Canonical Conditions for $K/2$ Degrees of Freedom

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***Abstract*—Stotz and Bölcskei, 2015, identified an explicit condition for $K/2$ degrees of freedom (DoF) in constant single-antenna interference channels (ICs). This condition is expressed in terms of linear independence—over the rationals—of monomials in the off-diagonal entries of the IC matrix and is satisfied for almost all IC matrices. There is, however, a prominent class of IC matrices that admits $K/2$ DoF but fails to satisfy this condition. The main contribution of the present paper is a more general condition for $K/2$ DoF (in fact for $1/2$ DoF for each user) that, inter alia, encompasses this example class. While the existing condition by Stotz and Bölcskei is of algebraic nature, the new condition is canonical in the sense of capturing the essence of interference alignment by virtue of being expressed in terms of a generic injectivity condition that guarantees separability of signal and interference.**

I. Introduction

Interference alignment as introduced by Cadambe and Jafar [1] is a signaling scheme that realizes $K/2$ degrees of freedom (DoF) in K-user interference channels (ICs) by exploiting time-frequency selectivity in the channel. It was found in [2], [3] that—surprisingly—full (i.e., $K/2$) DoF can also be achieved in single-antenna K-user ICs with constant IC matrix, i.e., in channels that do not exhibit any selectivity. A general formula for the DoF in constant single-antenna ICs was derived in [4]. This formula is in terms of Rényi information dimension [5] of the received signals, which is often difficult to evaluate. Recently, Stotz and Bölcskei [6] showed that the Wu-Shamai-Verdú DoF-formula [4] can be turned into a DoF-formula exclusively in terms of entropy. The resulting equivalent DoF-formula is then used in [6] to identify an explicit sufficient condition on the constant IC matrix to admit $K/2$ DoF. This condition, henceforth referred to as Condition $(*)$, is satisfied for almost all IC matrices. It turns out, however, that Condition $(*)$ is not necessary for $K/2$ DoF as illustrated in [6] by way of an example class of IC matrices, originally introduced in [3], that admit $K/2$ DoF but fail to satisfy Condition $(*)$.

While results on DoF pertain to high-snr asymptotics and are therefore of limited practical relevance, they do contribute to the understanding of complicated communication networks and can serve as sanity checks for (finite-snr) capacity results.

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Contributions. The main contribution of the present paper is a condition for full, i.e., $K/2$, DoF that is more general than Condition $(*)$. While Condition $(*)$ is of algebraic nature, the new Condition $(**)$ is canonical in the sense of capturing the essence of interference alignment by virtue of being expressed in terms of a generic injectivity condition that guarantees separability of signal and interference.

In this paper, instead of studying conditions for $K/2$ DoF, we pursue a slightly more refined analysis in the sense of investigating conditions that guarantee $1/2$ DoF per user. Clearly if every user achieves $1/2$ DoF, we have $K/2$ DoF in total. The opposite, however, is not true and restricting to the achievability of $1/2$ DoF per user allows us to exclude trivial counterexamples that do not satisfy Condition $(**)$ but still admit $K/2$ DoF. It turns out, in fact, that Condition $(*)$ guarantees $K/2$ DoF by guaranteeing $1/2$ DoF per user. This becomes evident by inspecting [6, Sec. VI]. Moreover, the example class in [6] showing that Condition $(*)$ is not necessary for $K/2$ DoF also demonstrates that Condition $(*)$ is not necessary for $1/2$ DoF per user.

The proof in [6] of Condition $(*)$ guaranteeing $K/2$ DoF is based on two ideas: separability of signal and interference and alignment cardinality. In [6] separability and alignment cardinality are both satisfied by requiring linear independence—over the rationals—of monomials in the off-diagonal channel coefficients. The resulting algebraic nature of Condition $(*)$ leads, however, to superfluous restrictions, which, in turn, are responsible for a popular example class of IC matrices [3] that admit $1/2$ DoF per user failing to satisfy Condition $(*)$. The main point of the present paper is to show how these restrictions can be removed through a non-algebraic condition for separability together with an independent condition on alignment cardinality. It remains open whether Condition $(**)$ is necessary for $1/2$ DoF per user.

