Diversity-Multiplexing Tradeoff in Two-User Fading Interference Channels

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***Abstract*— We analyze the two-user single-antenna fading interference channel with perfect receive channel state information (CSI) and no transmit CSI. The diversity-multiplexing tradeoff (DMT) region of a fixed-power-split Han and Kobayashi (HK)-type superposition coding scheme is considered and design criteria for the corresponding superposition codes are derived. We demonstrate that this scheme is DMT-optimal under strong and very strong interference by showing that it achieves a DMT region outer bound that we derive. In addition, we show that, under very strong interference, decoding interference while treating the intended signal as noise, subtracting the result out, and then decoding the desired signal, a process known as "stripping", achieves the optimal DMT region. Our proofs reveal code design criteria for achieving DMT optimality (in the cases where we can demonstrate it).**

I. Introduction

The interference channel (IC) models the situation where multiple transmitter-receiver pairs communicate over a shared medium. Apart from a few special cases [1], [2], [3], the capacity region of the IC remains unknown. Recently, for the interference-limited regime, Etkin *et al.* [4] characterized the capacity region of the Gaussian IC to within one bit. Later, Telatar and Tse [5] generalized this result to a wider class of ICs. The techniques used in [4], [5] rely on perfect channel state information (CSI) at the transmitter. Shang *et al.* derived the noisy-interference sum-rate capacity for Gaussian ICs in [6], while Raja *et al.* [7] characterized the capacity region of the two-user finite-state compound Gaussian IC to within one bit. Annapureddy and Veeravalli [8] showed that the sum capacity of the two-user Gaussian IC, under weak interference, is achieved by treating interference as noise.

Akuiyibo and Lévêque [9] derived an outer bound on the diversity-multiplexing tradeoff (DMT) region for the two-user fading IC based on the results of Etkin *et al.* [4]. In the present paper, we investigate the achievability of the outer bound reported in [9] and we analyze the DMT region of the fading IC realized by a fixed-power-split Han and Kobayashi (HK)-type superposition coding scheme. We restrict our attention to the two-user case throughout the paper. Furthermore, we assume that the receivers have perfect CSI whereas the transmitters only know the channel statistics. The schemes used in [4] make explicit use of transmit CSI and so does the scheme in [9], which immediately implies that the results reported in [9] serve as an outer bound on the DMT region achievable in the absence of transmit CSI, the case considered here. Our main contributions can be summarized as follows:

- We derive an outer bound on the IC DMT region that is tighter than the bound reported in [9] for certain interference regimes.
- For general interference levels, we compute the DMT region of a fixed-power-split HK-type superposition coding scheme and we provide design criteria for the corresponding superposition codes. We demonstrate that the fixed-power-split HK-type superposition coding scheme achieves the optimal DMT region of the two-user IC under *strong* and *very strong interference*.
- For *very strong interference*, we show that a *stripping decoder*, which decodes interference while treating the intended signal as noise, subtracts the result out, and then decodes the intended signal is DMT-optimal. We furthermore show that the optimal DMT region can be achieved if each of the two transmitters employs a code that is DMT-optimal on a single-input single-output (SISO) channel.

Notation: The superscripts ^{T} and ^{H} stand for transpose and conjugate transpose, respectively. x_i represents the ith element of the column vector $\mathbf{x}$, and $\lambda_{\min}(\mathbf{X})$ denotes the smallest eigenvalue of the matrix $\mathbf{X}$. $\mathbf{I}_N$ is the $N \times N$ identity matrix, and $\mathbf{0}$ denotes the all zeros matrix of appropriate size. All logarithms are to the base 2 and $(a)^+ = \max\{a, 0\}$. $\mathcal{A}$ and $\bar{\mathcal{A}}$ denote a set and its complement, respectively. $\mathcal{B}$ is a subset of $\mathcal{A}$ if $\mathcal{A} \supseteq \mathcal{B}$. $X \sim \mathcal{CN}(0, \sigma^2)$ stands for a circularly symmetric complex Gaussian random variable (RV) with variance σ^2. $f(\rho) \doteq g(\rho)$ denotes exponential equality of the functions $f(\cdot)$ and $g(\cdot)$, i.e.,

$$\lim_{\rho\to\infty} \frac{\log f(\rho)}{\log \rho} = \lim_{\rho\to\infty} \frac{\log g(\rho)}{\log \rho}.$$

The symbols $\dot{\geq}$, $\dot{\leq}$, $\dot{>}$, and $\dot{<}$ are defined analogously.

System model: We consider a fading IC with two transmitter-receiver pairs. The fading coefficient between transmitter i $(i = 1, 2)$ and receiver j $(j = 1, 2)$ is denoted by h_{ij} and is assumed to be $\mathcal{CN}(0, 1)$. Transmitter i ($\mathcal{T}_i$) chooses an N-dimensional codeword $\mathbf{x}_i \in \mathbb{C}^N$, $\|\mathbf{x}_i\|^2 \leq N$, from its codebook, and transmits $\check{\mathbf{x}}_i = \sqrt{P_i}\mathbf{x}_i$ in accordance with its transmit power constraint $\|\check{\mathbf{x}}_i\|^2 \leq NP_i$. In addition, we account for the attenuation of transmit signal i at receiver j ($\mathcal{R}_j$) through the real-valued coefficients $\eta_{ij} > 0$. Defining $\mathbf{y}_i$ and $\mathbf{z}_i \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_N)$ as the N-dimensional received signal vector and noise vector, respectively, at $\mathcal{R}_i$, the input-output relations are given by

$$\mathbf{y}_1 = \eta_{11}h_{11}\check{\mathbf{x}}_1 + \eta_{21}h_{21}\check{\mathbf{x}}_2 + \mathbf{z}_1 \quad (1)$$

$$\mathbf{y}_2 = \eta_{12}h_{12}\check{\mathbf{x}}_1 + \eta_{22}h_{22}\check{\mathbf{x}}_2 + \mathbf{z}_2. \quad (2)$$

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Part of this paper appeared in the Proc. of the IEEE Int. Symposium on Information Theory (ISIT), Seoul, South Korea, June 2009.

Setting $\eta_{11}^2 P_1 = \eta_{22}^2 P_2 = \text{SNR}$ and $\eta_{21}^2 P_2 = \eta_{12}^2 P_1 = \text{SNR}^\alpha$ with $\alpha \in [0, \infty]$ simplifies the exposition of our results, and the comparison to [4] and [9]. The resulting equivalent set of input-output relations is given by

$$\mathbf{y}_1 = \sqrt{\text{SNR}} h_{11} \mathbf{x}_1 + \sqrt{\text{SNR}^\alpha} h_{21} \mathbf{x}_2 + \mathbf{z}_1 \quad (3)$$

$$\mathbf{y}_2 = \sqrt{\text{SNR}^\alpha} h_{12} \mathbf{x}_1 + \sqrt{\text{SNR}} h_{22} \mathbf{x}_2 + \mathbf{z}_2. \quad (4)$$

We assume that both receivers know the signal-to-noise ratio (SNR) value SNR and the parameter α, and $\mathcal{R}_i$ $(i = 1, 2)$ knows $\mathbf{h}_i = [h_{1i}\ h_{2i}]^T$ perfectly, whereas the transmitters know the statistics of h_{ij} $(i, j = 1, 2)$, the SNR value, and the interference parameter α. The data rate of $\mathcal{T}_i$ scales with SNR according to $R_i = r_i \log \text{SNR}$, where the multiplexing rate r_i obeys $0 \leq r_i \leq 1$. As a result, for $\mathcal{T}_i$ to operate at multiplexing rate r_i, we need a sequence of codebooks $\mathcal{C}_i(\text{SNR}, r_i)$, one for each SNR, with $|\mathcal{C}_i(\text{SNR}, r_i)| = 2^{NR_i}$ codewords $\{\mathbf{x}_i^1, \mathbf{x}_i^2, \ldots, \mathbf{x}_i^{2^{NR_i}}\}$. In the following, we will need the multiplexing rate vector $\mathbf{r} = [r_1\ r_2]^T$.

Performance metrics: The error probability corresponding to maximum-likelihood (ML) decoding of $\mathcal{T}_i$ at $\mathcal{R}_i$ under the assumption that the correctly decoded interfering signal from $\mathcal{T}_j$ has been subtracted out is denoted by $\mathbb{P}[E_{ii}|\mathbf{h}_i]$, for $i, j = 1, 2$, and $i \neq j$. The corresponding average (with respect to (w.r.t.) the random channel) error probability is $P(E_{ii}) \triangleq \mathbb{E}_{\mathbf{h}_i}\{\mathbb{P}[E_{ii}|\mathbf{h}_i]\}$. The notation $\mathbf{x}_i^j \to \mathbf{x}_i^k$ designates the event of mistakenly decoding the transmitted codeword $\mathbf{x}_i^j$ for the codeword $\mathbf{x}_i^k$.

The average (w.r.t. the random channel) error probability corresponding to decoding of $\mathcal{T}_i$ at $\mathcal{R}_i$ incurred by a particular communication scheme χ is denoted by $P(E_i^\chi)$, for $i = 1, 2$. Throughout the paper, as done in [9], we use the performance metric $P(E^\chi) = \max\{P(E_1^\chi), P(E_2^\chi)\}$. The DMT region realized by communication scheme χ is then characterized by

$$d^\chi(\mathbf{r}) = -\lim_{\text{SNR} \to \infty} \frac{\log P(E^\chi)}{\log \text{SNR}}. \quad (5)$$

As discussed in [10], [11], the receiver that minimizes the error probability for decoding $\mathcal{T}_i$ at $\mathcal{R}_i$, for $i = 1, 2$, is the *individual ML receiver*, which we define next.

