On the "Critical Rate" in Ricean MIMO Channels

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***Abstract*— We analyze the outage characteristics of correlated Ricean fading, coherent multiple-input multiple-output (MIMO) channels. In particular, we establish the notion of a "critical rate", below which communication at zero outage is possible and above which the channel appears as Ricean fading. The critical rate is shown to depend on the "angle" between the subspace spanned by the Ricean component of the channel matrix and the subspace spanned by the correlation matrix of the Rayleigh fading component. A nonzero critical rate is possible only if the correlation matrix of the Rayleigh fading component is rank-deficient. Finally, we provide a complete characterization of the optimum diversity-multiplexing tradeoff for correlated Ricean fading MIMO channels taking into account the existence of the critical rate and thereby establishing the notion of a critical multiplexing gain.**

I. Introduction

Analytical approaches for computing the performance limits of multiple-input multiple-output (MIMO) wireless channels commonly rely on the Rayleigh fading assumption [1]–[3]. In practice, however, the channel often exhibits a static component, especially in fixed wireless scenarios [4], which suggests a Ricean fading distribution. Despite their practical relevance, capacity results for Ricean MIMO channels remain scarce. Capacity bounds for spatially correlated Ricean MIMO channels can be found in [5]. Properties of the capacity achieving input distribution have been reported in [6], [7]. Results on the eigenvalue distribution of Ricean MIMO channels can be found in [8], and an asymptotic (in the number of antennas) capacity analysis has been carried out in [9]. Most of the results available in the literature are based on the classical i.i.d. complex Gaussian assumption for the random component of the channel matrix. For orthogonal space-time block codes, it was demonstrated in [10] that a more general Ricean MIMO channel model allowing for correlated entries in the random component of the channel matrix gives rise to the notion of a *critical rate*. For data rates below the critical rate, the fading channel behaves like an additive white Gaussian noise (AWGN) channel and communication with zero outage is possible. Above the critical rate, the channel exhibits Ricean fading behavior. A nonzero critical rate is possible only if the correlation matrix $\mathbf{R}$ of the random component of the channel matrix is rank-deficient and the Ricean component spans dimensions in the null space of $\mathbf{R}$. The analysis in [10] only covers the case of Ricean MIMO channels in conjunction with orthogonal space-time block codes, which turn the MIMO channel into an effective single-input single-output channel.

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Contributions: The main theme of the present paper is to establish the notion of a critical rate for the general MIMO case. Our detailed contributions can be summarized as follows:

- We compute the critical rate for general Ricean MIMO channels as a function of the channel statistics.
- We provide a complete characterization of the optimum diversity-multiplexing (DM) tradeoff for Ricean MIMO channels taking into account the existence of the critical rate, thereby establishing the notion of a critical multiplexing gain.

Notation: The superscripts ^{T}, ^{H} and * stand for transposition, conjugate transposition and conjugation, respectively. $\mathbf{I}_N$ is the $N \times N$ identity matrix, $\mathbf{0}$ is the all-zeros matrix of appropriate size, and $\|\mathbf{x}\|$ stands for the Euclidean norm of the vector $\mathbf{x}$. $\text{diag}_{n=1}^{N}\{a_n\}$ is the $N \times N$ diagonal matrix with diagonal elements a_n. For an $M \times N$ matrix $\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_N]$, we define $\text{vec}\{\mathbf{A}\} = [\mathbf{a}_1^T \ \mathbf{a}_2^T \ \cdots \ \mathbf{a}_N^T]^T$ and $\|\mathbf{A}\|_F$ is the Frobenius norm of $\mathbf{A}$. $\mathcal{E}[\cdot]$ denotes the expectation operator. $\mathbb{R}_0^+$ stands for the set of all nonnegative real numbers. A multivariate, circularly symmetric, zero-mean, complex Gaussian random vector is a random vector $\mathbf{z} = \mathbf{x} + j\mathbf{y} \sim \mathcal{CN}(\mathbf{0}, \mathbf{\Sigma})$, where the real-valued random vectors $\mathbf{x}$ and $\mathbf{y}$ are jointly Gaussian, $\mathcal{E}[\mathbf{z}] = \mathbf{0}$, $\mathcal{E}[\mathbf{z}\mathbf{z}^H] = \mathbf{\Sigma}$, and $\mathcal{E}[\mathbf{z}\mathbf{z}^T] = \mathbf{0}$. $f(x)$ and $g(x)$ are said to be exponentially equal, denoted by $f(x) \doteq g(x)$, if $\lim_{x\to\infty} \frac{\log f(x)}{\log x} = \lim_{x\to\infty} \frac{\log g(x)}{\log x}$. Exponential inequality, denoted by $\dot{\leq}$ and $\dot{\geq}$, is defined analogously. Logarithms are to the base e unless stated otherwise.