Notation. Random variables are represented by uppercase letters from the end of the alphabet. Lowercase letters are used exclusively for deterministic quantities. Boldface uppercase letters indicate deterministic matrices. Sets are denoted by uppercase calligraphic letters. For $x \in \mathbb{R}$, we write $\lfloor x \rfloor$ for the largest integer not exceeding x. We use $\log$ to denote the logarithm to base 2 and $\ln$ stands for the natural logarithm. $\mathbb{E}[\cdot]$ denotes the expectation operator. $H(\cdot)$ stands for entropy and $h(\cdot)$ for differential entropy with base 2.

II. System model

We consider a constant single-antenna K-user IC, where $K \geqslant 2$, with IC matrix $\mathbf{H} = (h_{ij})_{1 \leqslant i,j \leqslant K} \in \mathbb{R}^{K \times K}$, and input-output relation

$$Y_i = \sqrt{\mathsf{snr}} \sum_{j=1}^{K} h_{ij} X_j + Z_i, \quad i = 1, ..., K, \tag{1}$$

where $X_i \in \mathbb{R}$ is the input at the i-th transmitter, $Y_i \in \mathbb{R}$ is the output at the i-th receiver, and $Z_i \in \mathbb{R}$ is noise of absolutely continuous distribution with $h(Z_i) > -\infty$ and $H(\lfloor Z_i \rfloor) < \infty$. The input signals at different transmitters are independent and noise is i.i.d. across users and channel uses.

The IC matrix $\mathbf{H}$ is assumed to be known perfectly at all transmitters and receivers and we take $h_{ii} \neq 0$, for $i = 1, ..., K$, which avoids degenerate situations where direct links between transmitter-receiver pairs are absent. We impose the average power constraint

$$\frac{1}{n} \sum_{k=1}^{n} \left(x_i^{(k)} \right)^2 \leqslant 1$$

on codewords $\left(x_i^{(1)} \ldots \, x_i^{(n)} \right)$ of block-length n transmitted by user $i = 1, ..., K$. The DoF of $\mathbf{H}$ are defined as

$$\mathsf{DoF}(\mathbf{H}) := \limsup_{\mathsf{snr} \to \infty} \frac{\overline{C}(\mathbf{H}; \mathsf{snr})}{\frac{1}{2} \log \mathsf{snr}}, \tag{2}$$

where $\overline{C}(\mathbf{H}; \mathsf{snr})$ is the sum-capacity [7] of the channel (1). We say that every user achieves (at least) $1/2$ DoF if there exist a sequence $\mathsf{snr}_k \to \infty$ and, for every user $i = 1, ..., K$, corresponding codes with rates $R_i(\mathsf{snr}_k)$ such that

$$\limsup_{k \to \infty} \frac{R_i(\mathsf{snr}_k)}{\frac{1}{2} \log \mathsf{snr}_k} \geqslant \frac{1}{2}. \tag{3}$$

III. Main Result

We denote the vector containing the off-diagonal coefficients of $\mathbf{H}$ by $\check{\mathbf{h}} \in \mathbb{R}^{K(K-1)}$, and let $f_1, f_2, ...$ be the monomials of all degrees[1] in $K(K-1)$ variables enumerated as follows: $f_1, ..., f_{\varphi(d)}$ are the monomials of degree not larger than d, where

$$\varphi(d) := \binom{K(K-1) + d}{d}.$$

In [6, Thm. 1] it was shown that the following Condition $(*)$—satisfied for almost all $\mathbf{H}$—is sufficient for $\mathsf{DoF}(\mathbf{H}) = K/2$, and, in fact, the proof can be directly reformulated to show that Condition $(*)$ is sufficient for $1/2$ DoF per user.

$$\text{For each } i = 1, ..., K, \text{ the set}$$
$$\{f_j(\check{\mathbf{h}}) : j \geqslant 1\} \cup \{h_{ii} f_j(\check{\mathbf{h}}) : j \geqslant 1\} \tag{$*$}$$
$$\text{is linearly independent over } \mathbb{Q}.$$

[1]The "degree" of a monomial is the sum of all exponents of the variables involved (sometimes called the total degree).