Definition 1: An *individual ML receiver* for $\mathcal{T}_i$ at $\mathcal{R}_j$, for $i, j = 1, 2$, carries out an ML detection of $\mathcal{T}_i$ while treating $\mathcal{T}_k$, for $k = 1, 2$, $k \neq i$, as noise with known structure (i.e., known codebook) [10], [11]. In the following, we denote the error probability of an individual ML receiver for decoding the signal of $\mathcal{T}_i$ at $\mathcal{R}_j$ by $\mathbb{P}\big[\mathcal{E}_{ij}^{IML}\big]$, for $i, j = 1, 2$. The corresponding average (w.r.t. the random channel) error probability is denoted by $P(E_{ij}^{IML}) \triangleq \mathbb{E}_{\mathbf{h}_j}\big\{\mathbb{P}\big[\mathcal{E}_{ij}^{IML}\big]\big\}$.

The DMT region realized by employing an *individual ML receiver* for decoding the signal from $\mathcal{T}_i$ at receiver $\mathcal{R}_i$, for $i = 1, 2$, is given by

$$d^{IML}(\mathbf{r}) = -\lim_{\text{SNR} \to \infty} \frac{\log \max\big\{P(E_{11}^{IML}), P(E_{22}^{IML})\big\}}{\log \text{SNR}}. \quad (6)$$

Since the *individual ML receiver* minimizes the error probability for decoding of $\mathcal{T}_i$ at $\mathcal{R}_i$, for $i = 1, 2$, we have that the DMT region $d^{IML}(\mathbf{r})$ is an outer bound on the DMT region realized by any communication scheme χ, i.e.,

$$d^{IML}(\mathbf{r}) \geq d^\chi(\mathbf{r}). \quad (7)$$

II. Achievable DMT for Joint Decoding

In an IC, the interfering signal from $\mathcal{T}_i$ at $\mathcal{R}_j$, $i, j = 1, 2$ with $i \neq j$, need not be decoded. A simple achievable rate region is, however, obtained if each receiver performs joint ML decoding of the messages from both transmitters. We formally define the joint ML decoder next.

Definition 2: A *joint ML decoder* at $\mathcal{R}_j$ $(j = 1, 2)$ carries out joint ML detection on the messages from both transmitters. For the *joint ML decoder* at $\mathcal{R}_j$, $j = 1, 2$, we declare an error if $\mathcal{T}_j$ is decoded in error[1]. The corresponding average error probability is $P\big(E_j^{JD}\big) \triangleq \mathbb{E}_{\mathbf{h}_j}\big\{\mathbb{P}\big[\mathcal{E}_j^{JD}\big]\big\}$.

The achievable DMT of the *joint ML decoder* is characterized next.

Theorem 1: The DMT corresponding to joint decoding at each receiver is given by

$$d^{JD}(\mathbf{r}) = \min_{i=1,2,3} d_i^{JD}(\mathbf{r}) \quad (8)$$

where

$$d_i^{JD}(\mathbf{r}) = (1 - r_i)^+, \quad \text{for} \quad i = 1, 2 \quad (9)$$
$$d_3^{JD}(\mathbf{r}) = (1 - r_1 - r_2)^+ + (\alpha - r_1 - r_2)^+ .$$

Denote[2] $j^* = \arg\min_{i=1,2,3} d_i^{JD}(\mathbf{r})$. Let $\Gamma_i(\mathbf{r}) = [\gamma_i^1(\mathbf{r})\ \gamma_i^2(\mathbf{r})]^T$ be functions[3] such that $d_{j^*}^{JD}(\mathbf{r}) = d_i^{JD}(\Gamma_i(\mathbf{r}))$, for $i = 1, 2, 3$. If a sequence (in SNR) of codebooks with block length $N \geq 2$ satisfies

$$\|\Delta \mathbf{x}_i\|^2 \dot{\geq} \text{SNR}^{-\gamma_i^i(\mathbf{r}) + \epsilon}, \quad (10)$$

$$\lambda_{\min}\big(\Delta \mathbf{X}_{ij} (\Delta \mathbf{X}_{ij})^H\big) \dot{\geq} \text{SNR}^{-\gamma_3^1(\mathbf{r}) - \gamma_3^2(\mathbf{r}) + \epsilon}, \quad (11)$$

for some[4] $\epsilon > 0$, for all pairs of codewords $\mathbf{x}_i^{n_i}, \mathbf{x}_i^{\tilde{n}_i} \in \mathcal{C}_i(\text{SNR}, r_i)$ s.t. $\mathbf{x}_i^{n_i} \neq \mathbf{x}_i^{\tilde{n}_i}$, and $\mathbf{x}_j^{n_j}, \mathbf{x}_j^{\tilde{n}_j} \in \mathcal{C}_j(\text{SNR}, r_j)$ s.t. $\mathbf{x}_j^{n_j} \neq \mathbf{x}_j^{\tilde{n}_j}$, for $i, j = 1, 2$, and $i \neq j$, where $\Delta \mathbf{x}_i = \mathbf{x}_i^{n_i} - \mathbf{x}_i^{\tilde{n}_i}$, $\Delta \mathbf{x}_j = \mathbf{x}_j^{n_j} - \mathbf{x}_j^{\tilde{n}_j}$, $\Delta \mathbf{X}_{ij} = [\Delta \mathbf{x}_i\ \Delta \mathbf{x}_j]$, and $\lambda_{\min}(\Delta \mathbf{X}_{ij} (\Delta \mathbf{X}_{ij})^H)$ denotes the smallest nonzero eigenvalue of $\Delta \mathbf{X}_{ij} (\Delta \mathbf{X}_{ij})^H$, then $P\big(E^{JD}\big)$ obeys

$$P\big(E^{JD}\big) \doteq \text{SNR}^{-d^{JD}(\mathbf{r})}. \quad (12)$$

Proof: The basic proof idea is to establish upper and lower bounds on $P\big(E^{JD}\big)$ and to show that the corresponding SNR exponents are identical.

We start by establishing a lower bound on $P(E^{JD})$. To this end, we define, for $i, j = 1, 2$, and $i \neq j$, the outage events corresponding to decoding of $\mathcal{T}_i$ at $\mathcal{R}_i$ after the interfering signal from $\mathcal{T}_j$ has been subtracted out and to joint decoding of $\mathcal{T}_i$ *and* $\mathcal{T}_j$ at $\mathcal{R}_i$, respectively, by

$$\mathcal{O}_{i1}^{JD} \triangleq \{\mathbf{h}_i : I(\mathbf{x}_i; \mathbf{y}_i | \mathbf{x}_j, \mathbf{h}_i) < R_i\} \quad (13)$$

$$\mathcal{O}_{i2}^{JD} \triangleq \{\mathbf{h}_i : I(\mathbf{x}_i, \mathbf{x}_j; \mathbf{y}_i | \mathbf{h}_i) < R_1 + R_2\}. \quad (14)$$

[1]This differs from the joint decoder considered usually, where an error is declared if either $\mathcal{T}_1$ or $\mathcal{T}_2$ or both are decoded in error.

[2]For notational convenience, we do not make the dependence of j^* on $\mathbf{r}$ explicit.

[3]We note that the functions $\Gamma_i(\mathbf{r})$ might not be unique.

[4]We note that ϵ is allowed to be different in (10) and (11).

The outage event corresponding to the case where we are unable to decode $\mathcal{T}_i$ at $\mathcal{R}_i$, $i=1,2$, is hence given by

$$\mathcal{O}_i^{JD} \triangleq \bigcup_{k=1}^{2} \mathcal{O}_{ik}^{JD}. \tag{15}$$

The corresponding outage probability yields a lower bound on the error probability of the *joint ML decoder*. As in [9], we define the total outage probability of the IC as

$$\mathbb{P}\big[\mathcal{O}^{JD}\big] \triangleq \max\big\{\mathbb{P}\big[\mathcal{O}_1^{JD}\big], \mathbb{P}\big[\mathcal{O}_2^{JD}\big]\big\}. \tag{16}$$

Using a standard argument along the lines of [10], [12], we can see that i.i.d. Gaussian codebooks for both transmitters result in no loss of optimality in terms of DMT performance. We can therefore evaluate (13) and (14) to

$$\begin{aligned}
&\mathcal{O}_{i1}^{JD}(\mathbf{r}) \triangleq \big\{\mathbf{h}_i : \log\big(1+\text{SNR}|h_{ii}|^2\big) < R_i\big\}\\
&\mathcal{O}_{i2}^{JD}(\mathbf{r}) \triangleq\\
&\quad\big\{\mathbf{h}_i : \log\big(1+\text{SNR}^{\alpha}|h_{ji}|^2+\text{SNR}|h_{ii}|^2\big) < R_1+R_2\big\},
\end{aligned}$$

for $i=1,2$, and $i\neq j$. In the following, we will also need the no-outage events defined as

$$\begin{aligned}
&\bar{\mathcal{O}}_{i1}^{JD}(\mathbf{r}) \triangleq \big\{\mathbf{h}_i : \log\big(1+\text{SNR}|h_{ii}|^2\big) \geq R_i\big\}\\
&\bar{\mathcal{O}}_{i2}^{JD}(\mathbf{r}) \triangleq\\
&\quad\big\{\mathbf{h}_i : \log\big(1+\text{SNR}^{\alpha}|h_{ji}|^2+\text{SNR}|h_{ii}|^2\big) \geq R_1+R_2\big\}
\end{aligned}$$

with $i,j=1,2$ and $i\neq j$. We can now establish the asymptotic behavior of $\mathcal{O}_i^{JD}$. By the union bound, we have

$$\mathbb{P}\big[\mathcal{O}_i^{JD}(\mathbf{r})\big] \leq \sum_{k=1}^{2} \mathbb{P}\big[\mathcal{O}_{ik}^{JD}(\mathbf{r})\big]. \tag{17}$$