II. System Model

We consider a point-to-point frequency-flat fading MIMO channel with M_T transmit and M_R receive antennas. The corresponding channel matrix is given by

$$\mathbf{H} = \overline{\mathbf{H}} + \mathbf{R}^{1/2}\mathbf{H}_w$$

where $\overline{\mathbf{H}} = \mathcal{E}[\mathbf{H}]$ represents the fixed (static) channel component and $\mathbf{R}^{1/2}\mathbf{H}_w$ with $\text{vec}\{\mathbf{H}_w\} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{M_R M_T})$ denotes the zero-mean random channel component. The $M_R \times M_R$

receive correlation matrix $\mathbf{R}$ is of rank r. The transmit antennas are assumed to fade in an uncorrelated fashion. The input-output relation is given by

$$\mathbf{y} = \sqrt{\rho}\,\mathbf{H}\mathbf{c} + \mathbf{z}$$

where ρ stands for the signal-to-noise ratio (SNR), $\mathbf{c} \sim \mathcal{CN}(\mathbf{0}, \frac{1}{M_T}\mathbf{I}_{M_T})$ denotes the M_T-dimensional transmit signal vector, and $\mathbf{z} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{M_R})$ represents the additive noise vector. Finally, throughout the paper, the receiver is assumed to have perfect knowledge of the channel, whereas the transmitter does not have any channel state information (CSI), but is aware of the channel statistics.

III. The Notion of a "Critical Rate"

Our first main result, summarized in Theorem 1 below, provides an expression for the critical rate in the general MIMO case and shows that a geometrical interpretation, similar to the one provided in [10], can be given.

Theorem 1: The mutual information of the Ricean MIMO channel with channel matrix $\mathbf{H} = \overline{\mathbf{H}} + \mathbf{R}^{1/2}\mathbf{H}_w$ can be decomposed as

$$I = \log\det\left(\mathbf{I}_{M_T} + \frac{\rho}{M_T}\mathbf{B}^H\mathbf{B}\right) + I_{\mathrm{f}}$$

where I_{f} is a nonnegative random variable and $\mathbf{B} = \mathbf{U}_\perp^H\overline{\mathbf{H}}$ with $\mathbf{U}_\perp$ containing the eigenvectors corresponding to the zero eigenvalues of $\mathbf{R}$.

Proof: Based on the eigendecomposition $\mathbf{R} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H$, we define $\mathbf{U} = [\mathbf{U}_\parallel \ \mathbf{U}_\perp]$, where $\mathbf{U}_\parallel$ and $\mathbf{U}_\perp$ contain the eigenvectors corresponding to the nonzero and zero eigenvalues, respectively, of $\mathbf{R}$. Analogously, we define

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_+ & \\ & \mathbf{0} \end{bmatrix}$$

so that $\mathbf{R} = \mathbf{U}_\parallel\mathbf{\Lambda}_+\mathbf{U}_\parallel^H$. Since $\mathbf{U}$ is a unitary $M_R \times M_R$ matrix, we can write

$$I = \log\det\left(\mathbf{I}_{M_R} + \frac{\rho}{M_T}\mathbf{U}^H\mathbf{H}\mathbf{H}^H\mathbf{U}\right). \tag{1}$$

Letting $\widetilde{\mathbf{H}}_w$ be an $r \times M_T$ matrix such that $\mathrm{vec}\{\widetilde{\mathbf{H}}_w\} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{rM_T})$ yields

$$\begin{aligned}\mathbf{U}^H\mathbf{H} &\sim \mathbf{U}^H\overline{\mathbf{H}} + \mathbf{U}^H\mathbf{U}_\parallel\mathbf{\Lambda}_+^{1/2}\widetilde{\mathbf{H}}_w\\ &= \begin{bmatrix} \mathbf{U}_\parallel^H\overline{\mathbf{H}} \\ \mathbf{U}_\perp^H\overline{\mathbf{H}} \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}_+^{1/2}\widetilde{\mathbf{H}}_w \\ \mathbf{0} \end{bmatrix}\end{aligned}$$

