Multiuser Space–Time/Frequency Code Design

Markus E. Gärtner and Helmut Bölcskei
Communication Technology Laboratory
ETH Zurich
8092 Zurich, Switzerland
E-mail: {gaertner, boelcskei}@nari.ee.ethz.ch

***Abstract*— A significant body of results on space-time and space-frequency coding for single-user channels is available in the literature. In contrast, space-time/frequency coding for multiple-access channels (MACs) seems largely unexplored. Building on the framework in Gallager, *IEEE Trans. IT*, 1985 for characterizing the dominant error event regions in single-antenna additive white Gaussian noise (AWGN) MACs, we derive rate-dependent space-time/frequency code design criteria for fading multiantenna MACs with perfect channel state information at the receiver. It is demonstrated that, depending on the transmission rate tuple, joint designs taking the presence of multiple users explicitly into account may be necessary. Our results furthermore allow to identify the rate regions where, for each user, employing codes designed for the single-user case is optimal. Finally, we show that the number of receive antennas has a significant impact on the dominant error event regions and hence, plays an important role in the code design criteria. As a byproduct of our analysis, we find that the classical code design criteria (based on pairwise error probabilities) are recovered using a completely different approach aimed at minimizing the probability of encountering a bad effective channel realization.**

I. Introduction

The design of space-time/frequency codes for single-user multiantenna channels has been studied in great detail [1]–[4]. Little is known, however, about space-time/frequency coding in multiple-access channels (MACs). Past work focuses mostly on employing single-user space-time codes for each of the users and separating the users in signal space [5] or canceling multiuser interference [6]. These approaches lead, however, to (significantly, if the number of users is high) reduced transmission rate [5] or suboptimum performance [6].

A systematic study of the general problem of space-time/frequency code design for MACs seems to be missing. Filling this gap is the main goal of the present paper. Our analysis is based on an idea by Gallager [7], used to characterize the error mechanisms in *two-user* additive white Gaussian noise (AWGN) MACs. Depending on the transmission rate tuple, it is shown in [7] that the dominant error event is either one of the two users or both users being in error. Taking our cue from the results in [7], we can conclude that the rate regions where single-user error events dominate can be dealt with using space-time/frequency codes designed for the single-user case, such as those in [1]–[3]. The rate region where the event of both users (or a subset of the users in the case of MACs with more than two users) being in error dominates requires, however, new design criteria as shown in this paper.

This work was supported in part by the Swiss National Science Foundation (SNF) under grant No. 200021-100025/1 and by the STREP project No. IST-026905 (MASCOT) within the sixth framework programme of the European Commission.

An important conceptual difference between the setup in [7] and the case considered here originates from the fading nature of the channel, which results in two sources of errors, namely errors due to additive noise (also present in the AWGN case) and errors due to the channel being in outage. Throughout the paper, we assume that the blocklengths are large enough for errors due to outage to dominate the error performance.

For the sake of simplicity of exposition, we restrict ourselves to the two-user case. Comments on the generalization of our results to the case of an arbitrary number of users will be made where appropriate. Furthermore, we assume that the receiver has perfect channel state information (CSI).

Contributions: Our detailed contributions can be summarized as follows:

- We characterize the dominant error event regions for fading multiantenna MACs and discuss their dependence on SNR and the number of antennas at the receiver.
- Based on the dominant error event regions, we establish space-time/frequency code design criteria for fading multiantenna MACs.
- We illustrate the value of code designs taking the multiuser nature of the problem explicitly into account.

Notation: The superscripts ^{T}, ^{H} and * stand for transposition, conjugate transposition and elementwise conjugation, respectively. $\mathbf{I}_N$ is the $N \times N$ identity matrix, $\mathbf{0}$ is the all-zeros matrix of appropriate size, $\|\mathbf{x}\|$ stands for the Euclidean norm of the vector $\mathbf{x}$ and $\|\mathbf{A}\|_F$ denotes the Frobenius norm of the matrix $\mathbf{A}$. For an $M \times N$ matrix $\mathbf{A} = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_N\end{bmatrix}$, we define $\mathrm{vec}\{\mathbf{A}\} = \begin{bmatrix}\mathbf{a}_1^T & \mathbf{a}_2^T & \cdots & \mathbf{a}_N^T\end{bmatrix}^T$. If $\mathscr{L}$ is a set, then $|\mathscr{L}|$ denotes its cardinality. $\mathcal{E}[\cdot]$ represents the expectation operator. A multivariate, circularly symmetric, complex Gaussian random vector is a random vector $\mathbf{z} = \mathbf{x} + j\mathbf{y} \sim \mathcal{CN}(\mathbf{0}, \mathbf{\Sigma})$, where the real-valued random vectors $\mathbf{x}$ and $\mathbf{y}$ are jointly Gaussian, $\mathcal{E}[\mathbf{z}] = \mathbf{0}$, $\mathcal{E}\big[\mathbf{z}\mathbf{z}^H\big] = \mathbf{\Sigma}$, and $\mathcal{E}\big[\mathbf{z}\mathbf{z}^T\big] = \mathbf{0}$. The notation $\mathbf{x} \preceq \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ with $\sum_{k=1}^N x_k = \sum_{k=1}^N y_k$ means that $\mathbf{y}$ majorizes $\mathbf{x}$, i.e., $\sum_{k=1}^n x_k \leq \sum_{k=1}^n y_k$ $(n = 1, 2, \ldots, N-1)$ for the elements of $\mathbf{x}$ and $\mathbf{y}$ being arranged in nonincreasing order. Logarithms are to the base e unless stated otherwise.

