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Ergodic Capacity and Outage Properties of CDMA in Multiple-Access Fading Channels

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Abstract

The purpose of this paper is to conduct an analysis similar to the one by Rupf and Massey in [1] for frequency-selective fading multiple-access channels (MACs) with perfect channel state information (CSI) at the receiver and no CSI at the transmitters. For fixed spreading gain N and the number of users $K \to \infty$, we show that spreading does not incur a loss in ergodic sum capacity. The corresponding optimum spreading sequences are precisely those that achieve the Welch-bound-equality (WBE). For $N \geq KL$, where L denotes the length of the channel impulse response, we find that shift-orthogonal sequence sets maximize the ergodic sum capacity of the fading CDMA channel. For $N > K$, we show that the ergodic sum capacity achieved by CDMA is strictly smaller than the ergodic sum capacity of the fading MAC. In the high-SNR regime, assuming $N \geq KL$, shift-orthogonal sequence sets are shown to also yield full sum diversity gain. In the low-SNR regime, CDMA is capable of achieving the same sum outage performance as the fading MAC, independently of the spreading gain. However, neither WBE sequences nor shift-orthogonal sequences turn out to be optimal in this case.

1 INTRODUCTION

In [1], Rupf and Massey analyzed the capacity of the additive white Gaussian noise (AWGN) symbol-synchronous code-division multiple-access (CDMA) channel. The purpose of this paper is to conduct a similar analysis for the frequency-selective fading multiple-access channel (MAC) with perfect channel state information (CSI) at the receiver and no CSI at the transmitters, imposing an equal-power constraint on the inputs.

In the fading case, multiple-access schemes that realize collision in signal space (such as CDMA) perform strictly better than orthogonal accessing schemes (such as FDMA) [2, 3] — an effect not present in AWGN MACs. From an information-theoretic point of view, when communicating over fading channels, collision in signal space reduces the Jensen penalty. Since spreading allows to vary the amount of collision in signal space, it is interesting to understand how (if at all) the fundamental results in [1] change in the fading case. In addition, fading channels give rise to the notions of ergodic capacity and outage capacity [4]. It is therefore important to understand which properties of spreading sequences govern the ergodic and the outage capacity performance.

Contributions. Our contributions are as follows:

- We show that for fixed spreading gain N and the number of users $K \to \infty$, the ergodic sum capacity achieved by CDMA equals the ergodic sum capacity of the fading MAC. Moreover, the spreading sequence sets achieving ergodic sum capacity are precisely those that satisfy the Welch-bound-equality (WBE). In the regime $N > K$, we show that the ergodic sum capacity realized by CDMA is strictly smaller than the ergodic sum capacity of the fading MAC. For $N \geq KL$, where L denotes the length of the channel impulse response, it is furthermore demonstrated that the corresponding optimum sequence sets induce orthogonal received signatures when the intersymbol interference (ISI) nature of the channel is taken into account.
- With respect to the achievable sum diversity order, shift-orthogonal sequence sets are found to be optimal for $N \geq KL$. In the low-SNR regime, we show that spreading does not entail a loss in terms of sum outage probability. The corresponding optimum sequence sets, however, neither fulfill the WBE condition nor are they shift-orthogonal.

Notation. The superscripts ^{T}, ^{H} and * stand for transposition, conjugate transposition and element-wise conjugation, respectively. $\mathbf{I}_N$ is the $N \times N$ iden-