It turns out, however, that Condition $(*)$ is not necessary, neither for $K/2$ DoF in total, nor for $1/2$ DoF per user. This is illustrated in [6, Sect. VIII] through the following example class first presented in [3].

Example 1: Consider IC matrices that have $h_{ii} \in \mathbb{R} \backslash \mathbb{Q}$, $i = 1, ..., K$, and $h_{ij} \in \mathbb{Q} \backslash \{0\}$, for $i, j = 1, ..., K$ with $i \neq j$. This example class was shown implicitly in [4, Thm. 7] to allow $1/2$ DoF per user, i.e., $K/2$ DoF in total. However, as two rational numbers are linearly dependent over $\mathbb{Q}$, these IC matrices violate Condition $(*)$.

Further examples of channels that have $1/2$ DoF per user but fail to satisfy Condition $(*)$ can be constructed by starting from an arbitrary $\mathbf{H}$ admitting $1/2$ DoF per user and setting one or more of the off-diagonal channel coefficients to zero. This operation simply eliminates interference and hence does not lead to a reduction in the number of DoF achieved by each user. It does, however, result in IC matrices that violate Condition $(*)$ as zero becomes an element of the two sets in Condition $(*)$, which, in turn, leads to linear dependence over $\mathbb{Q}$. This shows that the algebraic nature of Condition $(*)$ is brittle.

The main result of the present paper is a sufficient condition for $1/2$ DoF per user, which is more general than Condition $(*)$, incorporates the class of channels in Example 1, and is canonical in the following sense. While Condition $(*)$ is of algebraic nature, the new condition captures the essence of interference alignment by virtue of being expressed in terms of a generic injectivity condition that guarantees separability of signal and interference.

Before stating the new condition, we need the following definition. For $d \geqslant 0$ and $N \geqslant 1$, let

$$\mathcal{W}_{N,d} := \left\{ \sum_{i=1}^{\varphi(d)} a_i f_i(\check{\mathbf{h}}) \;:\; a_1, ..., a_{\varphi(d)} \in \{0, ..., N-1\} \right\}$$

and set

$$\mathcal{W} := \bigcup_{d \geqslant 0} \bigcup_{N \geqslant 1} \mathcal{W}_{N,d}. \tag{4}$$

Inspired by [6], the sets $\mathcal{W}_{N,d}$ will be used to build input distributions achieving $1/2$ DoF per user.

Theorem 1: Let $\mathcal{W}$ be as in (4) and suppose that, potentially after suitable scaling of rows and columns by non-zero constants, the IC matrix $\mathbf{H}$ satisfies the following Condition $(**)$:

$$\text{For each } i = 1, ..., K, \text{ the map}$$
$$\begin{aligned} \mathcal{W} \times \mathcal{W} &\to \mathcal{W} + h_{ii} \mathcal{W} \\ (w_1, w_2) &\mapsto w_1 + h_{ii} w_2 \end{aligned} \tag{$**$}$$
$$\text{is injective.}$$

Then, every user achieves $1/2$ DoF.

Remark 1: As mentioned above, we chose to study the achievability of $1/2$ DoF per user in order to exclude trivial examples of channel matrices admitting $K/2$ DoF but failing to satisfy Condition $(**)$. One such example would be as follows. Pick a channel matrix $\mathbf{H}$ such that $\mathsf{DoF}(\mathbf{H}) < K/2$, which necessarily means that not all users achieve $1/2$ DoF. Now, add user pairs that are interference-free and hence achieve 1 DoF until the resulting $\widetilde{K}$-user channel matrix $\widetilde{\mathbf{H}}$ satisfies $\mathsf{DoF}(\widetilde{\mathbf{H}}) \geqslant \widetilde{K}/2$. Clearly, the channel matrix $\widetilde{\mathbf{H}}$ admits $\widetilde{K}/2$ DoF without satisfying Condition $(**)$. We note that all examples of $\mathbf{H}$-matrices admitting full DoF without satisfying Condition $(**)$ we found were of such topological nature.