The exponential order of $\mathbb{P}\big[\mathcal{O}_i^{JD}(\mathbf{r})\big]$ is given by

$$\mathbb{P}\big[\mathcal{O}_i^{JD}(\mathbf{r})\big] \doteq \max_{k=1,2} \mathbb{P}\big[\mathcal{O}_{ik}^{JD}(\mathbf{r})\big]. \tag{18}$$

It is shown in [13] and [9] that

$$\mathbb{P}\big[\mathcal{O}_{i1}^{JD}(\mathbf{r})\big] \doteq \text{SNR}^{-d_{i1}^{JD}(\mathbf{r})} \tag{19}$$

$$\mathbb{P}\big[\mathcal{O}_{i2}^{JD}(\mathbf{r})\big] \doteq \text{SNR}^{-d_{i2}^{JD}(\mathbf{r})} \tag{20}$$

with

$$d_{i1}^{JD}(\mathbf{r}) = (1-r_i)^+ \tag{21}$$

$$d_{i2}^{JD}(\mathbf{r}) = (1-r_1-r_2)^+ + (\alpha-r_1-r_2)^+, \tag{22}$$

for $i=1,2$. We point out that (21) and (22) define four SNR exponents $d_{ij}^{JD}(\mathbf{r})$ for $i,j=1,2$. The DMT-characterization of the outage event corresponding to jointly decoding the signals from both transmitters at $\mathcal{R}_1$ is equal to the DMT-characterization of the outage event corresponding to jointly decoding the signals from both transmitters at $\mathcal{R}_2$. Hence, the corresponding SNR exponents of the outage probabilities of these events, namely, $d_{12}^{JD}(\mathbf{r})$ and $d_{22}^{JD}(\mathbf{r})$, are equal. From (18), it follows that

$$\mathbb{P}\big[\mathcal{O}_i^{JD}(\mathbf{r})\big] \doteq \max_{k=1,2} \mathbb{P}\big[\mathcal{O}_{ik}^{JD}(\mathbf{r})\big] \doteq \text{SNR}^{-\min\limits_{k=1,2} d_{ik}^{JD}(\mathbf{r})}. \tag{23}$$

Combining (16) and (23), we get

$$\mathbb{P}\big[\mathcal{O}^{JD}(\mathbf{r})\big] \doteq \max_{i=1,2} \text{SNR}^{-\min\limits_{k=1,2} d_{ik}^{JD}(\mathbf{r})} \tag{24}$$

$$\doteq \text{SNR}^{-d^{JD}(\mathbf{r})} \tag{25}$$

where

$$d^{JD}(\mathbf{r}) = \min_{i=1,2,3} d_i^{JD}(\mathbf{r}) \tag{26}$$

with

$$\begin{aligned}
&d_i^{JD}(\mathbf{r}) = (1-r_i)^+ \quad \text{for} \quad i=1,2 \qquad (27)\\
&d_3^{JD}(\mathbf{r}) = (1-r_1-r_2)^+ + (\alpha-r_1-r_2)^+.
\end{aligned}$$

With (23) we arrived at a lower bound on the error probability of the *joint ML decoder* at $\mathcal{R}_i$, $i=1,2$, and, hence, at an outer bound on the DMT region achievable by the *joint ML decoder*. We next find an upper bound on the error probability of the *joint ML decoder* that has the same exponential behavior as the lower bound above. We start by letting $\mathbf{x}_i^{n_i}$ and $\mathbf{x}_j^{n_j}$ with $n_i \in \{1,2,\ldots,2^{NR_i}\}$, $n_j \in \{1,2,\ldots,2^{NR_j}\}$ ($i,j=1,2$, and $i\neq j$) be the codewords transmitted by $\mathcal{T}_i$ and $\mathcal{T}_j$, respectively. The results of joint ML decoding of $\mathcal{T}_i$ and $\mathcal{T}_j$ at $\mathcal{R}_i$ (for $i,j=1,2$, and $i\neq j$) are denoted by $\mathbf{x}_i^{\tilde{n}_i}$ and $\mathbf{x}_j^{\tilde{n}_j}$, respectively, with $\tilde{n}_i \in \{1,2,\ldots,2^{NR_i}\}$, $\tilde{n}_j \in \{1,2,\ldots,2^{NR_j}\}$. We have, for $i,j=1,2$, with $i\neq j$, the error events corresponding to the message from $\mathcal{T}_i$ being decoded in error and the messages from $\mathcal{T}_i$ and $\mathcal{T}_j$ both being decoded in error, in both cases at $\mathcal{R}_i$ as

$$\mathcal{E}_{i1}^{JD} \triangleq \{\tilde{n}_i \neq n_i,\ \tilde{n}_j = n_j\} \tag{28}$$

$$\mathcal{E}_{i2}^{JD} \triangleq \{\tilde{n}_i \neq n_i,\ \tilde{n}_j \neq n_j\}. \tag{29}$$

We will also need the total error event defined as

$$\mathcal{E}_i^{JD} \triangleq \bigcup_{k=1,2} \mathcal{E}_{ik}^{JD}. \tag{30}$$

We denote $j^* = \arg\min_{i=1,2,3} d_i^{JD}(\mathbf{r})$. Let $\Gamma_i(\mathbf{r}) = [\gamma_i^1(\mathbf{r})\ \gamma_i^2(\mathbf{r})]^T$ be functions[5] such that $d_{j^*}^{JD}(\mathbf{r}) = d_i^{JD}(\Gamma_i(\mathbf{r}))$ for $i=1,2,3$. We recall that $d_{i2}^{JD}(\mathbf{r}) = d_3^{JD}(\mathbf{r})$, for $i=1,2$, by definition.

We next find an upper bound on the probability of the events $\mathcal{E}_{i1}^{JD}$, $i=1,2$, as follows:

$$\begin{aligned}
\mathbb{P}\big[\mathcal{E}_{i1}^{JD}\big] &= \mathbb{P}\big[\mathcal{E}_{i1}^{JD}, \mathcal{O}_{i1}^{JD}(\Gamma_i(\mathbf{r}))\big] + \mathbb{P}\big[\mathcal{E}_{i1}^{JD}, \bar{\mathcal{O}}_{i1}^{JD}(\Gamma_i(\mathbf{r}))\big]\\
&\leq \mathbb{P}\big[\mathcal{O}_{i1}^{JD}(\Gamma_i(\mathbf{r}))\big] + \mathbb{P}\big[\mathcal{E}_{i1}^{JD}|\bar{\mathcal{O}}_{i1}^{JD}(\Gamma_i(\mathbf{r}))\big] \qquad (31)
\end{aligned}$$

and for the events $\mathcal{E}_{i2}^{JD}$ according to:

$$\begin{aligned}
\mathbb{P}\big[\mathcal{E}_{i2}^{JD}\big] &= \mathbb{P}\big[\mathcal{E}_{i2}^{JD}, \mathcal{O}_{i2}^{JD}(\Gamma_3(\mathbf{r}))\big] + \mathbb{P}\big[\mathcal{E}_{i2}^{JD}, \bar{\mathcal{O}}_{i2}^{JD}(\Gamma_3(\mathbf{r}))\big]\\
&\leq \mathbb{P}\big[\mathcal{O}_{i2}^{JD}(\Gamma_3(\mathbf{r}))\big] + \mathbb{P}\big[\mathcal{E}_{i2}^{JD}|\bar{\mathcal{O}}_{i2}^{JD}(\Gamma_3(\mathbf{r}))\big]. \qquad (32)
\end{aligned}$$

We start by deriving an upper bound on the average pairwise error probability (PEP) of each error event $\mathcal{E}_{ik}^{JD}$, for $i=1,2$, and $k=1,2$. Let us first consider an $\mathcal{E}_{i2}^{JD}$-type event corresponding to the case where the messages from both transmitters are decoded in error. The probability of the joint ML decoder mistakenly deciding in favor of the codeword $\mathbf{X}_{ij}^{\tilde{n}_i\tilde{n}_j} = [\mathbf{x}_i^{\tilde{n}_i}\ \mathbf{x}_j^{\tilde{n}_j}]$ when $\mathbf{X}_{ij}^{n_in_j} = [\mathbf{x}_i^{n_i}\ \mathbf{x}_j^{n_j}]$ (with $\mathbf{x}_i^{n_i}, \mathbf{x}_i^{\tilde{n}_i} \in \mathcal{C}_i(\text{SNR}, r_i)$ and

[5] We note that the functions $\Gamma_i(\mathbf{r})$ might not be unique.