and hence

$$\mathbf{U}^H\mathbf{H}\mathbf{H}^H\mathbf{U} \sim \begin{bmatrix} \mathbf{A}\mathbf{A}^H & \mathbf{A}\mathbf{B}^H \\ \mathbf{B}\mathbf{A}^H & \mathbf{B}\mathbf{B}^H \end{bmatrix}$$

where $\mathbf{A} = \mathbf{U}_\parallel^H\overline{\mathbf{H}} + \mathbf{\Lambda}_+^{1/2}\widetilde{\mathbf{H}}_w$ and $\mathbf{B} = \mathbf{U}_\perp^H\overline{\mathbf{H}}$. Therefore, the mutual information in (1) can be expressed as

$$I = \log\det\left(\begin{bmatrix} \mathbf{I}_r + \frac{\rho}{M_T}\mathbf{A}\mathbf{A}^H & \frac{\rho}{M_T}\mathbf{A}\mathbf{B}^H \\ \frac{\rho}{M_T}\mathbf{B}\mathbf{A}^H & \mathbf{I}_{M_R-r} + \frac{\rho}{M_T}\mathbf{B}\mathbf{B}^H \end{bmatrix}\right).$$

Employing the Schur complement formula for determinants [11, Thm. 0.8.5], we get

$$I = \log\det\left(\mathbf{I}_{M_R-r} + \frac{\rho}{M_T}\mathbf{B}\mathbf{B}^H\right) + I_{\mathrm{f}} \tag{2}$$

where

$$\begin{aligned}I_{\mathrm{f}} = \log\det\Bigg(&\mathbf{I}_r + \frac{\rho}{M_T}\mathbf{A}\mathbf{A}^H\\ &-\frac{\rho^2}{M_T^2}\mathbf{A}\mathbf{B}^H\left(\mathbf{I}_{M_R-r} + \frac{\rho}{M_T}\mathbf{B}\mathbf{B}^H\right)^{-1}\mathbf{B}\mathbf{A}^H\Bigg).\end{aligned} \tag{3}$$

Applying $\det(\mathbf{I} + \mathbf{X}\mathbf{Y}) = \det(\mathbf{I} + \mathbf{Y}\mathbf{X})$ to the first term on the right-hand side (RHS) of (2) then yields the desired result. The expression for I_{f} in (3) can be simplified by using the matrix inversion lemma [11, Thm. 0.7.4] to obtain

$$I_{\mathrm{f}} = \log\det\left(\mathbf{I}_r + \frac{\rho}{M_T}\mathbf{A}\left(\mathbf{I}_{M_T} + \frac{\rho}{M_T}\mathbf{B}^H\mathbf{B}\right)^{-1}\mathbf{A}^H\right).$$

Noting that the matrix $\mathbf{A}\left(\mathbf{I}_{M_T} + \frac{\rho}{M_T}\mathbf{B}^H\mathbf{B}\right)^{-1}\mathbf{A}^H$ is positive semidefinite, it follows that $I_{\mathrm{f}} \geq 0$, which completes the proof. ∎

With the outage probability at rate R defined as

$$P_e(R) = \mathrm{P}(I \leq R)$$

Theorem 1 implies $P_e(R) = 0$ for $R \leq R_{\mathrm{crit}}$ where

$$R_{\mathrm{crit}} = \log\det\left(\mathbf{I}_{M_T} + \frac{\rho}{M_T}\mathbf{B}^H\mathbf{B}\right). \tag{4}$$

Equivalently, this result says that for rates below the critical rate, communication with zero outage is possible, i.e., the fading channel behaves like an AWGN channel. We infer from (4) that the critical rate is the capacity of an effective AWGN channel with the $(M_R - r) \times M_T$ transfer matrix $\mathbf{B}$; this effective AWGN channel is determined by the projection of the range space of $\overline{\mathbf{H}}$ onto the null space of $\mathbf{R}$. In other words, the effective MIMO channel with transfer matrix $\mathbf{B}$ is spanned by those dimensions of $\overline{\mathbf{H}}$ that are not impaired by fading. The critical rate is equal to zero in the following cases: i) $\mathrm{rank}\{\mathbf{R}\} = M_R$; ii) $\overline{\mathbf{H}} = \mathbf{0}$; iii) $\mathrm{rank}\{\mathbf{R}\} < M_R$, $\overline{\mathbf{H}} \neq \mathbf{0}$, and $\mathbf{U}_\perp^H\overline{\mathbf{H}} = \mathbf{B} = \mathbf{0}$, i.e., $\overline{\mathbf{H}}$ lies completely in the range space of $\mathbf{R}$. Similar to [10], one can express the critical rate as a function of the principal angles [12] between the subspaces spanned by $\overline{\mathbf{H}}$ and by $\mathbf{R}$.