Remark 2: Condition $(**)$ can equivalently be formulated as follows: $h_{ii}Q - P \neq 0$, for all $i = 1, ..., K$, and all polynomials P, Q in the h_{ij}, $i \neq j$, with integer coefficients. For the particular class of rank-deficient ICs (i.e., $\mathbf{H}$ is not full-rank) this condition was used in [8, Thm. 1] to prove achievability of $K/2$ DoF.

For fully connected $\mathbf{H}$, i.e., when all channel coefficients are non-zero, we have the upper bound $\mathsf{DoF}(\mathbf{H}) \leqslant K/2$ [9, Prop. 1]. When combined with the achievability of $1/2$ DoF per user in Theorem 1, this yields $\mathsf{DoF}(\mathbf{H}) = K/2$ for fully connected $\mathbf{H}$ satisfying Condition $(**)$ and therefore each user achieves exactly $1/2$ DoF.

The amendment on scaling of rows and columns in the formulation of Theorem 1 stems from the fact that $\mathsf{DoF}(\mathbf{H})$ is invariant under these operations [10, Lem. 3], which continues to hold for the DoF achieved by each user. To illustrate the usefulness of this amendment, consider the following example

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \sqrt{2} & 0 \\ 1 & 1 & 1 \end{pmatrix} \tag{5}$$

and note that it fails to satisfy Condition $(**)$ as here $\mathcal{W} = \mathbb{N}$ and $\mathbb{N} \times \mathbb{N} \to \mathbb{N} + h_{11}\mathbb{N} = \mathbb{N} + \mathbb{N}$ is not injective. However, after scaling of the first row and the third column by, e.g., $\sqrt{2}$ we obtain

$$\widetilde{\mathbf{H}} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 1 & \sqrt{2} & 0 \\ 1 & 1 & \sqrt{2} \end{pmatrix},$$

which turns out to satisfy Condition $(**)$ and therefore allows to conclude that $\mathbf{H}$ in (5) admits $1/2$ DoF per user.

We next show that Condition $(**)$ is more general than Condition $(*)$ in [6, Thm. 1], i.e., all IC matrices satisfying Condition $(*)$ also satisfy Condition $(**)$ and Condition $(**)$ encompasses IC matrices $\mathbf{H}$ that have $1/2$ DoF per user but fail to satisfy Condition $(*)$. We start by establishing inclusion. To see this, suppose that $w_1, w_2, \widetilde{w}_1, \widetilde{w}_2 \in \mathcal{W}$ are such that for some $i = 1, ..., K$, we have

$$w_1 + h_{ii}w_2 = \widetilde{w}_1 + h_{ii}\widetilde{w}_2, \tag{6}$$

or equivalently

$$(w_1 - \widetilde{w}_1) + h_{ii}(w_2 - \widetilde{w}_2) = 0. \tag{7}$$

To establish injectivity, as required by Condition $(**)$, we need to show that $(w_1, w_2) = (\widetilde{w}_1, \widetilde{w}_2)$. The first term on the left-hand side (LHS) of (7) is a $\mathbb{Z}$-linear combination of elements from the set $\{f_j(\check{\mathbf{h}}) : j \geqslant 1\}$, whereas the second term is a $\mathbb{Z}$-linear combination of elements from $\{h_{ii}f_j(\check{\mathbf{h}}) : j \geqslant 1\}$. It therefore follows from Condition $(*)$ that the two terms on the LHS of (7) must equal zero individually and hence $w_1 = \widetilde{w}_1$ and $w_2 = \widetilde{w}_2$, which implies that Condition $(**)$ is satisfied. In particular, by [6, Sect. III] this also implies that Condition $(**)$ is satisfied for almost all IC matrices $\mathbf{H}$.