$\mathbf{x}_j^{n_j}, \mathbf{x}_j^{\tilde{n}_j} \in \mathcal{C}_j(\text{SNR}, r_j)$, $i, j = 1, 2$, and $i \neq j$), with $\mathbf{x}_i^{\tilde{n}_i} \neq \mathbf{x}_i^{n_i}$ and $\mathbf{x}_j^{\tilde{n}_j} \neq \mathbf{x}_j^{n_j}$, was transmitted, can be upper-bounded according to

$$\mathbb{E}_{\mathbf{h}_i}\Big\{\mathbb{P}\Big[\mathbf{X}_{ij}^{n_i n_j} \rightarrow \mathbf{X}_{ij}^{\tilde{n}_i \tilde{n}_j}\Big]\Big\} \tag{33}$$

$$\leq \mathbb{E}_{\mathbf{h}_i}\left\{\exp\left[-\frac{\|\Delta\mathbf{X}_{ij}\tilde{\mathbf{h}}_i\|^2}{4}\right]\right\} \tag{34}$$

$$\leq \mathbb{E}_{\mathbf{h}_i}\left\{\exp\left[-\frac{\lambda_{\min}\|\tilde{\mathbf{h}}_i\|^2}{4}\right]\right\} \tag{35}$$

$$= \mathbb{E}_{\mathbf{h}_i}\left\{\exp\left[-\lambda_{\min}\frac{\text{SNR}|h_{ii}|^2 + \text{SNR}^{\alpha}|h_{ji}|^2}{4}\right]\right\} \tag{36}$$

where $\tilde{\mathbf{h}}_i = [\sqrt{\text{SNR}}h_{ii}\ \sqrt{\text{SNR}^{\alpha}}h_{ji}]^T$, for $i, j = 1, 2$ and $i \neq j$, and $\lambda_{\min}$ is the smallest nonzero eigenvalue of $\Delta\mathbf{X}_{ij}(\Delta\mathbf{X}_{ij})^H$.

Noting that the no-outage event $\bar{\mathcal{O}}_{i2}^{JD}(\Gamma_3(\mathbf{r}))$ entails $\text{SNR}|h_{ii}|^2 + \text{SNR}^{\alpha}|h_{ji}|^2 \geq \text{SNR}^{\gamma_3^1(\mathbf{r})+\gamma_3^2(\mathbf{r})} - 1$, Eq. (31) implies an upper bound on $\mathbb{P}\big[\mathcal{E}_{i2}^{JD}\big]$ according to:

$$\mathbb{E}_{\mathbf{h}_i}\{\mathbb{P}\big[\mathcal{E}_{i2}^{JD}\big]\} \ \dot{\leq} \tag{37}$$

$$\mathbb{P}\big[\mathcal{O}_{i2}^{JD}(\Gamma_3(\mathbf{r}))\big] + \text{SNR}^{N(r_1+r_2)}\exp\left[-\frac{\lambda_{\min}\text{SNR}^{\gamma_3^1(\mathbf{r})+\gamma_3^2(\mathbf{r})}}{4}\right].$$

Here, we used the definitions $R_i = r_i \log \text{SNR}$, $i = 1, 2$, and the fact that $\exp[-\frac{\lambda_{\min}}{4}(\text{SNR}^{\gamma_3^1(\mathbf{r})+\gamma_3^2(\mathbf{r})} - 1)] \doteq \exp[-\frac{\lambda_{\min}}{4}\text{SNR}^{\gamma_3^1(\mathbf{r})+\gamma_3^2(\mathbf{r})}]$. Since we have that $\lambda_{\min} \ \dot{\geq}\ \text{SNR}^{-\gamma_3^1(\mathbf{r})-\gamma_3^2(\mathbf{r})+\epsilon}$, with $\epsilon > 0$, by assumption, we obtain

$$\mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_{i2}^{JD}\big]\big\}$$

$$\dot{\leq}\ \mathbb{P}\big[\mathcal{O}_{i2}^{JD}(\Gamma_3(\mathbf{r}))\big] + \text{SNR}^{N(r_1+r_2)}\exp\left[-\frac{\text{SNR}^{\epsilon}}{4}\right] \tag{38}$$

$$\doteq \mathbb{P}\big[\mathcal{O}_{i2}^{JD}(\Gamma_3(\mathbf{r}))\big] \tag{39}$$

$$\doteq \text{SNR}^{-d_{j^*}^{JD}(\mathbf{r})} \tag{40}$$

as the second term on the right-hand-side (RHS) of (38) decays exponentially in SNR whereas the first term decays polynomially. Eq. (40) follows by definition (of the function $\Gamma_3(\mathbf{r})$).

A similar analysis for the $\mathcal{E}_{i1}^{JD}$-type error event, corresponding to $\mathcal{T}_i$ decoded in error at $\mathcal{R}_i$, results in

$$\mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathbf{x}_i^{n_i} \rightarrow \mathbf{x}_i^{\tilde{n}_i}\big]\big\} \leq \mathbb{E}_{\mathbf{h}_i}\left\{\exp\left[-\frac{\text{SNR}|h_{ii}|^2\|\Delta\mathbf{x}_i\|^2}{4}\right]\right\}, \tag{41}$$

with $\mathbf{x}_i^{n_i} \neq \mathbf{x}_i^{\tilde{n}_i}$, which, upon invoking (10) and using the fact that $\bar{\mathcal{O}}_{i1}^{JD}(\Gamma_i(\mathbf{r}))$ entails $\text{SNR}|h_{ii}|^2 \geq \text{SNR}^{\gamma_i^i} - 1$, yields

$$\mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_{i1}^{JD}\big]\big\}$$

$$\dot{\leq}\ \mathbb{P}\big[\mathcal{O}_{i1}^{JD}(\Gamma_i(\mathbf{r}))\big] + \text{SNR}^{Nr_i}\exp\left[-\frac{\text{SNR}^{\epsilon}}{4}\right] \tag{42}$$

$$\doteq \mathbb{P}\big[\mathcal{O}_{i1}^{JD}(\Gamma_i(\mathbf{r}))\big], \tag{43}$$

for $i = 1, 2$. To complete the proof, we note that

$$\mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_i^{JD}\big]\big\} \leq \sum_{k=1}^{2}\mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_{ik}^{JD}\big]\big\} \tag{44}$$

$$\dot{\leq}\ \mathbb{P}\big[\mathcal{O}_{i1}^{JD}(\Gamma_i(\mathbf{r}))\big] + \mathbb{P}\big[\mathcal{O}_{i2}^{JD}(\Gamma_3(\mathbf{r}))\big] \tag{45}$$

$$= 2\text{SNR}^{-d_{j^*}^{JD}(\mathbf{r})} \doteq \text{SNR}^{-\min_{k=1,2,3} d_k^{JD}(\mathbf{r})}.$$

Recalling that $P\big(E^{JD}\big) = \max_{i=1,2}\mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_i^{JD}\big]\big\}$, we upper-bound $P\big(E^{JD}\big)$ according to

$$P\big(E^{JD}\big) = \max_{i=1,2}\mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_i^{JD}\big]\big\} \tag{46}$$

$$\dot{\leq}\ \max_{i=1,2}\text{SNR}^{-\min_{j=1,2,3} d_j^{JD}(\mathbf{r})} \tag{47}$$

$$\doteq\ \text{SNR}^{-d^{JD}(\mathbf{r})}. \tag{48}$$

Since (48) gives an upper bound that matches exponentially the lower bound in (25), the proof is complete. ∎

Discussion: The *joint ML decoder* decodes the message from the interfering user $\mathcal{T}_j$ at $\mathcal{R}_i$, for $i, j = 1, 2$, with $i \neq j$, in its entirety together with the intended message from $\mathcal{T}_i$. We can relax this constraint and ask for "part" of the interfering signal $\mathcal{T}_j$ to be decoded at $\mathcal{R}_i$ only, for $i, j = 1, 2$, with $i \neq j$. This is the idea behind the Han-Kobayashi scheme, which we analyze in the next section.

Code Design Discussion: Based on a criterion established by Coronel *et al.* in [12, Section V], it is shown in [14] that there exist codes which satisfy the design criteria in (10) and (11) specified in this theorem for all $r_1 = r_2$ and $r_1 < 1.5$.

III. DMT Region of Two-Message Fixed-Power-Split Han-Kobayashi Scheme

The Han-Kobayashi (HK) rate region [15] remains the best known achievable rate region for the non-fading, Gaussian IC [3], [16]. The original HK strategy lets each transmitter split its message into two messages, allows each receiver to decode part of the interfering signal, and uses five auxiliary RVs Q, U_1, U_2, W_1, and W_2. The auxiliary RV U_i carries the private message of $\mathcal{T}_i$ destined for $\mathcal{R}_i$, whereas the auxiliary RV W_i carries the public message of $\mathcal{T}_i$ destined for both receivers. The RV Q is for time-sharing.

