Coding and Modulation for Underspread Fading Channels

Helmut Bölcskei
Communication Technology Laboratory
ETH Zürich
CH-8092 Zürich, Switzerland
e-mail: boelcskei@nari.ee.ethz.ch

Ralf Koetter
CSL, University of Illinois at Urbana-Champaign
1308 W. Main, Urbana, IL 61801
e-mail: koetter@uiuc.edu

Siddharta Mallik
CSL, University of Illinois at Urbana-Champaign
1308 W. Main, Urbana, IL 61801
e-mail: smallik@uiuc.edu

Abstract — **We study modulation and code design for underspread time-varying fading channels. For a coherent receiver employing maximum-likelihood (ML) decoding, we derive the code design criteria and express the maximum achievable diversity order in terms of the channel's scattering function.**

I. Underspread Fading Channels

We model the time-frequency selective fading channel $\mathbf{H}$ as a linear random system with input-output relation

$$y(t) = \int_\tau \int_\nu S_{\mathbf{H}}(\tau, \nu) x(t-\tau)\, e^{j2\pi\nu t}\, d\tau\, d\nu,$$

where $x(t)$ is the input signal, $y(t)$ is the output signal, and $S_{\mathbf{H}}(\tau,\nu) = \int_t h(t, t-\tau)\, e^{-j2\pi\nu t} dt$ is the channel's (*delay-Doppler*) *spreading function* with $h(t,t')$ denoting the random kernel of the linear operator $\mathbf{H}$. A wide-sense stationary uncorrelated scattering (WSSUS) channel is characterized by the scattering function $C_{\mathbf{H}}(\tau,\nu) \geq 0$ satisfying $\mathcal{E}\{S_{\mathbf{H}}(\tau,\nu)S^*_{\mathbf{H}}(\tau',\nu')\} = C_{\mathbf{H}}(\tau,\nu)\,\delta(\tau-\tau')\delta(\nu-\nu')$. The channel is said to be underspread if[1] [1]

$$C_{\mathbf{H}}(\tau,\nu) = 0 \qquad \text{for} \quad (\tau,\nu) \notin [-\tau_0,\tau_0] \times [-\nu_0,\nu_0]$$

with $\sigma_{\mathbf{H}} = 4\tau_0\nu_0 \leq 1$. *The underspread assumption is relevant as most mobile radio channels are underspread.*

It has been shown in [2] that the impulse response $h(t,t')$ of an underspread fading channel can be well approximated by setting

$$h(t,t') = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} L_{\mathbf{H}}(kT, lF)\, g_{k,l}(t)\gamma^*_{k,l}(t'), \qquad (1)$$

where $L_{\mathbf{H}}(t,f) = \int_\tau h(t,t-\tau)\, e^{-j2\pi f\tau}\, d\tau$, $g_{k,l}(t) = g(t-kT)\, e^{j2\pi lFt}$, $\gamma_{k,l}(t) = \gamma(t-kT)\, e^{j2\pi lFt}$, and $g(t)$ and $\gamma(t)$ are suitably chosen window functions depending on $C_{\mathbf{H}}(\tau,\nu)$. Furthermore, $T \leq \frac{1}{2\nu_0}$ and $F \leq \frac{1}{2\tau_0}$ and $\langle g_{k,l}, \gamma_{k',l'}\rangle = \int_t g_{k,l}(t)\gamma^*_{k',l'}(t)\, dt = \delta_{kk'}\delta_{ll'}$.

II. Modulation and Code Design

Based on the developments in the previous section, we suggest the use of pulse-shaped OFDM [2] as modulation scheme. We write the transmit signal as $x(t) = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{N-1} \sqrt{E_s} c_{k,l} g_{k,l}(t)$, where $c_{k,l}$ denotes the information bearing data symbols, E_s is an energy normalization factor, and N is the number of OFDM tones. With the received signal $r(t) = y(t) + n(t)$ and $n(t)$ additive white Gaussian noise, the receiver computes the inner products $\hat{c}_{k,l} = \langle r, \gamma_{k,l}\rangle$. Exploiting the biorthogonality of the basis functions $g_{k,l}(t)$ and $\gamma_{k,l}(t)$ and the orthonormality of the $\gamma_{k,l}(t)$, we obtain $\hat{c}_{k,l} = \sqrt{E_s}\, L_{\mathbf{H}}(kT, lF) c_{k,l} + n_{k,l}$, where $\mathcal{E}\{n_{k,l} n^*_{k',l'}\} = \sigma_n^2 \delta_{kk'}\delta_{ll'}$.