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tity matrix, $\mathbf{0}$ is the all-zeros matrix of appropriate size, $\|\mathbf{x}\|$ denotes the Euclidean norm of the vector $\mathbf{x}$, $\text{diag}_{n=0}^{N-1}\{x_n\}$ is the $N \times N$ diagonal matrix with main diagonal entries x_n, and $\text{Tr}\{\mathbf{A}\}$ stands for the trace of the matrix $\mathbf{A}$. If $\mathcal{L}$ is a set, then $|\mathcal{L}|$ denotes its cardinality. $\mathcal{E}[\cdot]$ represents the expectation operator. A multivariate, circularly symmetric, zero-mean, complex Gaussian random vector is a random vector $\mathbf{z} = \mathbf{x} + j\mathbf{y} \sim \mathcal{CN}(\mathbf{0}, \mathbf{\Sigma})$, where the real-valued random vectors $\mathbf{x}$ and $\mathbf{y}$ are jointly Gaussian, $\mathcal{E}[\mathbf{z}] = \mathbf{0}$, $\mathcal{E}\big[\mathbf{z}\mathbf{z}^H\big] = \mathbf{\Sigma}$, and $\mathcal{E}\big[\mathbf{z}\mathbf{z}^T\big] = \mathbf{0}$. Logarithms are to the base 2, unless otherwise stated. Mutual information and capacity are always specified in bits/chip.

Organization of the paper. The remainder of this paper is organized as follows. After introducing the channel and signal models in Section 2, we examine the ergodic sum capacity properties of fading CDMA channels and state the corresponding optimal sequence set design criteria in Section 3. Section 4 focuses on the outage properties of fading CDMA channels both for high and low SNR. We conclude in Section 5.

2 SYSTEM MODEL

2.1 Channel Model

In the following, K denotes the number of users. The vector-valued channel between the K single-antenna transmitters and the single-antenna receiver is frequency-selective with L taps. The corresponding vector-valued transfer function is given by

$$\mathbf{h}^T\big(e^{j2\pi\theta}\big) = \big[H_1\big(e^{j2\pi\theta}\big) \;\; H_2\big(e^{j2\pi\theta}\big) \;\; \cdots \;\; H_K\big(e^{j2\pi\theta}\big)\big] = \sum_{l=0}^{L-1} \mathbf{h}_l^T e^{-j2\pi l\theta}, \quad 0 \le \theta < 1.$$

Here, $H_i\big(e^{j2\pi\theta}\big)$ $(i = 1, 2, \ldots, K)$ stands for the single-input single-output (SISO) channel between the i^{th} user and the receiver. The $K \times 1$ complex-valued random vector $\mathbf{h}_l$ represents the l^{th} tap. The channel is assumed to be purely Rayleigh fading with $\mathbf{h}_l \sim \mathcal{CN}\big(\mathbf{0}, \sigma_l^2 \mathbf{I}_K\big)$ and uncorrelated across taps. The path gains σ_l^2, obtained from the power delay profile, are normalized such that $\sum_{l=0}^{L-1} \sigma_l^2 = 1$.

2.2 Signal Model

We assume i.i.d. (across time and across users) Gaussian codebooks with the i^{th} user $(i = 1, 2, \ldots, K)$ spreading its data symbols x_i using the spreading sequence $\mathbf{s}_i \in \mathbb{C}^N$, normalized such that $\|\mathbf{s}_i\|^2 = N$. The set of spreading sequences is denoted as $\mathcal{S} = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K\}$. We impose an equal-power constraint

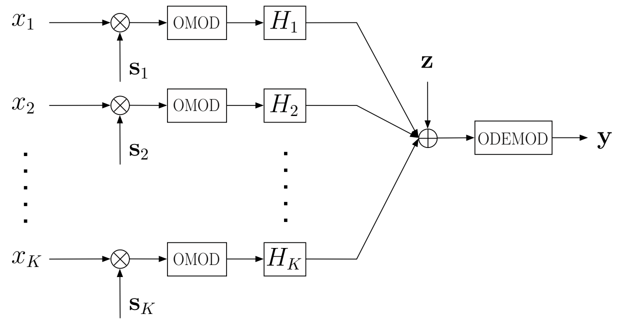

Figure 1: System model of the fading CDMA channel with each of the users employing OFDM modulation. OMOD stands for OFDM modulation and ODEMOD denotes OFDM demodulation.

according to $\mathcal{E}\big[|x_i|^2\big] = P/K$ for $i = 1, 2, \ldots, K$ with P denoting the total power (across users).

