the $N \times N$ full-rank matrix $\mathbf{G} = \begin{bmatrix} \mathbf{G}_t & \mathbf{G}_{t-K} & \ldots & \mathbf{G}_{t-K(P-1)} \end{bmatrix}$ be given. The matrix

$$\mathbf{A} = (\mathbf{G}\mathbf{G}^H)^{-1/2}\mathbf{G} \qquad (9)$$

satisfies $\mathbf{A}\mathbf{A}^H = \mathbf{A}^H\mathbf{A} = \mathbf{I}_N$ and has the same structure as $\mathbf{G}$, i.e.,

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_t & \mathbf{A}_{t-K} & \ldots & \mathbf{A}_{t-K(P-1)} \end{bmatrix} \qquad (10)$$

with

$$\mathbf{A}_t = \left(\sum_{i=0}^{P-1} \mathbf{G}_{t-iK} \mathbf{G}_{t-iK}^H \right)^{-1/2} \mathbf{G}_t.$$

Proof: From the full-rank property of $\mathbf{G}$ it follows that $\mathbf{G}\mathbf{G}^H$ is full-rank and hence $(\mathbf{G}\mathbf{G}^H)^{-1/2}$ exists and is full-rank as well. Using the factorization $\mathbf{G}\mathbf{G}^H = (\mathbf{G}\mathbf{G}^H)^{1/2}(\mathbf{G}\mathbf{G}^H)^{1/2}$, we obtain

$$\mathbf{A}\mathbf{A}^H = (\mathbf{G}\mathbf{G}^H)^{-1/2}\mathbf{G}\mathbf{G}^H(\mathbf{G}\mathbf{G}^H)^{-1/2} = \mathbf{I}_N.$$

Since $\mathbf{A}$ is a square matrix this implies $\mathbf{A}^H\mathbf{A} = \mathbf{I}_N$.

It remains to be shown that the resulting matrix $\mathbf{A}$ has the structure given by (10). We start by noting that $\mathbf{A}_{t-rK} = \mathbf{P}^{rK}\mathbf{A}_t$ with $\mathbf{P}$ denoting the permutation matrix which performs a cyclic shift downwards by one position. Now we need to show that the matrix $(\mathbf{G}\mathbf{G}^H)^{-1/2}$ commutes with $\mathbf{P}^{rK}$ for $r = 0, 1, \ldots, P-1$. We note that it suffices to show that $\mathbf{G}\mathbf{G}^H$ commutes with $\mathbf{P}^{rK}$ for $r = 0, 1, \ldots, P-1$. This implies that

every power of $\mathbf{G}\mathbf{G}^H$ and hence also $(\mathbf{G}\mathbf{G}^H)^{-1/2}$ commutes with $\mathbf{P}^{rK}$. Using $\mathbf{G}_{t-iK} = \mathbf{P}^{iK}\mathbf{G}_t$, we have

$$\mathbf{G}\mathbf{G}^H\mathbf{P}^{rK} = \left(\sum_{i=0}^{P-1} \mathbf{P}^{iK}\mathbf{G}_t\mathbf{G}_t^H\mathbf{P}^{-iK} \right) \mathbf{P}^{rK}$$

$$= \mathbf{P}^{rK}\mathbf{P}^{-rK} \left(\sum_{i=0}^{P-1} \mathbf{P}^{iK}\mathbf{G}_t\mathbf{G}_t^H\mathbf{P}^{(r-i)K} \right)$$

$$= \mathbf{P}^{rK} \left(\sum_{i=0}^{P-1} \mathbf{P}^{(i-r)K}\mathbf{G}_t\mathbf{G}_t^H\mathbf{P}^{-(i-r)K} \right)$$

$$= \mathbf{P}^{rK} \left(\sum_{i=0}^{P-1} \mathbf{P}^{iK}\mathbf{G}_t\mathbf{G}_t^H\mathbf{P}^{-iK} \right) = \mathbf{P}^{rK}\mathbf{G}\mathbf{G}^H,$$

where we made use of the fact that the summation ranges over an entire cycle of length N and hence includes all cyclic shifts by integer multiples of K. $\square$

Theorem 1 has a number of interesting implications. It tells us that we can start with an arbitrary matrix $\mathbf{G}_t$ chosen such that the corresponding stacked matrix $\mathbf{G}$ has full-rank, apply the algorithm described in the theorem and get a unitary matrix $\mathbf{A}$ *with the same structure as* $\mathbf{G}$. In practice, a full-rank matrix $\mathbf{G}$ can be obtained by picking $\mathbf{G}_t$ randomly. The resulting stacked matrix $\mathbf{G}$ has full rank with probability 1.