We conclude by noting that in the special case of a multiple-input single-output (MISO) channel where

$$I = \log\left(1 + \frac{\rho}{M_T}\|\mathbf{H}\|_F^2\right)$$

the expression for the critical rate in (4) can be shown to reduce to the critical rate computed in [10].

IV. Diversity-Multiplexing Tradeoff

In the following, we provide a complete characterization of the DM tradeoff [13] for Ricean MIMO channels. We shall see that the concept of a critical rate leads to the concept of a critical multiplexing gain.

Following [13], we define the diversity gain corresponding to a multiplexing gain of m as

$$d(m) = -\lim_{\rho\to\infty} \frac{\log \mathrm{P}(I \le m\log\rho)}{\log\rho}.$$

The optimality of the i.i.d. Gaussian input distribution with respect to the DM tradeoff can be established pursuing the same line of reasoning as in [13].

Before tackling the computation of the DM tradeoff curve for the general Ricean MIMO channel, we need the following Lemma, which states that the presence of a nonzero mean channel component does not alter the DM tradeoff behavior as long as the random channel component is i.i.d. complex Gaussian.

Lemma 2: The outage probability P_e of the $n \times k$ Ricean MIMO channel with channel matrix $\mathbf{H} = \overline{\mathbf{H}} + \mathbf{H}_w$, where $\mathbf{H}_w$ is such that $\mathrm{vec}\{\mathbf{H}_w\} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{nk})$, satisfies

$$P_e \doteq \mathrm{P}\left(\log\det\left(\mathbf{I}_n + \frac{\rho}{M_T}\mathbf{H}_w\mathbf{H}_w^H\right) \le m\log\rho\right).$$

Proof: The key idea of the proof is to show that setting the mean component of any column vector $\mathbf{h}_i$ $(i = 1, 2, \ldots, k)$ of $\mathbf{H} = \begin{bmatrix}\mathbf{h}_1 & \mathbf{h}_2 & \cdots & \mathbf{h}_k\end{bmatrix}$ to $\mathbf{0}$ preserves the exponential order of P_e. This result can be applied successively to all columns, thereby establishing the proof of the general statement. Without loss of generality, we provide the proof for the first column $\mathbf{h}_1$.

The mutual information of the Ricean MIMO channel with channel matrix $\mathbf{H}$ can be written as

$$I = \log\det\left(\mathbf{I}_n + \frac{\rho}{M_T}\sum_{i=1}^{k}\mathbf{h}_i\mathbf{h}_i^H\right). \tag{5}$$

Defining the nonsingular (w. p. 1) $n \times n$ matrix $\mathbf{M} = \mathbf{I}_n + \frac{\rho}{M_T}\sum_{i=2}^{k}\mathbf{h}_i\mathbf{h}_i^H$ allows to rewrite (5) as

$$I = \log\det(\mathbf{M}) + \log\left(1 + \frac{\rho}{M_T}\mathbf{h}_1^H\mathbf{M}^{-1}\mathbf{h}_1\right).$$

Inspired by the fact that the matrix $\mathbf{M}$ depends only on the random vectors $\mathbf{h}_2, \mathbf{h}_3, \ldots, \mathbf{h}_k$, we express the outage probability P_e in terms of a conditional probability according to

$$\begin{aligned} P_e &= \mathcal{E}_{\mathbf{h}_2,\mathbf{h}_3,\ldots,\mathbf{h}_k}\left[\mathrm{P}\left(I \le m\log\rho \,\middle|\, \mathbf{h}_2, \mathbf{h}_3, \ldots, \mathbf{h}_k\right)\right] \\ &= \mathcal{E}_{\mathbf{h}_2,\mathbf{h}_3,\ldots,\mathbf{h}_k}\Bigg[\mathrm{P}\Bigg(\mathbf{h}_1^H\mathbf{M}^{-1}\mathbf{h}_1 \\ &\qquad \le \frac{M_T}{\rho}\left(\rho^m\det(\mathbf{M}^{-1}) - 1\right)\Bigg|\, \mathbf{h}_2, \mathbf{h}_3, \ldots, \mathbf{h}_k\Bigg)\Bigg]. \end{aligned}$$