It remains to establish that there exist IC matrices $\mathbf{H}$ with $1/2$ DoF per user but failing to satisfy Condition $(*)$. To this end, we show that the IC matrices $\mathbf{H}$ in Example 1—known to have $1/2$ DoF per user—satisfy Condition $(**)$. To this end, suppose that for $w_1, w_2, \widetilde{w}_1, \widetilde{w}_2 \in \mathcal{W}$ and arbitrary $i \in \{1, ..., K\}$, we have

$$w_1 + h_{ii}w_2 = \widetilde{w}_1 + h_{ii}\widetilde{w}_2. \tag{8}$$

As the off-diagonal channel coefficients h_{ij} are all rational here, each element in $\mathcal{W}$ is rational, too. We therefore find that the LHS of

$$1 \cdot (w_1 - \widetilde{w}_1) + h_{ii} \cdot (w_2 - \widetilde{w}_2) = 0 \tag{9}$$

is a linear combination of $\{1, h_{ii}\}$ with coefficients in $\mathbb{Q}$. Since h_{ii} is irrational, for all $i = 1, ..., K$, the set $\{1, h_{ii}\}$ is linearly independent over $\mathbb{Q}$ and it follows that $w_1 - \widetilde{w}_1 = 0$ and $w_2 - \widetilde{w}_2 = 0$, i.e., $(w_1, w_2) = (\widetilde{w}_1, \widetilde{w}_2)$. This proves that the map in Condition $(**)$ is injective for all $i = 1, ..., K$ and hence establishes the desired result. We see that Condition $(**)$—in contrast to Condition $(*)$—allows for linear dependence over the rationals among the off-diagonal channel coefficients and hence does not suffer from the algebraic brittleness of Condition $(*)$ exposed by Example 1.

IV. Proof Idea and Auxiliary Results

In this section, we discuss the central ideas behind the proof of Theorem 1 as presented in Section V. The tenets of the proof are i) a condition guaranteeing the separability of signal and interference and ii) an alignment cardinality constraint for the codebook. In [6] both of these aspects are deduced from Condition $(*)$. Separability follows from linear independence across the two sets in Condition $(*)$ and the alignment cardinality constraint from linear independence of the elements in the first set in Condition $(*)$. The resulting algebraic nature of Condition $(*)$ leads, however, to superfluous restrictions, which are responsible, inter alia, for Example 1 failing to satisfy Condition $(*)$. Condition $(**)$ removes these constraints. The key to proving that Condition $(**)$ is sufficient for $1/2$ DoF per user is a decoupling of the issues of separability and alignment cardinality.

We begin by restating a result from [6] we will need below.

Proposition 1 ([6, Prop. 1]): Let $r \in (0,1)$ and let $W_1, ..., W_K$ be independent discrete random variables.[2] Then, for all $\mathbf{H}$, the i-th user achieves

$$\min\left\{\frac{H\left(\sum_{j=1}^{K} h_{ij}W_j\right)}{\log(1/r)}, 1\right\} - \min\left\{\frac{H\left(\sum_{j\neq i}^{K} h_{ij}W_j\right)}{\log(1/r)}, 1\right\} \tag{10}$$

[2]Throughout the paper "discrete random variable" refers to a random variable that takes finitely many values.

DoF, for $i = 1, ..., K$.

Remark 3: The original result [6, Prop. 1], in fact, states that the sum of the terms in (10) over $i = 1, ..., K$ constitutes a lower bound on $\mathsf{DoF}(\mathbf{H})$. An inspection of the arguments in [6] reveals, however, that, in fact, the stronger statement presented here holds.

The strategy for proving Theorem 1 is to use Proposition 1 with the W_i i.i.d. uniform on $\mathcal{W}_{N^d,d}$ for $d \geqslant 0$, $N > K-1$ and to show that (10) can be made to be arbitrarily close to $1/2$ for all $i = 1, ..., K$ concurrently. The first step in our proof is to guarantee that $H(\sum_{j\neq i}^K h_{ij}W_j)$ can not be too large relative to $H(W_i)$. This will be accomplished by first noting that $\sum_{j\neq i}^K h_{ij}W_j \in \mathcal{W}_{(K-1)N^d,d+1} \subseteq \mathcal{W}_{N^{d+1},d+1}$ and then showing that the ratio $\frac{\log|\mathcal{W}_{N^{d+1},d+1}|}{\log|\mathcal{W}_{N^d,d}|}$ is close to 1 for d large. We call this the alignment cardinality constraint, which, in the case of vector interference alignment [1], [10] corresponds to aligning interference in low-dimensional subspaces.