We assume a periodic signal model, or more precisely, the impact of the channel on the transmitted signal is described by circular convolution rather than linear convolution. Such an input-output relation is obtained, for instance, when each of the users employs orthogonal frequency-division multiplexing (OFDM) [5]. In particular, we choose the block length to equal the spreading gain N, i.e., the OFDM system employs N tones, and we assume perfect chip synchronization. A cyclic prefix [5] (guard interval) of length L_{CP} $(N > L_{\text{CP}} \ge L)$ guarantees that each of the frequency-selective fading channels decouples into a set of parallel flat-fading channels. The $N \times 1$ received signal vector is given by

$$\mathbf{y} = \sum_{i=1}^{K} \mathbf{\Lambda}_i \mathbf{s}_i x_i + \mathbf{z} \tag{1}$$

with the noise vector $\mathbf{z} \sim \mathcal{CN}(\mathbf{0}, N_0 \mathbf{I}_N)$ and $\mathbf{\Lambda}_i = \text{diag}_{n=0}^{N-1}\big\{H_i\big(e^{j2\pi\frac{n}{N}}\big)\big\}$. Throughout the paper, the receiver is assumed to have perfect knowledge of all users' channels and spreading sequences, whereas the transmitters do not have any CSI. The overall system model described in this section is depicted in Fig. 1.

3 ERGODIC SUM CAPACITY

3.1 Brief Review of the AWGN CDMA Channel

Before stating our results, we provide, for the sake of completeness, a brief summary of the key findings in [1] pertaining to the equal average-input-energy case. Organizing the spreading sequences into an $N \times K$ matrix $\mathbf{S} = \big[\mathbf{s}_1 \;\; \mathbf{s}_2 \;\; \cdots \;\; \mathbf{s}_K\big]$, it is shown in [1] that the sum capacity of the K-user AWGN CDMA channel reads as

$$C^{\text{CDMA}} = \begin{cases} \frac{K}{N}\log(1+\rho N), & \text{if } N > K \text{ for } \mathbf{S}^H\mathbf{S} = N\mathbf{I}_K \\ \log(1+\rho K), & \text{if } K \geq N \text{ for } \mathbf{S}\mathbf{S}^H = K\mathbf{I}_N \end{cases} \tag{2}$$

with the per-user SNR $\rho = P/(KN_0)$. The result in (2) shows that in the underspread case ($K \geq N$), the sum capacity of the AWGN MAC can be achieved, provided that the spreading sequences are chosen such that the rows of $\mathbf{S}$ are mutually orthogonal and have equal energy or equivalently, the spreading sequence set meets Welch's lower bound on total squared correlation [6]. In the overspread case ($N > K$), the sum capacity realized by CDMA is strictly smaller than the sum capacity of the AWGN MAC. The corresponding optimum spreading sequence set consists of mutually orthogonal $\mathbf{s}_i$ $(i = 1, 2, \ldots, K)$.

3.2 Brief Review of the Frequency-Selective Fading MAC

It is well known that the ergodic sum capacity of the frequency-selective fading MAC with no CSI at the transmitters and perfect CSI at the receiver is achieved by i.i.d. Gaussian codebooks and satisfies [4]

$$C_{erg}^{\text{MAC}} = \mathcal{E}\left[\log\left(1+\rho\chi_{2K}^2\right)\right] \tag{3}$$

with the same ρ as in (2) and χ_{2K}^2 denotes a χ^2-distributed random variable with $2K$ degrees of freedom. Obviously, the ergodic sum capacity in (3) represents an upper bound on the ergodic sum capacity of the fading CDMA channel.