Noting that the elements of the vector $\mathbf{h}_1$ are independent complex Gaussian random variables with unit variance, but possibly different means, it follows from [14, Eq. (3.1a.4)] that (conditioned on $\mathbf{h}_2, \mathbf{h}_3, \ldots, \mathbf{h}_k$) we have $\mathbf{h}_1^H\mathbf{M}^{-1}\mathbf{h}_1 = \sum_{i=1}^{n}\lambda_{M,i}^{-1}\chi_i$, where the χ_i $(i = 1, 2, \ldots, n)$ are independent noncentral χ^2-distributed random variables with noncentrality parameters $\overline{\chi}_i$ and two degrees of freedom each; $\lambda_{M,i}^{-1} > 0$ $(i = 1, 2, \ldots, n)$ denotes the ith eigenvalue of $\mathbf{M}^{-1}$.

Defining the set $\mathscr{A}_\mathbf{h}$ conditioned on $\mathbf{h}_2, \mathbf{h}_3, \ldots, \mathbf{h}_k$ as

$$\mathscr{A}_\mathbf{h} = \left\{\mathbf{v} \in \left(\mathbb{R}_0^+\right)^n : \sum_{i=1}^{n}\lambda_{M,i}^{-1}v_i \le \frac{M_T}{\rho}\left(\rho^m\det(\mathbf{M}^{-1}) - 1\right)\right\}$$

enables us to write

$$P_e = \mathcal{E}_{\mathbf{h}_2,\mathbf{h}_3,\ldots,\mathbf{h}_k}\left[\int_{\mathbf{v}\in\mathscr{A}_\mathbf{h}} p_{\chi_1,\chi_2,\ldots,\chi_n}(\mathbf{v})\,d\mathbf{v}\right]$$

where $p_{\chi_1,\chi_2,\ldots,\chi_n}(\mathbf{v})$ follows from [15, p. 76] as

$$p_{\chi_1,\chi_2,\ldots,\chi_n}(\mathbf{v}) = \prod_{i=1}^{n} e^{-(\overline{\chi}_i + v_i)}\left(\sum_{l=0}^{\infty}\frac{(4v_i\overline{\chi}_i)^l}{(l!)^2}\right). \tag{6}$$

Next, in order to reveal the exponential order behavior of P_e, we set $v_i = \rho^{-\alpha_i}$ $(i = 1, 2, \ldots, n)$ so that the term for $l = 0$ in the power series expansion of (6) can be seen to dominate the error probability and therefore

$$P_e \doteq \mathcal{E}_{\mathbf{h}_2,\mathbf{h}_3,\ldots,\mathbf{h}_k}\left[\int_{\boldsymbol{\alpha}:\rho^{-\boldsymbol{\alpha}}\in\mathscr{A}_\mathbf{h}} (\log\rho)^n \prod_{i=1}^{n} e^{-\overline{\chi}_i} e^{-\rho^{-\alpha_i}}\rho^{-\alpha_i}\,d\boldsymbol{\alpha}\right]$$

where the notation $\boldsymbol{\alpha} : \rho^{-\boldsymbol{\alpha}} \in \mathscr{A}_\mathbf{h}$ means $(\alpha_1, \alpha_2, \ldots, \alpha_n) : \begin{bmatrix}\rho^{-\alpha_1} & \rho^{-\alpha_2} & \cdots & \rho^{-\alpha_n}\end{bmatrix} \in \mathscr{A}_\mathbf{h}$.