Lemma 1: For $N > 1$, we have

$$\liminf_{d\to\infty} \frac{\log|\mathcal{W}_{N^{d+1},d+1}|}{\log|\mathcal{W}_{N^d,d}|} = 1. \tag{11}$$

Proof: Since $\mathcal{W}_{N,d} \subseteq \mathcal{W}_{N',d'}$ for $N \leqslant N'$ and $d \leqslant d'$, we have $\log|\mathcal{W}_{N^{d+1},d+1}| \geqslant \log|\mathcal{W}_{N^d,d}|$, which implies

$$\liminf_{d\to\infty} \frac{\log|\mathcal{W}_{N^{d+1},d+1}|}{\log|\mathcal{W}_{N^d,d}|} \geqslant 1.$$

We next establish (11) by way of contradiction. To this end, we suppose that

$$\liminf_{d\to\infty} \frac{\log|\mathcal{W}_{N^{d+1},d+1}|}{\log|\mathcal{W}_{N^d,d}|} > 1+\varepsilon,$$

for some $\varepsilon > 0$. Then, there exists a $d_0 \geqslant 0$ such that for all $d \in \mathbb{N}$

$$\frac{\log|\mathcal{W}_{N^{d_0+d+1},d_0+d+1}|}{\log|\mathcal{W}_{N^{d_0+d},d_0+d}|} > 1+\varepsilon. \tag{12}$$

Repeated application of (12) would then imply that

$$\begin{aligned}\log|\mathcal{W}_{N^{d_0},d_0}| &< \frac{\log|\mathcal{W}_{N^{d_0+1},d_0+1}|}{1+\varepsilon} \\ &< \dots \\ &< \frac{\log|\mathcal{W}_{N^{d_0+d},d_0+d}|}{(1+\varepsilon)^d}.\end{aligned}$$

Since $|\mathcal{W}_{N^{d_0+d},d_0+d}| \leqslant (N^{(d_0+d)})^{\varphi(d_0+d)}$, this yields

$$\log|\mathcal{W}_{N^{d_0},d_0}| < \frac{\varphi(d_0+d)(d_0+d)\log N}{(1+\varepsilon)^d}. \tag{13}$$

Without loss of generality, we may assume that $d_0 \geqslant 2$. It then follows from $N > 1$ that $N^{d_0} - 1 > 1$ and, since $\{0, ..., N^{d_0}-1\} \subseteq \mathcal{W}_{N^{d_0},d_0}$, this implies that $|\mathcal{W}_{N^{d_0},d_0}| > 1$. We will establish the contradiction $\log|\mathcal{W}_{N^{d_0},d_0}| = 0$. This is accomplished by showing that the right-hand side (RHS) of (13) tends to zero as $d \to \infty$. To this end, we first note that

$$\begin{aligned}\varphi(d_0+d) &= \frac{(K(K-1)+d_0+d)!}{(K(K-1))!(d_0+d)!} \qquad (14)\\ &= \frac{(K(K-1)+d_0+d)\cdot\ldots\cdot(d_0+d+1)}{(K(K-1))!} \qquad (15)\\ &\leqslant \frac{(K(K-1)+d_0+d)^{K(K-1)}}{(K(K-1))!} \qquad (16)\end{aligned}$$

and, since the largest power of d in (16) is $d^{K(K-1)}$, we find that

$$\varphi(d_0+d)(d_0+d) \leqslant d^{K(K-1)+2} \tag{17}$$

for sufficiently large d. On the other hand, from $e^x \geqslant x^{K(K-1)+3}/(K(K-1)+3)!$ for $x \geqslant 0$, we get

$$(1+\varepsilon)^d = e^{d\ln(1+\varepsilon)} \geqslant \frac{(d\ln(1+\varepsilon))^{K(K-1)+3}}{(K(K-1)+3)!}. \tag{18}$$