II. Channel and Signal Models

A. Channel Model

We assume that each of the two users is equipped with M_T transmit antennas and the receiver employs M_R antennas. The matrix-valued fading channel between the two users and the receiver is assumed frequency-selective fading with the $M_R \times 2M_T$ transfer function given by

$$\begin{aligned}\overline{\mathbf{H}}\big(e^{j2\pi\theta}\big) &= \big[\mathbf{H}_1\big(e^{j2\pi\theta}\big) \quad \mathbf{H}_2\big(e^{j2\pi\theta}\big)\big] \\ &= \sum_{l=0}^{L-1} \overline{\mathbf{H}}[l] e^{-j2\pi l\theta}, \quad 0 \leq \theta < 1.\end{aligned}$$

Here, $\mathbf{H}_i\big(e^{j2\pi\theta}\big) = \sum_{l=0}^{L-1} \mathbf{H}_i[l] e^{-j2\pi l\theta}$ $(i = 1, 2)$ represents the $M_R \times M_T$ channel between user i and the receiver and $\overline{\mathbf{H}}[l] = \big[\mathbf{H}_1[l] \ \ \mathbf{H}_2[l]\big]$. The channel is purely Rayleigh fading, i.e., $\mathrm{vec}\{\mathbf{H}_i[l]\} \sim \mathcal{CN}(\mathbf{0}, \sigma_l^2 \mathbf{I}_{M_T M_R})$ $\forall l$ for $i = 1, 2$, with the individual taps being uncorrelated (also across users). The path gains σ_l^2 are normalized such that $\sum_{l=0}^{L-1} \sigma_l^2 = 1$. Throughout the paper, the receiver is assumed to have perfect knowledge of both users' channels, whereas the transmitters do not have any CSI.

B. Signal Model

For simplicity, we assume an N-periodic signal model, which means that the impact of the channel on the transmitted signal is described by circular convolution rather than linear convolution. Such a signal model is obtained, for instance, when each of the users employs orthogonal frequency-division multiplexing (OFDM) [8] and the cyclic prefix (guard interval) length exceeds the channel order. Denoting the number of OFDM tones as N (where $N \geq 2M_T L$ is assumed throughout), the received vector signal on the nth tone is given by

$$\mathbf{y}^{(n)} = \sqrt{\frac{P}{2}} \overline{\mathbf{H}}\big(e^{j2\pi\frac{n}{N}}\big) \overline{\mathbf{c}}^{(n)} + \mathbf{z}^{(n)}, \quad n = 0, 1, \ldots, N-1$$

where $\overline{\mathbf{c}}^{(n)} = \big[(\mathbf{c}_1^{(n)})^T \ \ (\mathbf{c}_2^{(n)})^T\big]^T$ with the $M_T \times N$ codewords $\mathbf{C}_i = \big[\mathbf{c}_i^{(0)} \ \ \mathbf{c}_i^{(1)} \ \cdots \ \mathbf{c}_i^{(N-1)}\big]$ $(i = 1, 2)$ transmitted by user i and the noise vector (uncorrelated across tones and users) $\mathbf{z}^{(n)} \sim \mathcal{CN}(\mathbf{0}, N_0 \mathbf{I}_{M_R})$ $\forall n$.

The codewords $\mathbf{C}_i$ $(i = 1, 2)$ with $\mathcal{E}[\|\mathbf{C}_i\|_F^2] = N$ are chosen from the codebook $\mathscr{C}_i$ of size M_i according to a given probability assignment. We emphasize that although the users cannot cooperate in selecting their codewords, their codebooks can still be designed jointly, i.e., the design can take the multiuser nature of the problem explicitly into account (cf. the last paragraph in Section IV-A). This observation will turn out to be crucial later when dealing with rate tuples that lie in the region where the dominant error event corresponds to both users being in error.