In the following, we analyze the DMT region achieved by a two-message, fixed-power-split superposition HK scheme where $\mathcal{T}_i$ transmits the N-dimensional ($N \geq 2$) vector $\mathbf{x}_i = \mathbf{u}_i + \mathbf{w}_i$ with $\mathbf{u}_i$ and $\mathbf{w}_i$ representing the private and the public message, respectively. The power constraints for $\mathbf{u}_i$ and $\mathbf{w}_i$ are

$$\|\mathbf{u}_i\| \leq \sqrt{\frac{N}{\text{SNR}^{1-p_i}}},\ \ \|\mathbf{w}_i\| \leq \sqrt{N}\left(1 - \sqrt{\frac{1}{\text{SNR}^{1-p_i}}}\right)$$

so that $\|\mathbf{x}_i\| \leq \|\mathbf{u}_i\| + \|\mathbf{w}_i\| \leq \sqrt{N}$. Here, p_i accounts for the exponential order of the power allocated to the private message. The power split is assumed fixed and is independent of the channel realizations. When both the private and the public messages are allocated maximum power, we have $\frac{\|\mathbf{w}_i\|^2}{\|\mathbf{u}_i\|^2} \doteq \text{SNR}^{1-p_i}$. We restrict ourselves to $p_i < 1$ in order to satisfy the power constraint $\|\mathbf{u}_i\| + \|\mathbf{w}_i\| \leq \sqrt{N}$. Cases for which $p_i < 0$ do not contribute to the DMT region as the private message codebook has vanishing cardinality with increasing SNR, except for $p_i = -\infty$, which corresponds to having public messages only. In this case, already considered in Section II, only the public messages are decoded jointly at the receivers. In the remainder of the paper, we can therefore restrict ourselves to $0 \leq p_i < 1$. In the following, we will need the power-split SNR exponent vector $\mathbf{p} = [p_1 \ p_2]^T$.

We assume that $\mathcal{T}_i$ transmits at rate $R_i = r_i \log \text{SNR}$ where the rates for the private and the public messages, respectively, are $S_i = s_i \log \text{SNR}$ and $T_i = t_i \log \text{SNR}$ with $r_i = s_i + t_i$, $s_i, t_i \geq 0$, and $0 \leq r_i \leq 1$. The codebooks corresponding to the private and the public message parts are denoted as $\mathcal{C}^{\mathbf{u}_i}(\text{SNR}, s_i)$ and $\mathcal{C}^{\mathbf{w}_i}(\text{SNR}, t_i)$, respectively, and satisfy $|\mathcal{C}^{\mathbf{u}_i}(\text{SNR}, s_i)| = \text{SNR}^{N s_i}$ and $|\mathcal{C}^{\mathbf{w}_i}(\text{SNR}, t_i)| = \text{SNR}^{N t_i}$. Clearly, $\mathcal{C}^{\mathbf{x}_i}(\text{SNR}, r_i) = \mathcal{C}^{\mathbf{u}_i}(\text{SNR}, s_i) \times \mathcal{C}^{\mathbf{w}_i}(\text{SNR}, t_i)$ with $|\mathcal{C}^{\mathbf{x}_i}(\text{SNR}, r_i)| = \text{SNR}^{r_i}$. In the following, we will need the private message multiplexing rate vector $\mathbf{s} = [s_1 \ s_2]^T$.

Definition 3: A *joint ML decoder for the two-message, fixed-power-split HK scheme* at $\mathcal{R}_j$ $(j = 1, 2)$ carries out joint ML detection on the public messages from both transmitters and the private message from $\mathcal{T}_j$. Private messages will, of course, only be decoded if $p_i > -\infty$. An error is declared if the public or the private message from $\mathcal{T}_j$ is decoded in error. The corresponding error probability is denoted by $\mathbb{P}\big[\mathcal{E}_j^{HK}\big]$, for $j = 1, 2$, with the associated average error probability $P\big(E_j^{HK}\big) \triangleq \mathbb{E}_{\mathbf{h}_j}\left\{\mathbb{P}\big[\mathcal{E}_j^{HK}\big]\right\}$, $j = 1, 2$.

The SNR exponent of $P(E^{HK}) = \max\{P\big(E_1^{HK}\big), P\big(E_2^{HK}\big)\}$ and the conditions on the superposition codes for achieving this SNR exponent are characterized next.

Theorem 2: The DMT region for the two-message, fixed-power-split HK scheme is given by

$$d^{HK}(\mathbf{r}) = \max_{\mathbf{s}, \mathbf{p}} d(\mathbf{r}, \mathbf{s}, \mathbf{p}) \tag{49}$$

with the optimization carried out subject to the constraints

$$\begin{aligned} &s_i + t_i = r_i, \text{ with } 0 \leq s_i, t_i \\ &0 \leq p_i < 1, \ i = 1, 2 \end{aligned}$$

and

$$d(\mathbf{r}, \mathbf{s}, \mathbf{p}) = \min_{\substack{k=1,2 \\ l=1,2,\ldots,6}} d_{kl}(\mathbf{r}, \mathbf{s}, \mathbf{p})$$

where, for $i, j = 1, 2$, and $i \neq j$, we have

$$d_{i1}(\mathbf{r}, \mathbf{s}, \mathbf{p}) = \begin{cases} (p_i - s_i)^+, & \text{if } p_j < 1 - \alpha \\ (1 - \alpha - p_j + p_i - s_i)^+, & \text{if } p_j \geq 1 - \alpha \end{cases}$$

$$d_{i2}(\mathbf{r}, \mathbf{s}, \mathbf{p}) = \begin{cases} (1 - r_i + s_i)^+, & \text{if } p_j < 1 - \alpha \\ (2 - \alpha - p_j - r_i + s_i)^+, & \text{if } p_j \geq 1 - \alpha \end{cases}$$

$$d_{i3}(\mathbf{r}, \mathbf{s}, \mathbf{p}) = \begin{cases} (1 - r_i)^+, & \text{if } p_j < 1 - \alpha \\ (2 - \alpha - p_j - r_i)^+, & \text{if } p_j \geq 1 - \alpha \end{cases}$$

$$d_{i4}(\mathbf{r}, \mathbf{s}, \mathbf{p}) = \begin{cases} (p_i - s_i - r_j + s_j)^+ + (\alpha - s_i - r_j + s_j)^+, \\ \qquad \text{if } p_j < 1 - s_i - r_j + s_j \\ (p_i - s_i - r_j + s_j)^+, \\ \quad \text{if } p_j \geq 1 - s_i - r_j + s_j \text{ and } p_j < 1 - \alpha \\ (1 - \alpha - p_j + p_i - s_i - r_j + s_j)^+, \\ \quad \text{if } p_j \geq 1 - s_i - r_j + s_j \text{ and } p_j \geq 1 - \alpha \end{cases}$$

$$d_{i5}(\mathbf{r}, \mathbf{s}, \mathbf{p}) = \begin{cases} \left(1 - \sum_{k=1}^{2} r_k + \sum_{l=1}^{2} s_l\right)^+ + \left(\alpha - \sum_{k=1}^{2} r_k + \sum_{l=1}^{2} s_l\right)^+, \\ \quad \text{if } p_j < 1 - \sum_{k=1}^{2} r_k + \sum_{l=1}^{2} s_l, \\ \left(1 - \sum_{k=1}^{2} r_k + \sum_{l=1}^{2} s_l\right)^+, \\ \quad \text{if } p_j \geq 1 - \sum_{k=1}^{2} r_k + \sum_{l=1}^{2} s_l \text{ and } p_j < 1 - \alpha \\ \left(2 - \alpha - p_j - \sum_{k=1}^{2} r_k + \sum_{l=1}^{2} s_l\right)^+, \\ \quad \text{if } p_j \geq 1 - \sum_{k=1}^{2} r_k + \sum_{l=1}^{2} s_l \text{ and } p_j \geq 1 - \alpha \end{cases}$$

$$d_{i6}(\mathbf{r}, \mathbf{s}, \mathbf{p}) = \begin{cases} (1 - r_i - r_j + s_j)^+ + (\alpha - r_i - r_j + s_j)^+, \\ \qquad \text{if } p_j < 1 - r_i - r_j + s_j \\ (1 - r_i - r_j + s_j)^+, \\ \quad \text{if } p_j \geq 1 - r_i - r_j + s_j \text{ and } p_j < 1 - \alpha \\ (2 - \alpha - p_j - r_i - r_j + s_j)^+, \\ \quad \text{if } p_j \geq 1 - r_i - r_j + s_j \text{ and } p_j \geq 1 - \alpha \end{cases}.$$