3.3 Ergodic Sum Capacity of the Fading CDMA Channel

For perfect CSI at the receiver, the signal model in (1) yields the mutual information between the received signal vector $\mathbf{y}$ and the source symbol vector $\mathbf{x} = \left[x_1 \; x_2 \; \cdots \; x_K\right]^T$ as

$$I(\mathbf{y};\mathbf{x}|\mathcal{H}) = \frac{1}{N}\log\det\left(\mathbf{I}_N + \rho\sum_{i=1}^{K}\mathbf{\Lambda}_i\mathbf{s}_i\mathbf{s}_i^H\mathbf{\Lambda}_i^H\right) \tag{4}$$

where $\mathcal{H}$ stands for the set of channels $\left\{H_i\left(e^{j2\pi\theta}\right)\right\}_{i=1}^K$. Next, we note that $\mathbf{\Lambda}_i\mathbf{s}_i \sim \mathcal{CN}\left(\mathbf{0}, \mathbf{R}_i\right)$ with

$$\mathbf{R}_i = \mathcal{E}\left[\mathbf{\Lambda}_i\mathbf{s}_i\mathbf{s}_i^H\mathbf{\Lambda}_i^H\right] = \sum_{l=0}^{L-1}\sigma_l^2\mathbf{D}^l\mathbf{s}_i\mathbf{s}_i^H\mathbf{D}^{-l} \tag{5}$$

and $\mathbf{D} = \text{diag}_{n=0}^{N-1}\{e^{-j2\pi\frac{n}{N}}\}$. Defining the $N \times L$ matrices

$$\mathbf{G}_i = \left[\sigma_0\mathbf{s}_i \quad \sigma_1\mathbf{D}\mathbf{s}_i \quad \cdots \quad \sigma_{L-1}\mathbf{D}^{L-1}\mathbf{s}_i\right]$$

we can write (5) as $\mathbf{R}_i = \mathbf{G}_i\mathbf{G}_i^H$. Using $C_{erg}^{\text{CDMA}} = \mathcal{E}_{\mathcal{H}}\left[I(\mathbf{y};\mathbf{x}|\mathcal{H})\right]$, we obtain

$$C_{erg}^{\text{CDMA}} = \frac{1}{N}\mathcal{E}\left[\log\det\left(\mathbf{I}_N + \rho\sum_{i=1}^{K}\mathbf{G}_i\mathbf{w}_i\mathbf{w}_i^H\mathbf{G}_i^H\right)\right] \tag{6}$$

with $\mathbf{w}_i \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_L)$ being independent across i.

We can rewrite (6) in terms of time-domain versions of the $\mathbf{G}_i$ by defining $\mathbf{s}_i^t = \mathbf{F}^H\mathbf{s}_i$, with the $N \times N$ FFT matrix $[\mathbf{F}]_{m,n} = \left(1/\sqrt{N}\right)e^{-j2\pi mn/N}$ $(m, n = 0, 1, \ldots, N-1)$, and denoting the sequence obtained by cyclically shifting the elements of $\mathbf{s}_i^t$ downwards by l positions as $\mathbf{s}_i^{t-l}$. Noting that $\mathbf{F}^H\mathbf{G}_i = \mathbf{G}_i^t$ with

$$\mathbf{G}_i^t = \left[\sigma_0\mathbf{s}_i^t \quad \sigma_1\mathbf{s}_i^{t-1} \quad \cdots \quad \sigma_{L-1}\mathbf{s}_i^{t-L+1}\right] \tag{7}$$

we observe that C_{erg}^{CDMA} in (6) does not change if $\mathbf{G}_i$ is replaced by $\mathbf{G}_i^t$ on the right-hand side (RHS) of (6). We shall make use of this equivalence result later when interpreting our findings summarized below.

Based on (6), we are now ready to address the problem of finding spreading sequences $\mathbf{s}_i$ such that C_{erg}^{CDMA} is maximized. Answering this question calls for a distinction between three different cases: the *overspread* ($N \geq KL$), the *underspread* ($K \geq N$), and an *intermediate* case ($KL > N > K$). Obviously, the intermediate case vanishes in the frequency-flat fading channel where $L = 1$ and is hence a feature of the frequency-selective fading channel. Making statements of general nature in the intermediate case seems difficult. We will therefore limit our focus to the overspread and the (highly) underspread (N fixed, $K \to \infty$) cases.