III. Characterization of Error Events

A. Upper Bound on Error Probability

As already mentioned in the introduction, the space-time/frequency code design criteria will depend on the individual users' data rates R_i $(i = 1, 2)$. We therefore need to first relate the error probability to the rate tuple (R_1, R_2). The receiver employs joint maximum likelihood (ML) decoding according to

$$\widehat{\overline{\mathbf{C}}} = \arg\min_{\overline{\mathbf{C}}} \sum_{n=0}^{N-1} \left\| \mathbf{y}^{(n)} - \sqrt{\frac{P}{2}} \overline{\mathbf{H}}\big(e^{j2\pi\frac{n}{N}}\big) \overline{\mathbf{c}}^{(n)} \right\|^2$$

with $\overline{\mathbf{C}} = \big[\mathbf{C}_1^T \ \ \mathbf{C}_2^T\big]^T$ and $\widehat{\overline{\mathbf{C}}} = \big[\widehat{\mathbf{C}}_1^T \ \ \widehat{\mathbf{C}}_2^T\big]^T$. An error occurs, whenever the decision is made in favor of a codeword tuple $(\widehat{\mathbf{C}}_1, \widehat{\mathbf{C}}_2) \neq (\mathbf{C}_1, \mathbf{C}_2)$. In line with the reasoning in [7], we can identify three types of error events, namely errors of type 1 and 2, where only the codeword of user 1 or 2, respectively, is in error and of type 3, where both users' codewords are decoded erroneously. Denoting the corresponding error probabilities as $P_{ek|H}$ $(k = 1, 2, 3)$, the total average (w.r.t. the random channel) error probability is given by

$$P_e = P_{e1} + P_{e2} + P_{e3} \tag{1}$$

where $P_{ek} = \mathcal{E}_H[P_{ek|H}]$. Depending on the desired transmission rate tuple (R_1, R_2), one of the three terms in (1) dominates the total error probability P_e, leading to the dominant error event regions depicted in Fig. 1 and defined as follows: P_{ek} dominates in region $k = 1, 2, 3$. If P_{e1} or P_{e2} dominate, employing, for each of the two users, codes designed for single-user channels is sufficient. Moreover, these codes can be chosen independently of each other. If P_{e3} dominates, a joint design of the two users' codebooks is required. The notion of a joint design will be made precise in Section IV.

Exact expressions for the error probability in (1) as a function of (R_1, R_2) are difficult to obtain so that we resort to the standard upper bound in terms of error exponents. Setting $R_3 = R_1 + R_2$ in the remainder of the paper, we obtain

$$P_{ek|H} \leq e^{-N E_{rk}(R_k, \overline{\mathbf{H}})}, \quad k = 1, 2, 3$$

where the random coding exponent $E_{rk}(R_k, \overline{\mathbf{H}})$ $(k = 1, 2, 3)$ is given by

$$E_{rk}(R_k, \overline{\mathbf{H}}) = \max_{0 \leq \tau \leq 1} \max_{Q} \big(E_{0k}(\tau, Q, \overline{\mathbf{H}}) - \tau R_k\big)$$

with $E_{0k}(\tau, Q, \overline{\mathbf{H}})$ being defined (replacing $\mathrm{P}(\mathbf{Y}|\overline{\mathbf{C}})$ by $\mathrm{P}(\mathbf{Y}|\overline{\mathbf{C}}, \overline{\mathbf{H}})$) as in [7, Th. 2]. Furthermore, Q denotes the probability assignment on the transmitted codeword matrices.

The probability density function of $E_{rk}(R_k, \overline{\mathbf{H}})$ (note that $\overline{\mathbf{H}}$ is random) has a nonzero probability mass at $E_{rk}(R_k, \overline{\mathbf{H}}) = 0$, because the random coding exponent becomes zero for all channel realizations $\overline{\mathbf{H}}$ that do not support the desired rate R_k (due to outage). This observation, combined with the long blocklength assumption (the blocklength is assumed large enough so that errors due to additive noise can be ignored [9]) made earlier, results in (the outage probability)