We note that $\prod_{i=1}^{n} e^{-\overline{\chi}_i} = e^{-\|\mathcal{E}[\mathbf{h}_1]\|^2}$ reduces to a constant factor which does not affect the exponential order of P_e. Furthermore, the factor $(\log\rho)^n$ has no impact on the exponential order of P_e either, since it grows only logarithmically in ρ so that

$$P_e \doteq \mathcal{E}_{\mathbf{h}_2,\mathbf{h}_3,\ldots,\mathbf{h}_k}\left[\int_{\boldsymbol{\alpha}:\rho^{-\boldsymbol{\alpha}}\in\mathscr{A}_\mathbf{h}} \prod_{i=1}^{n} e^{-\rho^{-\alpha_i}}\rho^{-\alpha_i}\,d\boldsymbol{\alpha}\right]. \tag{7}$$

Next, we note that following the same steps as above for $\mathcal{E}[\mathbf{h}_1] = 0$ would lead to

$$p_{\chi_1,\chi_2,\ldots,\chi_n}(\mathbf{v}) = \prod_{i=1}^{n} e^{-v_i}$$

which upon applying the transformation $v_i = \rho^{-\alpha_i}$ $(i = 1, 2, \ldots, n)$ results in P_e satisfying (7). This shows that the presence of a nonzero mean does not affect the exponential behavior of P_e. As mentioned at the outset, successively applying this result to all columns of $\mathbf{H}$ completes the proof. ∎

Since $R_{\text{crit}} \doteq m_{\text{crit}} \log\rho$ with $m_{\text{crit}} = \text{rank}\{\mathbf{B}\}$, it follows by inspection that $d(m) = \infty$ for $m \leq m_{\text{crit}}$. We note that

$$\begin{aligned} P_e &= \mathrm{P}(I_{\mathrm{f}} + R_{\text{crit}} \leq m \log\rho) \\ &\doteq \mathrm{P}(I_{\mathrm{f}} + m_{\text{crit}} \log\rho \leq m \log\rho) \\ &= \mathrm{P}(I_{\mathrm{f}} \leq \Delta m \log\rho) \end{aligned} \tag{8}$$

where $\Delta m = m - m_{\text{crit}}$. It remains to determine $d(m)$ for $m > m_{\text{crit}}$, which, as a consequence of (8), can be done by characterizing the exponential behavior of

$$P_f = \mathrm{P}(I_{\mathrm{f}} \leq \Delta m \log\rho).$$

This will be accomplished by showing that P_f is exponentially equal to the outage probability of a zero-mean i.i.d. Gaussian MIMO channel with appropriate dimensions, for which the DM tradeoff curve has already been characterized in [13].

Lemma 3: The probability P_f satisfies

$$P_f \doteq \mathrm{P}\left(\log\det\left(\mathbf{I}_{M_T-b} + \frac{\rho}{M_T}\widehat{\mathbf{A}}^H\widehat{\mathbf{A}}\right) \leq \Delta m \log\rho\right)$$

where $b = \text{rank}\{\mathbf{B}\}$, $\widehat{\mathbf{A}} = \mathbf{\Lambda}_+^{-1/2}\mathbf{U}_\parallel^H\overline{\mathbf{H}}\mathbf{V}_\perp + \widehat{\mathbf{H}}_w$ with $\text{vec}\{\widehat{\mathbf{H}}_w\} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{r(M_T-b)})$ and $\mathbf{V}_\perp$ denotes the $M_T \times (M_T - b)$ matrix containing the eigenvectors of $\mathbf{B}^H\mathbf{B} = \mathbf{V}\mathbf{\Lambda}_B\mathbf{V}^H$ corresponding to zero eigenvalues.

Proof: The proof idea is to establish an upper and a lower bound on P_f, which are exponentially equal. We start by rewriting I_{f} as

$$I_{\mathrm{f}} = \log\det\left(\mathbf{I}_r + \mathbf{A}\mathbf{V}\,\text{diag}_{i=1}^{M_T}\left\{\frac{\rho}{M_T + \rho\lambda_{B,i}}\right\}\mathbf{V}^H\mathbf{A}^H\right)$$

where $\lambda_{B,i} = \lambda_i(\mathbf{B}^H\mathbf{B})$ $(i = 1, 2, \ldots, M_T)$. Next, we split up the $M_T \times M_T$ matrix $\mathbf{V}$ according to

$$\mathbf{V} = \begin{bmatrix}\mathbf{V}_\parallel & \mathbf{V}_\perp\end{bmatrix}$$

where $\mathbf{V}_\parallel$ denotes the matrix containing the eigenvectors of $\mathbf{B}^H\mathbf{B}$ corresponding to the b nonzero eigenvalues. With the above definitions, we can write

$$I_{\mathrm{f}} = \log\det\Bigg(\mathbf{I}_r + \frac{\rho}{M_T}\mathbf{A}\mathbf{V}_\perp\mathbf{V}_\perp^H\mathbf{A}^H + \mathbf{A}\mathbf{V}_\parallel\,\text{diag}_{i=1}^{b}\left\{\frac{\rho}{M_T + \rho\lambda_{B,i}}\right\}\mathbf{V}_\parallel^H\mathbf{A}^H\Bigg).$$