Theorem 1. *The ergodic sum capacity C_{erg}^{CDMA} of the frequency-selective fading CDMA channel satisfies*

$$C_{erg}^{CDMA} = \begin{cases} \dfrac{K}{N}\mathcal{E}\left[\log\left(1+\rho N\sum_{l=0}^{L-1}\sigma_l^2\xi_l\right)\right], & N \geq KL \\ \log\left(1+\dfrac{P}{N_0}\right), & N \text{ fixed, } K \to \infty \end{cases} \tag{8}$$

where the ξ_l are independent, χ^2-distributed random variables with two degrees of freedom. In the overspread case ($N \geq KL$), the optimal choice of spreading sequences is such that $\mathbf{G}_i^H\mathbf{G}_j = \mathbf{0}$ for $i \neq j$ and $\mathbf{G}_i^H\mathbf{G}_i = N\,\text{diag}_{l=0}^{L-1}\{\sigma_l^2\}$. In the highly underspread case (N fixed, $K \to \infty$), the optimum sequence set $\mathcal{S}$ satisfies $\sum_{i=1}^K\mathbf{G}_i\mathbf{G}_i^H = K\mathbf{I}_N$.

Proof: **Overspread case.** We start by proving the result for the overspread case ($N \geq KL$). Using the matrix identity [7, Thm. 1.3.20] $\det(\mathbf{I}+\mathbf{AB}) = \det(\mathbf{I}+\mathbf{BA})$, we can rewrite (6) to obtain

$$C_{erg}^{\text{CDMA}} = \frac{1}{N}\,\mathcal{E}[\log\det(\mathbf{I}_K + \rho\mathbf{W})] \tag{9}$$

with the $K \times K$ Hermitian random matrix $\mathbf{W}$ given by $[\mathbf{W}]_{i,j} = \mathbf{w}_i^H \mathbf{G}_i^H \mathbf{G}_j \mathbf{w}_j$. Applying Hadamard's inequality to (9), it follows that

$$C_{erg}^{\text{CDMA}} \leq \frac{1}{N}\sum_{i=1}^{K} \mathcal{E}\big[\log\big(1 + \rho\,\mathbf{w}_i^H \mathbf{G}_i^H \mathbf{G}_i \mathbf{w}_i\big)\big] \tag{10}$$

with equality if and only if the $K \times K$ matrix $\mathbf{W}$ is diagonal with probability 1 (w.p.1). This condition is met if $\mathbf{G}_i^H \mathbf{G}_j = 0$ for $i \neq j$, i.e., when the spreading sequences along with their cyclically shifted versions are mutually orthogonal. Since we have a total of K users and $N \geq KL$, this mutual orthogonality can indeed be ensured, provided that the sequence set is designed properly. Since $\mathcal{E}\big[\log\big(1 + \rho\,\mathbf{w}_i^H \mathbf{G}_i^H \mathbf{G}_i \mathbf{w}_i\big)\big]$ is Schur-concave in the eigenvalues $\lambda_l(\mathbf{G}_i^H \mathbf{G}_i)$, it follows with [8, Thm. 1] that the RHS in (10) is maximized if

$$\lambda_l(\mathbf{G}_i^H \mathbf{G}_i) = N\sigma_l^2, \quad l = 0, 1, \ldots, L-1 \tag{11}$$

for $i = 1, 2, \ldots, K$. Here, we used the fact that the vector of eigenvalues of a Hermitian matrix $\mathbf{A}$ majorizes the vector of the diagonal entries of $\mathbf{A}$ [9, Thm. 9.B.1]. This completes the proof for the overspread case.

Highly underspread case. For fixed N and $K \to \infty$, using [10, Thm. 1.8.D], we can infer

$$\frac{1}{K}\sum_{i=1}^{K} \mathbf{G}_i \mathbf{w}_i \mathbf{w}_i^H \mathbf{G}_i^H \xrightarrow{w.p.1} \frac{1}{K}\sum_{i=1}^{K} \mathbf{G}_i \mathbf{G}_i^H$$

which upon application of [10, Thm. 1.7] yields

$$I(\mathbf{y};\mathbf{x}|\mathcal{H}) \xrightarrow{w.p.1} \frac{1}{N}\log\det\left(\mathbf{I}_N + \rho\sum_{i=1}^{K} \mathbf{G}_i \mathbf{G}_i^H\right). \tag{12}$$