$$P_{ek} = \mathrm{P}\big(E_{rk}(R_k, \overline{\mathbf{H}}) = 0\big). \tag{2}$$

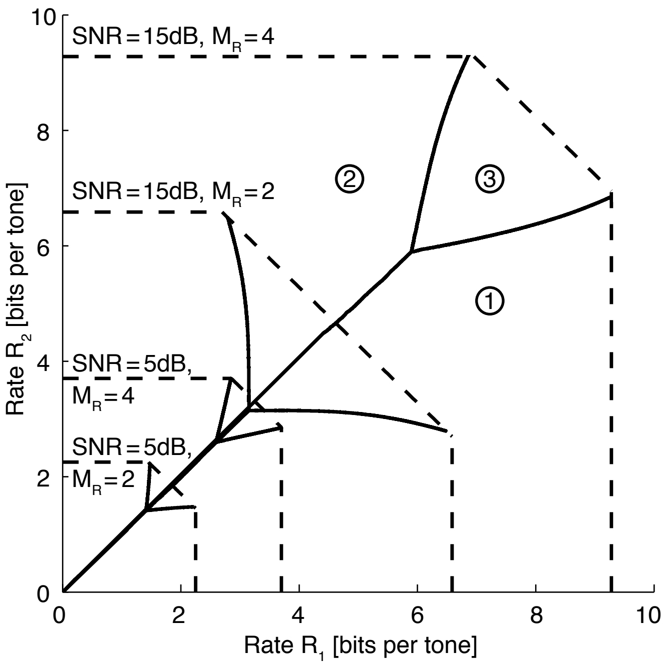

Fig. 1. Dominant error event regions for the two-user MAC with $M_T = 2$ and $L = 1$. Dashed lines represent the ergodic capacity regions.

B. The Case of Gaussian Codebooks

The purpose of this subsection is to discuss the dependence of the dominant error event regions on M_R and $\text{SNR} = P/N_0$. For Gaussian codebooks (which are capacity achieving), $M_T = 2$, $L = 1$, and various choices of M_R and SNR, Fig. 1 shows the dominant error event regions (obtained by means of Monte-Carlo simulations from (2)).

We observe that for fixed SNR, increasing M_R results in a reduction of the relative size of the region where both users are in error (region 3). This is due to the fact that for large M_R there are more spatial degrees of freedom and hence, imposing "separation" through appropriate joint code design is required only for a small set of (high) rates. From Fig. 1, we can infer that this effect is much less pronounced for low SNR, where the concept of spatial separation is not relevant [10]. In summary, we can conclude that increasing M_R results in choosing single-user codes for each of the two users being optimal in a larger fraction of the entire rate region. In case of a general number of users, error events corresponding to any subset of users being in error can dominate and code design has to account for that (cf. the last paragraph in Section IV-A). Finally, we note that L has hardly any impact on the shape of the dominant error event regions.

IV. Code Design Criteria

As suggested by the results in the previous section, the code design criteria depend on the transmission rate tuple (R_1, R_2). In order to state the general design guideline, we first need to establish the corresponding rate-dependent code design criteria.

We assume that the codeword matrices are composed of elements drawn from finite scalar constellations. Defining $M_1 = |\mathscr{C}_1|$, $M_2 = |\mathscr{C}_2|$ and $M_3 = M_1 M_2$, the corresponding maximum transmission rates R_k^0 (in nats per tone) are given by $R_k^0 = (1/N)\log M_k$ $(k = 1, 2, 3)$.

The expression in (2) does not lead to analytically tractable design criteria. Instead, we resort to cutoff rates by choosing $\tau = 1$, which yields

$$E_{rk}(R_k, \overline{\mathbf{H}}) \geq E_{0k}(1, Q, \overline{\mathbf{H}}) - R_k \tag{3}$$

for arbitrary Q. Using (3) in (2), we obtain

$$P_{ek} \leq \mathrm{P}\left(\frac{1}{\widetilde{M}_k} \sum_{\substack{\mathbf{C}_k, \mathbf{E}_k \\ \mathbf{C}_k \neq \mathbf{E}_k}} e^{-\frac{\rho}{4}\sum_{n=0}^{N-1} \left\| \mathbf{H}_k\left(e^{j2\pi\frac{n}{N}}\right)\left(\mathbf{c}_k^{(n)} - \mathbf{e}_k^{(n)}\right)\right\|^2} \geq e^{N(R_k^0 - R_k)} \right), \quad k = 1, 2, 3 \tag{4}$$

where we define $\rho = \frac{P}{2N_0}$, $\widetilde{M_1} = M_1 - 1$, $\widetilde{M_2} = M_2 - 1$, $\widetilde{M_3} = \widetilde{M_1}\widetilde{M_2}$, $\mathbf{H}_3 = \overline{\mathbf{H}}$, $\mathbf{C}_3 = \overline{\mathbf{C}}$ and $\mathbf{E}_3 = \overline{\mathbf{E}} = \left[\mathbf{E}_1^T \;\; \mathbf{E}_2^T\right]^T$. The notation $\mathbf{C}_3 \neq \mathbf{E}_3$ means that both $\mathbf{C}_1 \neq \mathbf{E}_1$ and $\mathbf{C}_2 \neq \mathbf{E}_2$. Furthermore, we define $\mathbf{C}_k = \left[\mathbf{c}_k^{(0)} \;\; \mathbf{c}_k^{(1)} \;\; \cdots \;\; \mathbf{c}_k^{(N-1)}\right]$ and $\mathbf{E}_k = \left[\mathbf{e}_k^{(0)} \;\; \mathbf{e}_k^{(1)} \;\; \cdots \;\; \mathbf{e}_k^{(N-1)}\right]$ for $k = 1, 2, 3$. The upper bound in (4) requires tightening the corresponding bound in [11, Th. 5.6.1], which is achieved by proper choice of Q (details are omitted due to length constraints). We are now ready to state