Define the codeword difference vectors $\Delta\mathbf{u}_i = \sqrt{\text{SNR}^{1-p_i}}(\mathbf{u}_i^{i_i^u} - \mathbf{u}_i^{\hat{i}_i^u})$, $\Delta\mathbf{w}_i = \mathbf{w}_i^{i_i^w} - \mathbf{w}_i^{\hat{i}_i^w}$, and $\Delta\mathbf{x}_i = \mathbf{x}_i^{i_i^x} - \mathbf{x}_i^{\hat{i}_i^x}$ with $\mathbf{u}_i^{i_i^u}, \mathbf{u}_i^{\hat{i}_i^u} \in \mathcal{C}^{\mathbf{u}_i}(\text{SNR}, s_i)$, $\mathbf{w}_i^{i_i^w}, \mathbf{w}_i^{\hat{i}_i^w} \in \mathcal{C}^{\mathbf{w}_i}(\text{SNR}, t_i)$ and $\mathbf{x}_i^{i_i^x}, \mathbf{x}_i^{\hat{i}_i^x} \in \mathcal{C}^{\mathbf{x}_i}(\text{SNR}, r_i)$ with $\mathbf{u}_i^{i_i^u} \neq \mathbf{u}_i^{\hat{i}_i^u}$, $\mathbf{w}_i^{i_i^w} \neq \mathbf{w}_i^{\hat{i}_i^w}$, $\mathbf{x}_i^{i_i^x} \neq \mathbf{x}_i^{\hat{i}_i^x}$, for $i = 1, 2$. Further, define $\Delta\mathbf{A}_{ij} = [\Delta\mathbf{u}_i \ \Delta\mathbf{w}_j]$, $\Delta\mathbf{B}_{ij} = [\Delta\mathbf{w}_i \ \Delta\mathbf{w}_j]$, and $\Delta\mathbf{C}_{ij} = [\Delta\mathbf{x}_i \ \Delta\mathbf{w}_j]$, for $i, j = 1, 2$, and $i \neq j$. Denote the optimizing values of $\mathbf{s}$, $\mathbf{t}$, and $\mathbf{p}$ obtained by solving (49) as $\mathbf{s}^*$, $\mathbf{t}^*$, and $\mathbf{p}^*$, respectively. We let

$$[k^* \ l^*] = \operatorname*{arg\,min}_{\substack{k=1,2 \\ l=1,2,3,4,5,6}} d_{kl}(\mathbf{r}, \mathbf{s}, \mathbf{p}). \tag{50}$$

Further, let the functions[6] $\Upsilon_{nm}(\mathbf{r}) = [\upsilon_{nm}^1(\mathbf{r}) \ \upsilon_{nm}^2(\mathbf{r})]^T$ and $\Psi_{nm}(\mathbf{s}^*) = [\psi_{nm}^1(\mathbf{s}^*) \ \psi_{nm}^2(\mathbf{s}^*)]^T$ be such that

$$d_{k^*l^*}(\mathbf{r}, \mathbf{s}^*, \mathbf{p}^*) = d_{nm}(\Upsilon_{nm}(\mathbf{r}), \Psi_{nm}(\mathbf{s}^*), \mathbf{p}^*),$$

for $n = 1, 2$, and $m = 1, 2, \ldots, 6$. If there exists a sequence (in SNR) of superposition codes satisfying

$$\begin{aligned} \|\Delta\mathbf{u}_i\|^2 &\stackrel{.}{\geq} \text{SNR}^{-\psi_{i1}^i(\mathbf{s}^*)+\epsilon} \\ \|\Delta\mathbf{w}_i\|^2 &\stackrel{.}{\geq} \text{SNR}^{-\upsilon_{i2}^i(\mathbf{r})+\psi_{i2}^i(\mathbf{s}^*)+\epsilon} \\ \|\Delta\mathbf{x}_i\|^2 &\stackrel{.}{\geq} \text{SNR}^{-\upsilon_{i3}^i(\mathbf{r})+\epsilon} \\ \lambda_{\min}(\Delta\mathbf{A}_{ij}(\Delta\mathbf{A}_{ij})^H) &\stackrel{.}{\geq} \text{SNR}^{-\psi_{i4}^i(\mathbf{s}^*)-\upsilon_{j4}^j(\mathbf{r})+\psi_{j4}^j(\mathbf{s}^*)+\epsilon} \\ \lambda_{\min}(\Delta\mathbf{B}_{ij}(\Delta\mathbf{B}_{ij})^H) &\stackrel{.}{\geq} \text{SNR}^{-\sum_{k=1}^{2}\upsilon_{k5}^k(\mathbf{r})+\sum_{j=1}^{2}\psi_{j5}^j(\mathbf{s}^*)+\epsilon} \\ \lambda_{\min}(\Delta\mathbf{C}_{ij}(\Delta\mathbf{C}_{ij})^H) &\stackrel{.}{\geq} \text{SNR}^{-\upsilon_{i6}^i(\mathbf{r})-\upsilon_{j6}^j(\mathbf{r})+\psi_{j6}^j(\mathbf{s}^*)+\epsilon}, \end{aligned} \tag{51}$$

[6] We note that the functions $\Upsilon_{nm}(\mathbf{r})$ and $\Psi_{nm}(\mathbf{s}^*)$ might not be unique.

for every pair of codewords in each codebook for $i, j = 1, 2$, $i \neq j$, and for some[7] $\epsilon > 0$, then we have

$$P(E^{HK}) \doteq \text{SNR}^{-d_{HK}(\mathbf{r})}. \quad (52)$$

Proof: Both public messages are to be decoded at both receivers, whereas the private message of each transmitter is to be decoded *only* at the intended receiver. As stated before and discussed in [13], there is no loss of optimality in assuming i.i.d. Gaussian inputs in obtaining an upper bound on the DMT. Hence, we restrict ourselves to the case where all codebooks are i.i.d. Gaussian, i.e.,

$$\mathbf{u}_i \sim \mathcal{CN}(\mathbf{0}, \text{SNR}^{p_i - 1}\mathbf{I}_N) \quad (53)$$

$$\mathbf{w}_i \sim \mathcal{CN}\left(\mathbf{0}, \left(1 - \sqrt{1/(\text{SNR}^{1-p_i})}\right)^2 \mathbf{I}_N\right) \quad (54)$$

with $0 \leq p_i < 1$. Since we are interested in the high-SNR asymptotics only, we can take

$$\mathbf{w}_i \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_N). \quad (55)$$

The set of achievable rates $\{S_i, T_i, T_j\}$ for $i, j = 1, 2$, $i \neq j$ at $\mathcal{R}_i$, given the channel realization $\mathbf{h}_i$, can be characterized as

$$\mathcal{R}^i_{HK} \triangleq \{S_i, T_i, T_j\} \ :$$

$$S_i \leq \log\left(1 + \frac{\text{SNR}^{p_i}|h_{ii}|^2}{1 + \text{SNR}^{\alpha + p_j - 1}|h_{ji}|^2}\right) \quad (56)$$

$$T_i \leq \log\left(1 + \frac{\text{SNR}|h_{ii}|^2}{1 + \text{SNR}^{\alpha + p_j - 1}|h_{ji}|^2}\right) \quad (57)$$

$$T_j \leq \log\left(1 + \frac{\text{SNR}^{\alpha}|h_{ji}|^2}{1 + \text{SNR}^{\alpha + p_j - 1}|h_{ji}|^2}\right) \quad (58)$$

$$S_i + T_i \leq \log\left(1 + \frac{\text{SNR}|h_{ii}|^2}{1 + \text{SNR}^{\alpha + p_j - 1}|h_{ji}|^2}\right) \quad (59)$$

$$S_i + T_j \leq \log\left(1 + \frac{\text{SNR}^{p_i}|h_{ii}|^2 + \text{SNR}^{\alpha}|h_{ji}|^2}{1 + \text{SNR}^{\alpha + p_j - 1}|h_{ji}|^2}\right) \quad (60)$$

$$T_i + T_j \leq \log\left(1 + \frac{\text{SNR}|h_{ii}|^2 + \text{SNR}^{\alpha}|h_{ji}|^2}{1 + \text{SNR}^{\alpha + p_j - 1}|h_{ji}|^2}\right) \quad (61)$$

$$S_i + T_i + T_j \leq \log\left(1 + \frac{\text{SNR}|h_{ii}|^2 + \text{SNR}^{\alpha}|h_{ji}|^2}{1 + \text{SNR}^{\alpha + p_j - 1}|h_{ji}|^2}\right) \quad (62)$$

$$S_i, T_i, T_j \geq 0. \quad (63)$$

For a set $\mathcal{S}$ of quadruples $\{S_1, T_1, S_2, T_2\}$, let $\prod(\mathcal{S})$ be the set of rate pairs (R_1, R_2) such that $R_1 = S_1 + T_1$ and $R_2 = S_2 + T_2$. Then, the set

$$\mathcal{R}^* \triangleq \prod\left(\mathcal{R}^1_{HK} \bigcap \mathcal{R}^2_{HK}\right) \quad (64)$$

is an achievable rate region for the IC. Specifically, the region is achieved by the HK scheme as defined above with fixed power split $\mathbf{p}$. By definition, no decoding error is made at $\mathcal{R}_i$ if the private and the public message of $\mathcal{T}_i$ are decoded correctly but the public message of $\mathcal{T}_j$, $j \neq i$, is decoded incorrectly [17]. As $\mathcal{R}_i$ is not interested in the messages from $\mathcal{T}_j$, it does not make sense to declare an outage if the channel between $\mathcal{T}_j$ and $\mathcal{R}_i$, for $i, j = 1, 2$, $i \neq j$, is not good enough to support the transmission rate T_j. Hence, the outage event corresponding to decoding the public message of the unintended transmitter, (58), and its counterpart for $\mathcal{R}_j$ are unnecessary from the point of view of the respective receivers. We therefore declare an outage for $\mathcal{R}_i$ through

$$\mathcal{O}_i(\mathbf{r}, \mathbf{s}, \mathbf{p}) \triangleq \bigcup_{j=1}^{6} \mathcal{O}_{ij}(\mathbf{r}, \mathbf{s}, \mathbf{p}) \quad (65)$$