It now follows that the RHS of (12) is maximized if

$$\sum_{i=1}^{K} \mathbf{G}_i \mathbf{G}_i^H = K\mathbf{I}_N$$

which eventually yields $C_{erg}^{\text{CDMA}} = \log(1 + P/N_0)$ in the large K limit. □

Before discussing the implications of Theorem 1, we relate the criterion $\sum_{i=1}^{K} \mathbf{G}_i \mathbf{G}_i^H = K\mathbf{I}_N$ to the WBE condition in [1]. The following lemma is proved in [11].

Lemma 1. *The set of spreading sequences $\mathcal{S}$ satisfies*

$$\sum_{i=1}^{K} \mathbf{G}_i \mathbf{G}_i^H = K\mathbf{I}_N \text{ if and only if}$$
$$\sum_{i=1}^{K} \mathbf{s}_i \mathbf{s}_i^H = \mathbf{S}\mathbf{S}^H = K\mathbf{I}_N.$$

We are now ready to state

Theorem 2. *The ergodic sum capacity C_{erg}^{CDMA} of the frequency-selective fading CDMA channel satisfies*

$$C_{erg}^{CDMA} < C_{erg}^{MAC} \quad \text{for } N > K$$
$$C_{erg}^{CDMA} \leq C_{erg}^{MAC} \quad \text{for } K \geq N$$

where equality in the latter case can be achieved if $K \to \infty$ for N fixed and $\mathcal{S}$ satisfies the WBE condition.

Proof: We start with the case $K \geq N$ and note that applying Hadamard's inequality to (4) yields

$$C_{erg}^{\text{CDMA}} \leq \frac{1}{N}\sum_{n=0}^{N-1} \mathcal{E}\left[\log\left(1 + \rho\sum_{i=1}^{K} |s_{i,n} H_i(e^{j2\pi\frac{n}{N}})|^2\right)\right] \tag{13}$$

where $\mathbf{s}_i = \begin{bmatrix} s_{i,0} & s_{i,1} & \cdots & s_{i,N-1}\end{bmatrix}^T$. Noting that $|H_i(e^{j2\pi\frac{n}{N}})|^2$ is χ^2-distributed with two degrees of freedom for all $i = 1, 2, \ldots, K$, $n = 0, 1, \ldots, N-1$ and that the individual users' channels were assumed independent, it follows from [8, Thm. 1] in conjunction with the normalization $\|\mathbf{s}_i\|^2 = N$ $(i = 1, 2, \ldots, K)$ that

$$C_{erg}^{\text{CDMA}} \leq \mathcal{E}\big[\log(1 + \rho\chi_{2K}^2)\big] = C_{erg}^{\text{MAC}}. \tag{14}$$

Still assuming $K \geq N$, (3) together with Theorem 1 results in

$$C_{erg}^{\text{MAC}} = \log(1 + \rho K) = \log\left(1 + \frac{P}{N_0}\right) = C_{erg}^{\text{CDMA}} \tag{15}$$

in the large K limit, which completes the proof of the underspread case.

For $N > K$ (intermediate and overspread cases), the $N \times N$ matrix $(1/K)\sum_{i=1}^{K} \mathbf{G}_i \mathbf{w}_i \mathbf{w}_i^H \mathbf{G}_i^H$ is rank-deficient w.p.1, which combined with the argument leading to (14) yields

$$C_{erg}^{\text{CDMA}} < \mathcal{E}\big[\log(1 + \rho\chi_{2K}^2)\big] = C_{erg}^{\text{MAC}}. \quad \square$$