Theorem 1: The error probability P_{ek} $(k = 1, 2, 3)$ is upper-bounded according to

$$P_{ek} \leq \sum_{\substack{\mathbf{C}_k, \mathbf{E}_k \\ \mathbf{C}_k \neq \mathbf{E}_k}} \mathrm{P}\left(\frac{1}{N}\sum_{n=0}^{N-1} \chi_n \lambda_n(\mathbf{R}_k) \leq \frac{4}{\rho} R_k \right)$$

where the χ_n are independent χ^2-distributed random variables with $2M_R$ degrees of freedom each and $\lambda_n(\mathbf{R}_k)$ stands for the nth eigenvalue of the matrix

$$\mathbf{R}_k = \mathbf{G}(\mathbf{C}_k, \mathbf{E}_k)\mathbf{G}^H(\mathbf{C}_k, \mathbf{E}_k) \tag{5}$$

with the stacked codeword difference matrix

$$\mathbf{G}(\mathbf{C}_k, \mathbf{E}_k) = \left[\sigma_0(\mathbf{C}_k - \mathbf{E}_k)^T \quad \sigma_1 \mathbf{D}(\mathbf{C}_k - \mathbf{E}_k)^T \quad \cdots \quad \cdots \quad \sigma_{L-1}\mathbf{D}^{L-1}(\mathbf{C}_k - \mathbf{E}_k)^T\right]$$

where $\mathbf{D} = \mathrm{diag}_{n=0}^{N-1}\{e^{-j2\pi n/N}\}$.

Remark: Note that (5) can equivalently be expressed in the "time domain" by defining $\mathbf{C}_k^t = \mathbf{C}_k\mathbf{F}$ and $\mathbf{E}_k^t = \mathbf{E}_k\mathbf{F}$ with the $N \times N$ FFT matrix $[\mathbf{F}]_{m,n} = (1/\sqrt{N})e^{j2\pi mn/N}$ $(m, n = 0, 1, \ldots, N-1)$ and noting that due to unitary equivalence $\lambda_n(\mathbf{R}_k) = \lambda_n(\mathbf{R}_k^t)$ $(n = 0, 1, \ldots, N-1)$, where $\mathbf{R}_k^t = \mathbf{G}^t(\mathbf{C}_k^t, \mathbf{E}_k^t)\left(\mathbf{G}^t(\mathbf{C}_k^t, \mathbf{E}_k^t)\right)^H$ with

$$\mathbf{G}^t(\mathbf{C}_k^t, \mathbf{E}_k^t) = \left[\sigma_0(\mathbf{C}_k^t - \mathbf{E}_k^t)^T \quad \sigma_1(\mathbf{C}_k^{t-1} - \mathbf{E}_k^{t-1})^T \quad \cdots \quad \cdots \quad \sigma_{L-1}(\mathbf{C}_k^{t-L+1} - \mathbf{E}_k^{t-L+1})^T\right]$$

where $\mathbf{C}_k^{t-l}$ and $\mathbf{E}_k^{t-l}$ denote the matrices obtained by cyclically shifting the columns of $\mathbf{C}_k^t$ and $\mathbf{E}_k^t$ by l positions to the right.

Proof: We start by noting that

$$\mathrm{P}\left(\sum_{i=1}^{n} e^{-x_i} \geq c\right) \leq \sum_{i=1}^{n} \mathrm{P}\left(x_i \leq \log \frac{n}{c}\right)$$

which, when applied to (4), yields

$$P_{ek} \leq \sum_{\substack{\mathbf{C}_k,\mathbf{E}_k\\ \mathbf{C}_k\neq\mathbf{E}_k}} \mathrm{P}\left(\frac{1}{N}\sum_{n=0}^{N-1}\left\|\mathbf{H}_k\left(e^{j2\pi\frac{n}{N}}\right)\left(\mathbf{c}_k^{(n)}-\mathbf{e}_k^{(n)}\right)\right\|^2 \leq \frac{4}{\rho}R_k\right).$$