VI. Simulation Results

In this section, we provide simulation results that quantify some of the analytical results derived in the paper. We simulated a space-frequency coded MIMO-OFDM system with $M_T = M_R = 2$, uniform power delay profile, and $N = 32$ tones. The signal-to-noise ratio is defined as SNR $= 10 \log_{10} \left(\frac{M_T E_s}{\sigma_n^2} \right)$. In all simulations an ML decoder was used and the channel was normalized to satisfy[6] $\sum_l \mathcal{E}\{\|\mathbf{H}_l\|_F^2\} = M_T M_R$.

Our simulation example studies the impact of the choice of $\mathbf{A}_t$ on the frame error rate[7]. For $K = N/2$, $\mathbf{A}_t^H \mathbf{A}_t = \mathbf{I}_K$, and $\mathbf{A}_{t-1}^H \mathbf{A}_t = \mathbf{0}_K$, Fig. 1a) shows the frame error rate for QPSK-based spatial multiplexing and various channel orders L. We note that the spatial diversity order achieved by spatial multiplexing is $d = 2$ (recall that $M_R = 2$). Our code was designed such that second-order frequency-diversity is guaranteed. Fig. 1a) shows that the code indeed leverages the additionally available frequency-diversity for $L = 2$ and doubles the spatial diversity order achieved by spatial multiplexing, which results in an overall diversity order of 4. For $L = 3, 4$ additional frequency-diversity is available but not exploited by our code construction. Whether or not frequency-diversity gain beyond the one guaranteed by construction can be achieved depends on the inner space-time code and the scalar constellation employed. Obviously, QPSK-based spatial multiplexing combined with the modulation matrix chosen is not capable of leveraging diversity gain beyond the

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[6]$\|\mathbf{A}\|_F^2 = \sum_{i,j} |[\mathbf{A}]_{i,j}|^2$ denotes the squared Frobenius norm of the matrix $\mathbf{A}$.

[7]A frame was chosen to be one spatial OFDM symbol, i.e., NM_T data symbols.

one guaranteed by construction of the outer code. This is not surprising as pure spatial multiplexing does not involve coding over frequency or space. If we wanted to increase the guaranteed diversity order, we could do so by sacrificing transmission rate through a reduction of K and ensuring that $\mathbf{A}_{t-l}^H \mathbf{A}_t = \mathbf{0}_K$ for $l = 2$ and/or $l = 3$ as well. Fig. 1b) shows the performance of our space-frequency code in combination with the Alamouti scheme [14] (employing QPSK) as inner code. Note that the Alamouti scheme achieves 4-th order spatial diversity. Clearly the presence of a second tap in the channel doubles the achievable diversity order to $d = 8$ which, as is evident from Fig. 1b), is realized by our space-frequency code. Summarizing, we note that our code construction indeed doubles the diversity order achieved by the inner space-time code.
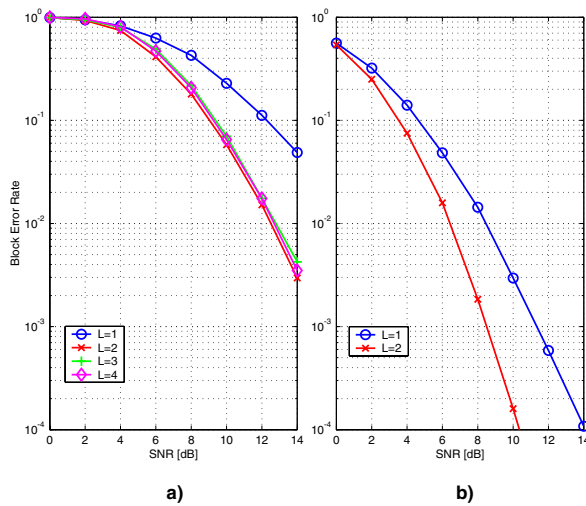

Fig. 1. Performance of space-frequency codes for varying delay spread. Inner space-time code used: a) spatial multiplexing, and b) Alamouti scheme.

VII. CONCLUSION

We proposed a new class of space-frequency codes which allows to trade transmission rate for diversity gain in a flexible way. The construction consists of a linear (over the complex field) outer code which guarantees a certain amount of frequency-diversity and an inner space-time code which realizes spatial diversity. In particular, the new scheme can be shown to yield guaranteed multiplexing-diversity tradeoffs

independently of the inner space-time code and the scalar constellation used. We emphasize, however, that an inner code and constellation independent guarantee of frequency-diversity comes at the cost of reduced transmission rate. However, using large scalar constellations and high-rate inner space-time codes will generally result in an overall scheme that exhibits small rate loss. We furthermore provided an algorithm for designing the modulation matrices for certain special cases.

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