Summarizing the results in this section, we can draw the following parallels to the findings in [1]. In the underspread regime ($K \geq N$), CDMA is capable of achieving the ergodic sum capacity of the fading MAC at least for N fixed and $K \to \infty$ when $\mathcal{S}$ satisfies the WBE condition. In the AWGN case, the same condition on $\mathcal{S}$ guarantees that CDMA achieves the sum capacity of the AWGN MAC for any (finite) $K \geq N$. In the intermediate and overspread regimes ($N > K$), the sum capacity achieved by CDMA is strictly smaller than the sum capacity of the MAC both in the fading and in the AWGN case. Furthermore, for $N > K$ in the AWGN case and for $N \geq KL$ in the fading case, the sum capacity maximizing sequence design leads to orthogonal sequences. We emphasize, however, that in the frequency-selective fading case, orthogonality of the $\mathbf{s}_i$ $(i = 1, 2, \ldots, K)$ alone is not sufficient to maximize the ergodic sum capacity of the CDMA channel. Theorem 1 combined with (7) shows that we need to ensure orthogonality of the $\mathbf{s}_i$ along with their cyclically shifted versions, i.e., ${\mathbf{G}_i^t}^H \mathbf{G}_j^t = \mathbf{0}$ for $i \neq j$. This shift-orthogonality essentially guarantees that the individual users can be perfectly separated in the receiver despite the presence of ISI. Clearly, in the frequency-flat fading case, orthogonality of the $\mathbf{s}_i$ alone maximizes the (ergodic) sum capacity like in the AWGN case.

4 OUTAGE PROBABILITY ANALYSIS

In this section, we describe our results on the outage properties of CDMA in frequency-selective fading MACs. We start by defining the sum outage probability at sum rate R as $p_{out}(R) = \mathrm{P}(I(\mathbf{y}; \mathbf{x}|\mathcal{H}) \leq R)$ and the sum diversity order as

$$d(R) = -\lim_{\rho \to \infty} \frac{\log(p_{out}(R))}{\log(\rho)}.$$

In the following, we only consider finite rates R that do not scale with SNR. Moreover, in the high-SNR regime, we assume uniform power allocation across tones.

4.1 The High-SNR Case

The proof of the following result is detailed in [11].

Theorem 3. *The sum diversity order of the frequency-selective fading MAC (without spreading) is given by*

$$d_{MAC} = K|\mathcal{L}| \tag{16}$$

where $\mathcal{L} = \{l : \sigma_l^2 > 0\}$.

Theorem 3 states the intuitive fact that the sum diversity order of the fading MAC is given by the total number of users, reflecting the multiuser diversity gain, multiplied by the frequency diversity order $|\mathcal{L}|$, which is simply the number of taps with nonzero variance. We shall next state a result on the achievable sum diversity order of the overspread fading CDMA channel.

Theorem 4. *In the overspread case ($N \geq KL$), the sum diversity order realized by CDMA under frequency-selective fading equals the sum diversity order of the frequency-selective fading MAC, i.e.,*

$$d_{CDMA} = d_{MAC} = K|\mathcal{L}|$$

which is achieved by shift-orthogonal sequences, i.e., $\mathbf{G}_i^H \mathbf{G}_j = \mathbf{0}$ *for* $i \neq j$ *and* $\mathbf{G}_i^H \mathbf{G}_i = N \operatorname{diag}_{l=0}^{L-1}\{\sigma_l^2\}$.

The proof of Theorem 4 is provided in [11]. Employing shift-orthogonal sequences in the overspread case guarantees that delayed replicas of the transmitted signals can be perfectly separated at the receiver and hence, it is evident that full diversity order is achieved. Combining Theorem 4 with Theorem 1, we can conclude that in the overspread case, shift-orthogonal sequence sets maximize both ergodic sum capacity and achievable sum diversity order. In the underspread and intermediate cases, it seems difficult to make general statements on the impact of spreading on the achievable sum diversity order.

4.2 The Low-SNR Case

The concept of diversity order, as described in the previous section, is not applicable to the low-SNR regime. We therefore resort to direct computation of the outage probability as a function of rate. Unlike in the high-SNR regime, however, a uniform power allocation across tones does not lead to minimum outage probability. The following lemma formalizes this result with the proof being provided in [11].