A straightforward manipulation reveals that the quantity

$$\sum_{n=0}^{N-1}\left\|\mathbf{H}_k\left(e^{j2\pi\frac{n}{N}}\right)\left(\mathbf{c}_k^{(n)}-\mathbf{e}_k^{(n)}\right)\right\|^2$$

has the same distribution as the quadratic form $\mathbf{h}_w^H(\mathbf{R}_k \otimes \mathbf{I}_{M_R})\mathbf{h}_w$ with $\mathbf{h}_w \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{M_R N})$. Since each nonzero eigenvalue of the matrix $(\mathbf{R}_k \otimes \mathbf{I}_{M_R})$ has multiplicity M_R, the proof is complete. ■

We shall next consider the high and low-SNR cases, which allows us to formulate tangible code design criteria.

A. High-SNR Design Criteria

We start with the following

Corollary 2: Define

$$P_{ek}(\mathbf{R}) = \mathrm{P}\left(\frac{1}{N}\sum_{n=0}^{N-1}\chi_n\lambda_n(\mathbf{R}) \leq \frac{4}{\rho}R_k\right) \qquad (6)$$

and let $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathbb{R}^N$ be vectors containing the eigenvalues of the positive-semidefinite matrices $\mathbf{R}_1$ and $\mathbf{R}_2$, respectively. If ρ is large, the following holds

$$\boldsymbol{\lambda}_1 \preceq \boldsymbol{\lambda}_2 \Rightarrow P_{ek}(\mathbf{R}_1) \leq P_{ek}(\mathbf{R}_2).$$

Proof: The corollary is a direct consequence of the Schur-convexity of $P_{ek}(\mathbf{R})$ in the vector of eigenvalues $\boldsymbol{\lambda}$ of $\mathbf{R}$ for large ρ [12, Th. 2]. ■

As a consequence of Corollary 2, it can be shown[1] that (keeping R_k fixed)

$$\lim_{\rho\to\infty}\frac{\log P_{ek}(\mathbf{R})}{\log\rho} = -\operatorname{rank}\{\mathbf{R}\}M_R$$

which implies that the high-SNR error probability P_{ek} is dominated by the codeword difference matrices $\mathbf{R}_k$ with minimum rank. In the high-SNR regime, the space-time/frequency code design criteria minimizing $P_e = P_{e1} + P_{e2} + P_{e3}$ can now be summarized as follows:

1) Given the rate tuple (R_1, R_2), determine the type ($k = 1, 2, 3$) of error event dominating the overall error probability P_e.
2) If the dominant error event is of type k ($k = 1, 2, 3$), the corresponding error probability P_{ek} is minimized by codes that fulfill:
   a) *Rank criterion:* For every codeword pair $(\mathbf{C}_k, \mathbf{E}_k)$ with $\mathbf{C}_k \neq \mathbf{E}_k$ the rank of the corresponding codeword difference matrix $\mathbf{R}_k$ shall be maximized.
   b) *Eigenvalue criterion:*[2] For every codeword pair $(\mathbf{C}_k, \mathbf{E}_k)$ with $\mathbf{C}_k \neq \mathbf{E}_k$ the vector of eigenvalues of the corresponding matrix $\mathbf{R}_k$ shall be majorized by any other possible choice of eigenvalues.

Before discussing the implications of the design criteria above, we note that space-time/frequency codes designed for the single-user case according to 2a) and 2b) are also optimal w.r.t. the classical (rate-independent) space-time/frequency code design criteria [1]–[3]. This can be seen by noting that the product of eigenvalues (i.e., the determinant of $\mathbf{R}_k$) is a Schur-concave function [13, Th. 3.C.1] and hence, an optimum vector of eigenvalues according to 2b) is optimal w.r.t. the determinant criterion in [1] as well. It is interesting to observe that a completely different derivation (as compared to [1]–[3]) aiming at minimizing the probability of encountering a bad effective channel[3] realization results in essentially the same criteria as those based on pairwise error probability [1]–[3].

Joint code design: If the rate tuple (R_1, R_2) lies in the dominant error event region 1 or 2, we design codes according to the criteria 2a) and 2b) for each of the two users. The two codebooks can furthermore be chosen independently of each other. If (R_1, R_2) lies in region 3, the design criteria 2a) and 2b) have to be applied to the sum of the two codeword difference matrices $\mathbf{R}_3 = \mathbf{R}_1 + \mathbf{R}_2$ (joint design). Note, however, that the users will not cooperate in selecting their codewords. In the case of a general number of users, the design rule is to apply 2a) and 2b) to the sum of the codeword difference matrices corresponding to the subset of users leading to the dominant error event.