Since both $\mathbf{A}\mathbf{V}_\perp\mathbf{V}_\perp^H\mathbf{A}^H$ and $\mathbf{A}\mathbf{V}_\parallel\,\text{diag}_{i=1}^{b}\{\rho/(M_T + \rho\lambda_{B,i})\}\mathbf{V}_\parallel^H\mathbf{A}^H$ are positive semidefinite, we can invoke a Theorem by Weyl [11, Thm. 4.3.1] to lower-bound I_{f} according to

$$I_{\mathrm{f}} \geq \log\det\left(\mathbf{I}_r + \frac{\rho}{M_T}\mathbf{A}\mathbf{V}_\perp\mathbf{V}_\perp^H\mathbf{A}^H\right). \tag{9}$$

On the other hand, I_{f} can be upper-bounded using $\frac{\rho}{M_T+\rho\lambda_{B,i}} \leq \frac{1}{\lambda_{B,i}}$ and resorting again to the result by Weyl [11, Thm. 4.3.1] to obtain

$$I_{\mathrm{f}} \leq \log\det\Bigg(\mathbf{I}_r + \frac{\rho}{M_T}\mathbf{A}\mathbf{V}_\perp\mathbf{V}_\perp^H\mathbf{A}^H + \mathbf{A}\mathbf{V}_\parallel\,\text{diag}_{i=1}^{b}\left\{\lambda_{B,i}^{-1}\right\}\mathbf{V}_\parallel^H\mathbf{A}^H\Bigg). \tag{10}$$

The last term inside the argument of the RHS of (10) is independent of ρ. Therefore, in the $\rho \to \infty$ limit, the upper and lower bounds (9) and (10) coincide, which yields

$$P_f \doteq \mathrm{P}\left(\log\det\left(\mathbf{I}_r + \frac{\rho}{M_T}\mathbf{A}\mathbf{V}_\perp\mathbf{V}_\perp^H\mathbf{A}^H\right) \leq \Delta m \log\rho\right).$$

The matrix product $\mathbf{A}\mathbf{V}_\perp$ can be expressed as

$$\mathbf{A}\mathbf{V}_\perp \sim \mathbf{U}_\parallel^H\overline{\mathbf{H}}\mathbf{V}_\perp + \mathbf{\Lambda}_+^{1/2}\widehat{\mathbf{H}}_w$$

where we exploited the fact that $\widetilde{\mathbf{H}}_w\mathbf{V}_\perp \sim \widehat{\mathbf{H}}_w$ with $\text{vec}\{\widehat{\mathbf{H}}_w\} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{r(M_T-b)})$ due to the orthonormality of the columns of $\mathbf{V}_\perp$. Defining

$$\widehat{\mathbf{A}} = \mathbf{\Lambda}_+^{-1/2}\mathbf{U}_\parallel^H\overline{\mathbf{H}}\mathbf{V}_\perp + \widehat{\mathbf{H}}_w$$

and employing the relation $\log\det(\mathbf{I} + \mathbf{X}\mathbf{Y}) = \log\det(\mathbf{I} + \mathbf{Y}\mathbf{X})$ yields

$$P_f \doteq \mathrm{P}\left(\log\det\left(\mathbf{I}_{M_T-b} + \frac{\rho}{M_T}\widehat{\mathbf{A}}^H\mathbf{\Lambda}_+\widehat{\mathbf{A}}\right) \leq \Delta m \log\rho\right).$$

Note that the $r \times r$ real-valued diagonal matrix $\mathbf{\Lambda}_+$ is nonsingular. We denote the smallest diagonal element of $\mathbf{\Lambda}_+$ by $\lambda_{\min}$ and the largest by $\lambda_{\max}$. Due to the concavity of $\log\det(\mathbf{I}+\mathbf{P})$ over the convex set of positive definite Hermitian matrices $\mathbf{P}$ [11, Thm. 7.6.7], we obtain the following bounds

$$P_f \dot{\leq} \mathrm{P}\left(\log\det\left(\mathbf{I}_{M_T-b} + \frac{\lambda_{\min}\rho}{M_T}\widehat{\mathbf{A}}^H\widehat{\mathbf{A}}\right) \leq \Delta m \log\rho\right)$$

$$P_f \dot{\geq} \mathrm{P}\left(\log\det\left(\mathbf{I}_{M_T-b} + \frac{\lambda_{\max}\rho}{M_T}\widehat{\mathbf{A}}^H\widehat{\mathbf{A}}\right) \leq \Delta m \log\rho\right).$$