where, for $i, j = 1, 2$, and $i \neq j$, we have

$$\mathcal{O}_{i1}(\mathbf{r}, \mathbf{s}, \mathbf{p}) \triangleq \left\{\mathbf{h}_i : \log\left(1 + \frac{\text{SNR}^{p_i}|h_{ii}|^2}{1 + \text{SNR}^{\alpha + p_j - 1}|h_{ji}|^2}\right) < S_i\right\} \quad (66)$$

$$\mathcal{O}_{i2}(\mathbf{r}, \mathbf{s}, \mathbf{p}) \triangleq \left\{\mathbf{h}_i : \log\left(1 + \frac{\text{SNR}|h_{ii}|^2}{1 + \text{SNR}^{\alpha + p_j - 1}|h_{ji}|^2}\right) < T_i\right\} \quad (67)$$

$$\mathcal{O}_{i3}(\mathbf{r}, \mathbf{s}, \mathbf{p}) \triangleq \left\{\mathbf{h}_i : \log\left(1 + \frac{\text{SNR}|h_{ii}|^2}{1 + \text{SNR}^{\alpha + p_j - 1}|h_{ji}|^2}\right) < S_i + T_i\right\} \quad (68)$$

$$\mathcal{O}_{i4}(\mathbf{r}, \mathbf{s}, \mathbf{p}) \triangleq \left\{\mathbf{h}_i : \log\left(1 + \frac{\text{SNR}^{p_i}|h_{ii}|^2 + \text{SNR}^{\alpha}|h_{ji}|^2}{1 + \text{SNR}^{\alpha + p_j - 1}|h_{ji}|^2}\right) < S_i + T_j\right\} \quad (69)$$

$$\mathcal{O}_{i5}(\mathbf{r}, \mathbf{s}, \mathbf{p}) \triangleq \left\{\mathbf{h}_i : \log\left(1 + \frac{\text{SNR}|h_{ii}|^2 + \text{SNR}^{\alpha}|h_{ji}|^2}{1 + \text{SNR}^{\alpha + p_j - 1}|h_{ji}|^2}\right) < T_i + T_j\right\} \quad (70)$$

$$\mathcal{O}_{i6}(\mathbf{r}, \mathbf{s}, \mathbf{p}) \triangleq \left\{\mathbf{h}_i : \log\left(1 + \frac{\text{SNR}|h_{ii}|^2 + \text{SNR}^{\alpha}|h_{ji}|^2}{1 + \text{SNR}^{\alpha + p_j - 1}|h_{ji}|^2}\right) < S_i + T_i + T_j\right\}. \quad (71)$$

It is shown in [9] that $\mathbb{P}[\mathcal{O}_{ik}(\mathbf{r}, \mathbf{s}, \mathbf{p})] \doteq \text{SNR}^{-d_{ik}(\mathbf{r}, \mathbf{s}, \mathbf{p})}$, $i = 1, 2$, $k = 1, 2, \ldots, 6$, where, for $i, j = 1, 2$, and $i \neq j$,

$$d_{i1}(\mathbf{r}, \mathbf{s}, \mathbf{p}) = \begin{cases} (p_i - s_i)^+, & \text{if } p_j < 1 - \alpha \\ (1 - \alpha - p_j + p_i - s_i)^+, & \text{if } p_j \geq 1 - \alpha \end{cases} \quad (72)$$

$$d_{i2}(\mathbf{r}, \mathbf{s}, \mathbf{p}) = \begin{cases} (1 - r_i + s_i)^+, & \text{if } p_j < 1 - \alpha \\ (2 - \alpha - p_j - r_i + s_i)^+, & \text{if } p_j \geq 1 - \alpha \end{cases} \quad (73)$$

$$d_{i3}(\mathbf{r}, \mathbf{s}, \mathbf{p}) = \begin{cases} (1 - r_i)^+, & \text{if } p_j < 1 - \alpha \\ (2 - \alpha - p_j - r_i)^+, & \text{if } p_j \geq 1 - \alpha \end{cases} \quad (74)$$

$$d_{i4}(\mathbf{r}, \mathbf{s}, \mathbf{p}) = \begin{cases} (p_i - s_i - r_j + s_j)^+ + (\alpha - s_i - r_j + s_j)^+, & \text{if } p_j < 1 - s_i - r_j + s_j \\ (p_i - s_i - r_j + s_j)^+, & \text{if } p_j \geq 1 - s_i - r_j + s_j \text{ and } p_j < 1 - \alpha \\ (1 - \alpha - p_j + p_i - s_i - r_j + s_j)^+, & \text{if } p_j \geq 1 - s_i - r_j + s_j \text{ and } p_j \geq 1 - \alpha \end{cases} \quad (75)$$

[7] We note that the ϵ's in (51) are allowed to be different.

$$d_{i5}(\mathbf{r},\mathbf{s},\mathbf{p}) = \begin{cases} \left(1-\sum_{k=1}^{2} r_k+\sum_{l=1}^{2} s_l\right)^+ + \left(\alpha-\sum_{k=1}^{2} r_k+\sum_{l=1}^{2} s_l\right)^+, \\ \quad \text{if } p_j < 1-\sum_{k=1}^{2} r_k+\sum_{l=1}^{2} s_l, \\ \left(1-\sum_{k=1}^{2} r_k+\sum_{l=1}^{2} s_l\right)^+, \\ \quad \text{if } p_j \geq 1-\sum_{k=1}^{2} r_k+\sum_{l=1}^{2} s_l \text{ and } p_j < 1-\alpha \\ \left(2-\alpha-p_j-\sum_{k=1}^{2} r_k+\sum_{l=1}^{2} s_l\right)^+, \\ \quad \text{if } p_j \geq 1-\sum_{k=1}^{2} r_k+\sum_{l=1}^{2} s_l \text{ and } p_j \geq 1-\alpha \end{cases} \tag{76}$$

$$d_{i6}(\mathbf{r},\mathbf{s},\mathbf{p}) = \begin{cases} (1-r_i-r_j+s_j)^+ + (\alpha-r_i-r_j+s_j)^+, \\ \qquad \text{if } p_j < 1-r_i-r_j+s_j \\ (1-r_i-r_j+s_j)^+, \\ \quad \text{if } p_j \geq 1-r_i-r_j+s_j \text{ and } p_j < 1-\alpha \\ (2-\alpha-p_j-r_i-r_j+s_j)^+, \\ \quad \text{if } p_j \geq 1-r_i-r_j+s_j \text{ and } p_j \geq 1-\alpha. \end{cases} \tag{77}$$

We define the total outage probability of the IC as

$$\mathbb{P}[\mathcal{O}(\mathbf{r},\mathbf{s},\mathbf{p})] \triangleq \max\{\mathbb{P}[\mathcal{O}_1(\mathbf{r},\mathbf{s},\mathbf{p})], \mathbb{P}[\mathcal{O}_2(\mathbf{r},\mathbf{s},\mathbf{p})]\}\,. \tag{78}$$

We note that this definition is compatible with our previous definitions. The outage probability of the two-message, fixed-power-split HK scheme is obtained by minimizing

$$\mathbb{P}\big[\mathcal{O}^{HK}(\mathbf{r})\big] \triangleq \min_{\mathbf{s},\mathbf{p}} \mathbb{P}[\mathcal{O}(\mathbf{r},\mathbf{s},\mathbf{p})] \tag{79}$$

subject to

$$r_i = s_i + t_i \tag{80}$$

$$s_i, t_i \geq 0 \tag{81}$$

$$0 \leq p_i < 1, \quad \text{for } i = 1,2. \tag{82}$$

We will next show that $\mathbb{P}\big[\mathcal{O}^{HK}(\mathbf{r})\big]$ obeys the following exponential behavior in SNR

$$\mathbb{P}\big[\mathcal{O}^{HK}(\mathbf{r})\big] \doteq \text{SNR}^{-d^{HK}(\mathbf{r})} \tag{83}$$

where

$$d^{HK}(\mathbf{r}) = \max_{\mathbf{s},\mathbf{p}} \min\{d_1(\mathbf{r},\mathbf{s},\mathbf{p}), d_2(\mathbf{r},\mathbf{s},\mathbf{p})\} \tag{84}$$

subject to

$$\begin{aligned} & s_i + t_i = r_i \\ & 0 \leq s_i \leq r_i \\ & 0 \leq t_i \leq r_i \\ & 0 \leq p_i < 1, \quad \text{for } i = 1,2, \end{aligned} \tag{85}$$

and where

$$d_1(\mathbf{r},\mathbf{s},\mathbf{p}) = \min_{i=1,2,\ldots,6} d_{1i}(\mathbf{r},\mathbf{s},\mathbf{p}) \tag{86}$$

$$d_2(\mathbf{r},\mathbf{s},\mathbf{p}) = \min_{i=1,2,\ldots,6} d_{2i}(\mathbf{r},\mathbf{s},\mathbf{p}). \tag{87}$$