Combining (17) and (18) we find that

$$\begin{aligned}&\lim_{d\to\infty} \frac{\varphi(d_0+d)(d_0+d)}{(1+\varepsilon)^d} \\ &\leqslant \lim_{d\to\infty} \frac{d^{K(K-1)+2}(K(K-1)+3)!}{(d\ln(1+\varepsilon))^{K(K-1)+3}} \\ &= 0,\end{aligned}$$

which, when used in (13), completes the proof. ■

Remark 4: Lemma 1 is inspired by [8, Lem. 3], which establishes conditions on rank-deficient IC matrices $\mathbf{H}$ to have $\mathsf{DoF}(\mathbf{H}) \geqslant K/2$.

We emphasize that the alignment cardinality constraint (11) was established without invoking Condition ($**$), which shows that the issues of separability and alignment cardinality can, indeed, be dealt with independently of each other.

The second step in our proof is concerned with the separability of signal and interference and uses the injectivity of the map in Condition ($**$) to establish

$$H\left(h_{ii}W_i + \sum_{j\neq i}^K h_{ij}W_j\right) = H(h_{ii}W_i) + H\left(\sum_{j\neq i}^K h_{ij}W_j\right). \tag{19}$$

After taking care of some minor technicalities, this will allow us to show that (10) can be arbitrarily close (from below) to $1/2$, for $i = 1, ..., K$.

V. Proof of Theorem 1

We are now ready to proceed to the formal proof of our main result, Theorem 1.

We begin by noting that the arguments in [10, Lem. 3] showing that $\mathsf{DoF}(\mathbf{H})$ is invariant under scaling of rows and columns extend to prove to per-user DoF invariance. This implies that it suffices to prove the statement for $\mathbf{H}$ satisfying Condition ($**$). As discussed in Section IV, the first step of the proof is to meet the alignment cardinality constraint.

Specifically, we guarantee that $H(\sum_{j\neq i}^{K} h_{ij}W_j)$ can not be too large relative to $H(W_i)$. This will be accomplished by controlling the cardinality of $\mathcal{W}_{N^{d+1},d+1}$ relative to that of $\mathcal{W}_{N^d,d}$. We choose $N > K-1$. As $K-1 \geqslant 1$, we can apply Lemma 1 to find a subsequence $\{\mathcal{W}_{N^{d_n},d_n}\}_{n\geqslant 0}$ of $\{\mathcal{W}_{N^d,d}\}_{d\geqslant 0}$ such that

$$\lim_{n\to\infty} \frac{\log|\mathcal{W}_{N^{d_n+1},d_n+1}|}{\log|\mathcal{W}_{N^{d_n},d_n}|} = 1. \tag{20}$$

We choose discrete random variables $W_1,...,W_K$ i.i.d. uniform on $\mathcal{W}_{N^{d_n},d_n}$ and apply Proposition 1 with $r = |\mathcal{W}_{N^{d_n},d_n}|^{-2}$ to get that the i-th user achieves

$$\min\left\{\frac{H\left(\sum_{j=1}^{K} h_{ij}W_j\right)}{2\log|\mathcal{W}_{N^{d_n},d_n}|},1\right\} - \min\left\{\frac{H\left(\sum_{j\neq i}^{K} h_{ij}W_j\right)}{2\log|\mathcal{W}_{N^{d_n},d_n}|},1\right\} \tag{21}$$

DoF, for $i=1,...,K$, for all $n\in\mathbb{N}$. Note that $\sum_{j\neq i}^{K} h_{ij}W_j \in \mathcal{W}_{(K-1)N^{d_n},d_n+1} \subseteq \mathcal{W}_{N^{d_n+1},d_n+1}$ as $N>K-1$. It follows from the cardinality bound for entropy that

$$\frac{H\left(\sum_{j\neq i}^{K} h_{ij}W_j\right)}{2\log|\mathcal{W}_{N^{d_n},d_n}|} \leqslant \frac{\log|\mathcal{W}_{N^{d_n+1},d_n+1}|}{2\log|\mathcal{W}_{N^{d_n},d_n}|} \xrightarrow{n\to\infty} \frac{1}{2}, \tag{22}$$

where we used (20).