B. Low-SNR Design Criteria

In the low-SNR regime, using the approximation $e^{-x} \approx 1 - x$ on the LHS of the argument of $\mathrm{P}(\cdot)$ in (4), we can state

Theorem 3: For small ρ (keeping R_k fixed), the error probability P_{ek} ($k = 1, 2, 3$) satisfies[4]

$$P_{ek} \lessapprox \mathrm{P}\left(\sum_{n=0}^{N-1}\chi_n\lambda_n(\mathbf{R}_k) \leq \frac{4}{\rho}e^{NR_k^0}\left(1-e^{-NR_k}\right)\right) \qquad (7)$$

where the χ_n denote independent χ^2-distributed random variables with $2M_R$ degrees of freedom each and $\lambda_n(\mathbf{R}_k)$ stands for the nth eigenvalue of

$$\mathbf{R}_k = \frac{1}{\widetilde{M}_k}\sum_{\substack{\mathbf{C}_k,\mathbf{E}_k\\ \mathbf{C}_k\neq\mathbf{E}_k}}\mathbf{G}(\mathbf{C}_k,\mathbf{E}_k)\mathbf{G}^H(\mathbf{C}_k,\mathbf{E}_k). \qquad (8)$$

Noting that for small ρ the RHS of (7) is Schur-concave in the eigenvalues $\lambda_n(\mathbf{R}_k)$, it follows that in the low-SNR regime,

[1] The proof consists of establishing lower and upper bounds on the right-hand side (RHS) of (6) and showing their asymptotic (in ρ) tightness.

[2] In the case where not all $\mathbf{R}_k$ have full rank, the optimization in this step extends only over the $\mathbf{R}_k$ with minimum rank.

[3] By effective channel we mean the physical channel in combination with the space-time/frequency code.

[4] Note that the inequality in (4) becomes approximate as a result of the small-x approximation of e^{-x}.

optimum code design amounts to minimizing the rank of $\mathbf{R}_k$ in (8), which implies that the rank of each term on the RHS of (8) should be minimized with all terms having the same range space. We note that this result cannot be obtained by specializing the classical code design criteria in [1]–[3] to the low-SNR case.

V. Numerical Example

In the following, we present a numerical example for the high-SNR regime, highlighting the importance of joint code designs. We consider a two-user system with $M_T = M_R = 2$, $N = 4$ and $L = 1$. If $x_1(i), x_2(i)$ for $i = 1, 2, 3, 4$ are the independently chosen QPSK data symbols of user 1 and 2, respectively, then the codeword matrices for user 1 take the form

$$\mathbf{C}_1 = \begin{bmatrix} x_1(1) & x_1(2) & x_1(3) & x_1(4) \\ -x_1^*(2) & x_1^*(1) & -x_1^*(4) & x_1^*(3) \end{bmatrix} \tag{9}$$

which is simply a concatenation of two Alamouti-type [14] codewords. Choosing $\mathbf{C}_2$ to be of the same form as $\mathbf{C}_1$, both users' codes achieve maximum diversity order in single-user channels. However, such a design does not take the type-3 error events into account, since the minimum rank of $\overline{\mathbf{C}} - \overline{\mathbf{E}}$ (and hence of $\mathbf{R}_3 = \mathbf{R}_1 + \mathbf{R}_2$) is two only, e.g., there are codeword difference matrices that are of the form

$$\overline{\mathbf{C}} - \overline{\mathbf{E}} = \begin{bmatrix} \times & 0 & 0 & 0 \\ 0 & \times & 0 & 0 \\ \times & 0 & 0 & 0 \\ 0 & \times & 0 & 0 \end{bmatrix}$$

where $\times$ denotes a nonzero entry. If instead, we use a joint design obtained by composing the codewords $\mathbf{C}_2$ as (obtained by swapping columns 2 and 3 in (9) and multiplying column 3 of the resulting matrix by $e^{-j\pi/8}$)

$$\mathbf{C}_2 = \begin{bmatrix} x_2(1) & x_2(3) & e^{-j\frac{\pi}{8}} x_2(2) & x_2(4) \\ -x_2^*(2) & -x_2^*(4) & e^{-j\frac{\pi}{8}} x_2^*(1) & x_2^*(3) \end{bmatrix}$$

while $\mathbf{C}_1$ remains unchanged, the resulting codeword difference matrices $\overline{\mathbf{C}} - \overline{\mathbf{E}}$ will have a minimum rank of three (as opposed to two in the original design), which results in a higher diversity order for the type-3 error events. Fig. 2 shows the corresponding error probabilities P_{e3} (obtained through Monte-Carlo simulations). Clearly, the joint design exhibits superior performance.