Lemma 2. *In the low-SNR regime, the sum outage probability* $p_{out}^{MAC}(R) = \mathrm{P}(I(\mathbf{y}; \mathbf{x}|\mathcal{H}) \leq R)$ *of the MAC under frequency-selective fading is minimized, irrespective of R, by having each user allocate its fraction of the total power to only one tone. The corresponding sum outage probability is given by*

$$p_{out}^{MAC}(R) = \mathrm{P}\left(\chi_{2K}^2 \leq \frac{2^{RN} - 1}{\rho N}\right). \tag{17}$$

Our results on the impact of spreading in the low-SNR regime are summarized in

Theorem 5. *In the low-SNR regime, the sum outage probability* $p_{out}^{CDMA}(R)$ *of the frequency-selective fading CDMA channel for arbitrary N and K is minimized by any sequence set that meets the condition* $\operatorname{rank}\{\mathbf{G}_i\} = 1$ *for* $i = 1, 2, \ldots, K$. *The resulting sum outage probability moreover satisfies*

$$p_{out}^{CDMA}(R) = p_{out}^{MAC}(R).$$

Proof: For small ρ, we have $\log\det(\mathbf{I}+\rho\mathbf{A}) \approx \log(1+\rho\,\mathrm{Tr}\{\mathbf{A}\})$, which using (4) yields

$$I(\mathbf{y};\mathbf{x}|\mathcal{H}) \approx \frac{1}{N}\log\left(1+\rho\sum_{i=1}^{K}\sum_{l=0}^{L-1}\lambda_{i,l}\xi_{i,l}\right)$$

where the $\xi_{i,l}$ are independent, χ^2-distributed random variables with two degrees of freedom and $\lambda_{i,l} = \lambda_l(\mathbf{G}_i^H\mathbf{G}_i)$, $l = 0,1,\ldots,L-1$. Consequently, we obtain

$$p_{out}^{\mathrm{CDMA}}(R) = \mathrm{P}\left(\sum_{i=1}^{K}\sum_{l=0}^{L-1}\lambda_{i,l}\xi_{i,l} \le \frac{2^{RN}-1}{\rho}\right)$$

which is Schur-concave in the vector of eigenvalues $\lambda_{i,l}$ for ρ sufficiently small [12, Thm. 2]. The sum outage probability is hence minimized if $\mathrm{rank}\{\mathbf{G}_i\} = 1$ for $i = 1,2,\ldots,K$, which implies

$$\lambda_{i,l} = \begin{cases}\mathrm{Tr}\{\mathbf{G}_i^H\mathbf{G}_i\} = N, & \text{for } l = 0\\ 0, & \text{for } l = 1,2,\ldots,L-1.\end{cases}$$

The corresponding sum outage probability is finally obtained as

$$p_{out}^{\mathrm{CDMA}}(R) = \mathrm{P}\left(\chi_{2K}^2 \le \frac{2^{RN}-1}{\rho N}\right)$$

which by comparison with (17) yields $p_{out}^{\mathrm{CDMA}}(R) = p_{out}^{\mathrm{MAC}}(R)$. □

Theorem 5 allows us to conclude for the low-SNR regime that i) spreading does not have an impact on the outage properties of the fading MAC and ii) neither WBE sequences nor shift-orthogonal sequences are optimal from a sum outage probability point of view.

5 CONCLUSIONS

We analyzed the impact of spreading on the ergodic sum capacity and the sum outage probability in frequency-selective fading multiple-access channels with perfect receive CSI and no CSI at the transmitters. Our results indicate that CDMA is capable of achieving the ergodic sum capacity of the fading MAC in the large number of users (K) limit for fixed spreading gain N. The corresponding optimum sequence sets are precisely those that satisfy the Welch-bound-equality. For $N > K$, we showed that CDMA is strictly inferior to the nonspread case in terms of ergodic sum capacity. The maximum sum diversity order of the fading MAC was found to be achievable by CDMA for $N \ge KL$, employing shift-orthogonal sequence sets. In the low-SNR regime, spreading does not incur a loss in terms of outage probability while neither WBE nor shift-orthogonal sequence sets prove optimal.

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