Since scaling (by the constants $\lambda_{\min}$ and $\lambda_{\max}$) of the SNR does not have an impact on the exponential (in ρ) behavior of P_f, we can conclude that

$$P_f \doteq \mathrm{P}\left(\log\det\left(\mathbf{I}_{M_T-b} + \frac{\rho}{M_T}\widehat{\mathbf{A}}^H\widehat{\mathbf{A}}\right) \leq \Delta m \log\rho\right)$$

which proves the lemma. ∎

The findings of Lemma 3 show that for $m > m_{\text{crit}}$ the DM tradeoff is equivalent to that of an $r \times (M_T - m_{\text{crit}})$-dimensional Ricean MIMO channel, whose random channel component consists of i.i.d. complex Gaussian elements. By combining Lemma 2 and Lemma 3, we have finally proven the following result.

Theorem 4: The DM tradeoff curve for the general Ricean MIMO channel $\mathbf{H} = \overline{\mathbf{H}} + \mathbf{R}^{1/2}\mathbf{H}_w$ with the rank-m_{crit} matrix $\mathbf{B} = \mathbf{U}_\perp^H\overline{\mathbf{H}}$ and $m_{\max} = \min\{M_T, r + m_{\text{crit}}\}$ is given by the piecewise-linear function connecting the points $(m, d(m))$, $m = 0, 1, \ldots, m_{\max}$, where

$$d(m) = \begin{cases}\infty, & m \leq m_{\text{crit}} \\ (r + m_{\text{crit}} - m)(M_T - m), & m_{\text{crit}} < m \leq m_{\max}.\end{cases}$$

The DM tradeoff curve given by Theorem 4 essentially consists of two parts. For multiplexing gains up to the critical multiplexing gain $m_{\text{crit}} = \text{rank}\{\mathbf{B}\}$, the maximum achievable diversity order is infinity, since the channel behaves like an AWGN channel. For multiplexing gains above m_{crit}, the tradeoff curve equals that of an $r \times (M_T - m_{\text{crit}})$ Rayleigh fading MIMO channel. Note that for full-rank $\mathbf{R}$, i.e., $\text{rank}\{\mathbf{R}\} = M_R$, we have $m_{\text{crit}} = 0$ and the tradeoff curve in Theorem 4 is simply that of an $M_R \times M_T$ Rayleigh fading MIMO channel.

Numerical example: An example of a DM tradeoff curve for a Ricean MIMO channel with $M_R = M_T = 4$ is depicted in Fig. 1. We assumed $r = \text{rank}\{\mathbf{R}\} = 2$ and $\text{rank}\{\overline{\mathbf{H}}\} = 2$. Furthermore, the matrices $\mathbf{R}$ and $\overline{\mathbf{H}}$ are chosen such that $m_{\text{crit}} = \text{rank}\{\mathbf{B}\} = 1$. Consequently, for $m \leq 1$, the channel behaves like an AWGN channel.

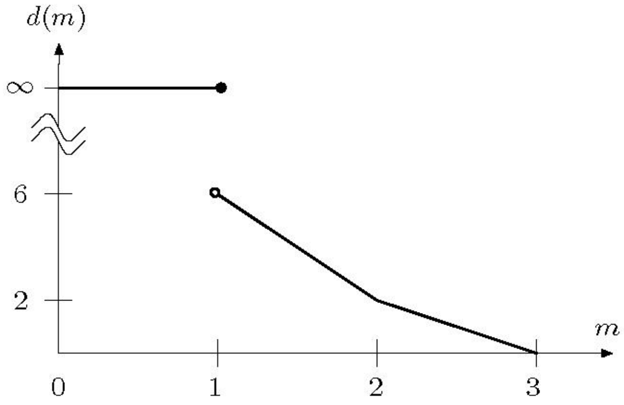

Fig. 1. Diversity-multiplexing tradeoff curve for a Ricean MIMO channel with $M_R = M_T = 4$, $r = 2$ and $m_{\text{crit}} = 1$.

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