Code design criteria. The bit stream to be transmitted is encoded into blocks of size $M \times N$, where M denotes the number of OFDM symbols. Stacking the transmitted vectors $\mathbf{c}_k$ $(k = 0, 1, ..., M-1)$ according to $\mathbf{c} = [\mathbf{c}_0^T\ \mathbf{c}_1^T\ \ldots\ \mathbf{c}_{M-1}^T]^T$, assuming ML decoding with perfect channel knowledge, the Chernoff upper bound on the expected (with respect to the channel) probability of mistaking $\mathbf{c}^{(i)}$ for another code vector say $\mathbf{c}^{(j)}$ is given by

$$P(\mathbf{c}^{(i)} \to \mathbf{c}^{(j)} | \mathbf{c}^{(i)}) \leq \prod_{l=0}^{r^{(i,j)}-1} \left(\frac{1}{1 + \frac{E_s}{4\sigma_n^2}\lambda_l(\mathbf{Y}_{i,j})} \right), \qquad (2)$$

where $\lambda_l(\mathbf{Y}^{(i,j)})$ denotes the eigenvalues of the $MN \times MN$ matrix[2]

$$\mathbf{Y}^{(i,j)} = \mathbf{R} \odot \left[\mathbf{y}^{(i,j)}(\mathbf{y}^{(i,j)})^H\right],$$

$\mathbf{y}^{(i,j)} = \mathbf{c}^{(i)} - \mathbf{c}^{(j)}$, $\mathbf{R} = \mathcal{E}\{\mathbf{h}\mathbf{h}^H\}$ with $\mathbf{h} = [\mathbf{h}_0^T\ \mathbf{h}_1^T\ \ldots\ \mathbf{h}_{M-1}^T]^T$ and $\mathbf{h}_k = [L_{\mathbf{H}}(kT,0)\ L_{\mathbf{H}}(kT,F)\ \ldots\ L_{\mathbf{H}}(kT,(N-1)F)]^T$ is the channel's correlation matrix, and $r^{(i,j)}$ stands for the rank of $\mathbf{Y}^{(i,j)}$. With $\mathcal{E}\{L_{\mathbf{H}}(t,f)\, L^*_{\mathbf{H}}(t',f')\} = R_{\mathbf{H}}(t-t', f-f')$ we note the Fourier correspondence

$$R_{\mathbf{H}}(\Delta t, \Delta f) = \int_\tau \int_\nu C_{\mathbf{H}}(\tau,\nu) e^{j2\pi(\nu\Delta t - \tau\Delta f)}\, d\tau d\nu. \qquad (3)$$

Based on (2) and using simple identities involving Hadamard products it follows that *the maximum achievable diversity order with any code is given by the rank of the channel's correlation matrix, i.e.,* $d_{max} = \text{rank}(\mathbf{R})$.

In order to minimize the PEP upper bound the code should be designed such that the minimum rank of $\mathbf{Y}^{(i,j)}$ over all pairs $i \neq j$ is maximized and in the high SNR case if full diversity gain is our goal the product of the nonzero eigenvalues of $\mathbf{Y}^{(i,j)}$ over all pairs $i \neq j$ is maximized. Invoking the eigenvalue decomposition of the channel's correlation matrix $\mathbf{R} = \sum_{l=0}^{r-1} \sigma_l^2 \mathbf{v}_l \mathbf{v}_l^H$ with r denoting the rank of $\mathbf{R}$, we can find an alternative representation for $\mathbf{Y}^{(i,j)}$ as $\mathbf{Y}^{(i,j)} = \mathbf{G}(\mathbf{c}^{(i)}, \mathbf{c}^{(j)})\, \mathbf{G}^H(\mathbf{c}^{(i)}, \mathbf{c}^{(j)})$ with the $NM \times r^{(i,j)}$ matrix (note that $r^{(i,j)} \leq r$)

$$\begin{aligned}\mathbf{G}(\mathbf{c}^{(i)}, \mathbf{c}^{(j)}) &= \Big[\sqrt{\sigma_0^2}\,\mathbf{v}_0 \odot (\mathbf{c}^{(i)} - \mathbf{c}^{(j)}) \ \ldots \\ &\qquad \sqrt{\sigma^2_{r^{(i,j)}-1}}\,\mathbf{v}_{r^{(i,j)}-1} \odot (\mathbf{c}^{(i)} - \mathbf{c}^{(j)})\Big].\end{aligned}$$

In order to achieve full diversity gain, the matrix $\mathbf{G}(\mathbf{c}^{(i)}, \mathbf{c}^{(j)})$ has to have rank d_{max} for every pair of codewords $\mathbf{c}^{(i)}$ and $\mathbf{c}^{(j)}$. We emphasize that the code design criteria depend strongly on the channel's scattering function.

References

[1] J. G. Proakis, *Digital Communications.* New York: McGraw-Hill, 3rd ed., 1995.

[2] W. Kozek, *Matched Weyl-Heisenberg Expansions of Nonstationary Environments.* PhD thesis, Vienna University of Technology, March 1997.

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[1] We assume that the scattering function is centered about $\tau = 0$.

[2] $\odot$ denotes the Hadamard product.