For the second step of the proof concerned with separability according to

$$H\left(h_{ii}W_i + \sum_{j\neq i} h_{ij}W_j\right) = H\left(h_{ii}W_i, \sum_{j\neq i} h_{ij}W_j\right), \tag{23}$$

for $i=1,...,K$, we apply the chain rule to find

$$H\left(h_{ii}W_i, \sum_{j\neq i} h_{ij}W_j\right) \tag{24}$$

$$= H\left(h_{ii}W_i, \sum_{j\neq i} h_{ij}W_j, h_{ii}W_i + \sum_{j\neq i} h_{ij}W_j\right) \tag{25}$$

$$= H\left(h_{ii}W_i + \sum_{j\neq i} h_{ij}W_j\right) + H\left(h_{ii}W_i, \sum_{j\neq i} h_{ij}W_j \,\middle|\, h_{ii}W_i + \sum_{j\neq i} h_{ij}W_j\right). \tag{26}$$

Next, we note that the injectivity of the map in Condition $(**)$ implies

$$H\left(h_{ii}W_i, \sum_{j\neq i} h_{ij}W_j \,\middle|\, h_{ii}W_i + \sum_{j\neq i} h_{ij}W_j\right) = 0, \tag{27}$$

which when combined with (24)–(26) establishes (23). From (23) and the independence of the W_i it now follows that

$$\frac{H\left(\sum_{j=1}^{K} h_{ij}W_j\right) - H\left(\sum_{j\neq i}^{K} h_{ij}W_j\right)}{2\log|\mathcal{W}_{N^{d_n},d_n}|} = \frac{H(W_i)}{2\log|\mathcal{W}_{N^{d_n},d_n}|} \tag{28}$$

$$= \frac{1}{2}, \tag{29}$$

where we used that W_i is uniform on $\mathcal{W}_{N^{d_n},d_n}$, for all $i = 1,...,K$. This allows us to conclude that, for all $n\in\mathbb{N}$, we have

$$\min\left\{\frac{H\left(\sum_{j=1}^{K} h_{ij}W_j\right)}{2\log|\mathcal{W}_{N^{d_n},d_n}|},1\right\} - \min\left\{\frac{H\left(\sum_{j\neq i}^{K} h_{ij}W_j\right)}{2\log|\mathcal{W}_{N^{d_n},d_n}|},1\right\} \geqslant 1 - \frac{\log|\mathcal{W}_{N^{d_n+1},d_n+1}|}{2\log|\mathcal{W}_{N^{d_n},d_n}|}, \tag{30}$$

which follows, as either the first minimum on the LHS of (30) coincides with the non-trivial term in which case by (28), (29) the second minimum coincides with the non-trivial term as well, and therefore the LHS of (30) equals $1/2 \geqslant 1 - \frac{\log|\mathcal{W}_{N^{d_n+1},d_n+1}|}{2\log|\mathcal{W}_{N^{d_n},d_n}|}$, or the first minimum coincides with 1 in which case we upper-bound the second minimum by $\frac{H\left(\sum_{j\neq i}^{K} h_{ij}W_j\right)}{2\log|\mathcal{W}_{N^{d_n},d_n}|} \leqslant \frac{\log|\mathcal{W}_{N^{d_n+1},d_n+1}|}{2\log|\mathcal{W}_{N^{d_n},d_n}|}$ using (22). The proof is completed by noting that, by (22), the RHS of (30) approaches $\frac{1}{2}$ as $n\to\infty$. ■

VI. Concluding Remarks

In Theorem 1 we have shown that Condition (**) is sufficient for each user to achieve $1/2$ DoF. We were not able to identify examples of channel matrices $\mathbf{H}$ admitting $1/2$ DoF per user but failing to satisfy Condition $(**)$. Our hope is that Condition $(**)$ is, in fact, also necessary for $1/2$ DoF per user, but this remains an open problem.

Finally, it would be interesting to apply the methodology developed in this paper to further investigate the influence of rank-deficiency of the IC matrix on DoF and to extend our insights to the MIMO setting.

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