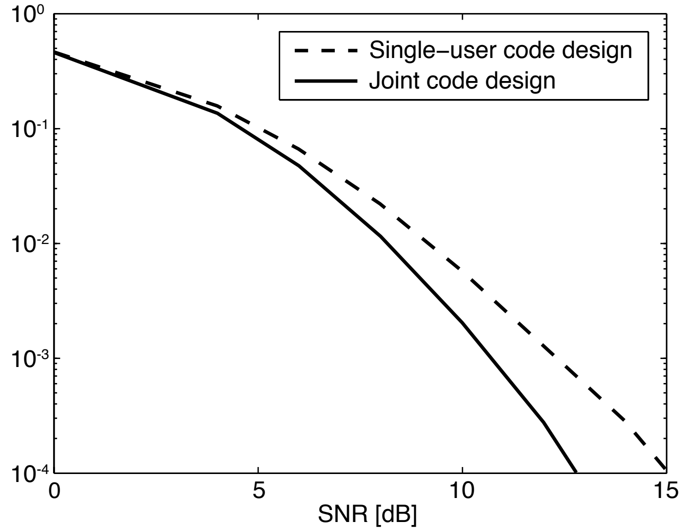

Fig. 2. P_{e3} for single-user and joint code design as a function of SNR.

VI. Concluding Remarks

We found space-time/frequency code design criteria for fading multiantenna MACs with perfect CSI at the receiver, using the concept of dominant error event regions first introduced in [7]. The essence of our design criteria is to recognize that, depending on the transmission rate tuple, joint code designs may or may not be necessary. It was shown that joint designs essentially require applying the classical design criteria [1]–[3] to a sum of codeword difference matrices, with the specific sum depending on the subset of users leading to the dominant error event for the given transmission rate tuple. Systematic joint code designs constitute an interesting area of research. Finally, we note that as a side result of our analysis we showed that the classical (based on pairwise error probability) code design criteria are recovered from a criterion that essentially aims at minimizing the probability of encountering a bad effective channel realization.

References

[1] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communication: Performance criterion and code construction," *IEEE Trans. Inform. Theory*, vol. 44, no. 2, pp. 744–765, Mar. 1998.

[2] H. Bölcskei and A. J. Paulraj, "Space-frequency coded broadband OFDM systems," in *Proc. IEEE Wireless Commun. Netw. Conf. (WCNC)*, vol. 1, Chicago, IL, Sept. 2000, pp. 1–6.

[3] B. Lu and X. Wang, "Space-time code design in OFDM systems," in *Proc. IEEE Global Telecommun. Conf. (GLOBECOM)*, vol. 2, San Francisco, CA, Nov. 2000, pp. 1000–1004.

[4] A. Paulraj, R. Nabar, and D. Gore, *Introduction to Space-Time Wireless Communications*. Cambridge, UK: Cambridge Press, 2003.

[5] B. K. Ng and E. S. Sousa, "On bandwidth-efficient multiuser-space-time signal design and detection," *IEEE J. Select. Areas Commun.*, vol. 20, no. 2, pp. 320–329, Feb. 2002.

[6] A. F. Naguib, N. Seshadri, and A. R. Calderbank, "Applications of space-time block codes and interference suppression for high capacity and high data rate wireless systems," in *Proc. Asilomar Conf. Signals, Syst., Comput.*, vol. 2, Pacific Grove, CA, Nov. 1998, pp. 1803–1810.

[7] R. G. Gallager, "A perspective on multiaccess channels," *IEEE Trans. Inform. Theory*, vol. 31, no. 2, pp. 124–142, Mar. 1985.

[8] A. Peled and A. Ruiz, "Frequency domain data transmission using reduced computational complexity algorithms," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process. (ICASSP)*, vol. 5, Denver, CO, 1980, pp. 964–967.

[9] D. N. C. Tse and P. Viswanath, *Fundamentals of Wireless Communication*. Cambridge, UK: Cambridge Press, 2005.

[10] S. Visuri and H. Bölcskei, "Multiple access strategies for frequency-selective MIMO channels," *IEEE Trans. Inform. Theory*, 2006, to appear.

[11] R. G. Gallager, *Information Theory and Reliable Communication*. New York: Wiley & Sons, 1968.

[12] H. Boche and E. A. Jorswieck, "Outage probability of multiple antenna systems: Optimal transmission and impact of correlation," in *Proc. Int. Zurich Seminar Commun.*, Zürich, Switzerland, Feb. 2004, pp. 116–119.

[13] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*. New York: Academic Press, 1979.

[14] S. M. Alamouti, "A simple transmitter diversity scheme for wireless communications," in *IEEE J. Select. Areas Commun.*, vol. 16, no. 8, Oct. 1998, pp. 1451–1458.