To see this, we note that $\mathbb{P}\big[\mathcal{O}^{HK}(\mathbf{r})\big]$ can be bounded as follows

$$\begin{aligned} & \min_{\mathbf{s},\mathbf{p}} \max\{\mathbb{P}[\mathcal{O}_{1k}(\mathbf{r},\mathbf{s},\mathbf{p})]\,, \mathbb{P}[\mathcal{O}_{2l}(\mathbf{r},\mathbf{s},\mathbf{p})]\} \leq \mathbb{P}\big[\mathcal{O}^{HK}(\mathbf{r})\big] \\ & \leq \min_{\mathbf{s},\mathbf{p}} \max\left\{\sum_{i=1}^{6} \mathbb{P}[\mathcal{O}_{1i}(\mathbf{r},\mathbf{s},\mathbf{p})]\,, \sum_{j=1}^{6} \mathbb{P}[\mathcal{O}_{2j}(\mathbf{r},\mathbf{s},\mathbf{p})]\right\} \end{aligned} \tag{88}$$

where the inequality holds for $k = 1,2,\ldots,6$ and $l = 1,2,\ldots,6$. In the high-SNR limit the RHS of (88) is dominated by the SNR exponent given by

$$\max_{\mathbf{s},\mathbf{p}} \min\left\{\min_{i=1,2,\ldots,6} d_{1i}(\mathbf{r},\mathbf{s},\mathbf{p}), \min_{j=1,2,\ldots,6} d_{2j}(\mathbf{r},\mathbf{s},\mathbf{p})\right\}. \tag{89}$$

The upper and lower bounds on $\mathbb{P}\big[\mathcal{O}^{HK}(\mathbf{r})\big]$ can be made to have the same SNR exponent upon selection of the appropriate values for k and l on the left-hand-side of (88). We arrived at a lower bound on the error probability of the *joint ML decoder for the two-message, fixed-power-split HK scheme* which gives an outer bound on the DMT region. Next, we are going to upper-bound this error probability to obtain an inner bound on the DMT region. This would allow us to "sandwich" the DMT region.

Following [10], we decompose the error probability of the *joint ML decoder for the two-message, fixed-power-split HK scheme* at $\mathcal{R}_i$ into seven disjoint error events. As noted earlier, we do not declare an error if only the public message of $\mathcal{T}_j$ is decoded in error at $\mathcal{R}_i$ with $i \neq j$. Denoting the decisions on the private and public message of $\mathcal{T}_i$ and the public message of $\mathcal{T}_j$ at $\mathcal{R}_i$ by $\mathbf{u}_i^{\hat{\imath}_i^u}$, $\mathbf{w}_i^{\hat{\imath}_i^w}$, and $\mathbf{w}_j^{\hat{\imath}_j^w}$, respectively, we end up with the following six error events when the transmitted codewords are $\mathbf{u}_i^{\imath_i^u}$, $\mathbf{w}_i^{\imath_i^w}$, and $\mathbf{w}_j^{\imath_j^w}$, for $i,j = 1,2$, and $i \neq j$:

$$\mathcal{E}_{i1}^{HK} \triangleq \left\{\hat{\imath}_i^u \neq \imath_i^u,\ \hat{\imath}_i^w = \imath_i^w,\ \hat{\imath}_j^w = \imath_j^w\right\} \tag{90}$$

$$\mathcal{E}_{i2}^{HK} \triangleq \left\{\hat{\imath}_i^u = \imath_i^u,\ \hat{\imath}_i^w \neq \imath_i^w,\ \hat{\imath}_j^w = \imath_j^w\right\} \tag{91}$$

$$\mathcal{E}_{i3}^{HK} \triangleq \left\{\hat{\imath}_i^u \neq \imath_i^u,\ \hat{\imath}_i^w \neq \imath_i^w,\ \hat{\imath}_j^w = \imath_j^w\right\} \tag{92}$$

$$\mathcal{E}_{i4}^{HK} \triangleq \left\{\hat{\imath}_i^u \neq \imath_i^u,\ \hat{\imath}_i^w = \imath_i^w,\ \hat{\imath}_j^w \neq \imath_j^w\right\} \tag{93}$$

$$\mathcal{E}_{i5}^{HK} \triangleq \left\{\hat{\imath}_i^u = \imath_i^u,\ \hat{\imath}_i^w \neq \imath_i^w,\ \hat{\imath}_j^w \neq \imath_j^w\right\} \tag{94}$$

$$\mathcal{E}_{i6}^{HK} \triangleq \left\{\hat{\imath}_i^u \neq \imath_i^u,\ \hat{\imath}_i^w \neq \imath_i^w,\ \hat{\imath}_j^w \neq \imath_j^w\right\}. \tag{95}$$

The total error event at $\mathcal{R}_i$ is simply the union of the above events, i.e.,

$$\mathcal{E}_i^{HK} \triangleq \bigcup_{k=1}^{6} \mathcal{E}_{ik}^{HK}. \tag{96}$$

We let

$$[k^*\ l^*] = \arg\min_{\substack{k=1,2 \\ l=1,2,3,4,5,6}} d_{kl}(\mathbf{r},\mathbf{s},\mathbf{p}). \tag{97}$$

Further, let the functions[8] $\Upsilon_{nm}(\mathbf{r}) = [\upsilon_{nm}^1(\mathbf{r})\ \upsilon_{nm}^2(\mathbf{r})]^T$ and $\Psi_{nm}(\mathbf{s}^*) = [\psi_{nm}^1(\mathbf{s}^*)\ \psi_{nm}^2(\mathbf{s}^*)]^T$ be such that

$$d_{k^*l^*}(\mathbf{r},\mathbf{s}^*,\mathbf{p}^*) = d_{nm}(\Upsilon_{nm}(\mathbf{r}), \Psi_{nm}(\mathbf{s}^*), \mathbf{p}^*)$$

for $n = 1,2$ and $m = 1,2,\ldots,6$.

[8]We note that the functions $\Upsilon_{nm}(\mathbf{r})$ and $\Psi_{nm}(\mathbf{s}^*)$ might not be unique.

Next, we derive an upper bound on $\mathbb{P}\big[\mathcal{E}_i^{HK}\big]$ and show that the SNR exponent of this bound matches the SNR exponent of the outage probability $\mathbb{P}\big[\mathcal{O}^{HK}(\mathbf{r})\big]$. We start by deriving an upper bound on $\mathbb{P}\big[\mathcal{E}_{ik}^{HK}\big]$ according to

$$\begin{aligned}\mathbb{P}\big[\mathcal{E}_{ik}^{HK}\big] &= \mathbb{P}\big[\mathcal{E}_{ik}^{HK}, \mathcal{O}_{ik}(\Upsilon_{ik}(\mathbf{r}), \Psi_{ik}(\mathbf{s}^*), \mathbf{p}^*)\big] + \\ &\qquad \mathbb{P}\big[\mathcal{E}_{ik}^{HK}, \bar{\mathcal{O}}_{ik}(\Upsilon_{ik}(\mathbf{r}), \Psi_{ik}(\mathbf{s}^*), \mathbf{p}^*)\big] \quad (98)\\ &\le \mathbb{P}[\mathcal{O}_{ik}(\Upsilon_{ik}(\mathbf{r}), \Psi_{ik}(\mathbf{s}^*), \mathbf{p}^*)] + \\ &\qquad \mathbb{P}\big[\mathcal{E}_{ik}^{HK} | \bar{\mathcal{O}}_{ik}(\Upsilon_{ik}(\mathbf{r}), \Psi_{ik}(\mathbf{s}^*), \mathbf{p}^*)\big], \quad (99)\end{aligned}$$

for $i = 1, 2$, and $k = 1, 2, \ldots, 6$. Next, we derive an upper bound on $\mathbb{P}\big[\mathcal{E}_{ik}^{HK} | \bar{\mathcal{O}}_{ik}(\Upsilon_{ik}(\mathbf{r}), \Psi_{ik}(\mathbf{s}^*), \mathbf{p}^*)\big]$. To this end, we note that, for the event $\mathcal{E}_{i1}^{HK}$, the receiver can cancel out the contributions of $\mathbf{w}_i$ and $\mathbf{w}_j$ as they have been decoded correctly. The resulting equivalent signal model is then

$$\mathbf{y} = \sqrt{\text{SNR}} h_{ii} \mathbf{u}_i + \sqrt{\text{SNR}^\alpha} h_{ji} \mathbf{u}_j + \mathbf{z}. \quad (100)$$

Treating $\mathbf{u}_j$ as noise with $\mathbf{u}_j \sim \mathcal{CN}(\mathbf{0}, \text{SNR}^{-(1-p_j)} \mathbf{I}_N)$ results in an upper bound on the error probability as the worst noise (in terms of mutual information) under a covariance constraint is Gaussian [18]. The equivalent noise $\mathbf{n} = \mathbf{z} + \sqrt{\text{SNR}^\alpha} h_{ji} \mathbf{u}_j$ is therefore Gaussian with $\mathbf{n} \sim \mathcal{CN}(\mathbf{0}, (1 + \text{SNR}^{-(1-p_j)+\alpha} |h_{ji}|^2) \mathbf{I}_N)$. Recall that we assumed that $\mathcal{R}_j$ knows h_{ji} perfectly. We are now in a position to upper-bound the PEP according to

$$\begin{aligned}&\mathbb{E}_{\mathbf{h}_i}\{\mathbb{P}[\mathbf{u}_i \to \tilde{\mathbf{u}}_i]\} \\ &\quad \le \mathbb{E}_{\mathbf{h}_i}\left\{\exp\left[-\frac{\|h_{ii}(\mathbf{u}_i - \tilde{\mathbf{u}}_i)\|^2 \text{SNR}}{4(1 + \text{SNR}^{-(1-p_j)+\alpha} |h_{ji}|^2)}\right]\right\}.\end{aligned}$$

Since $\Delta \mathbf{u}_i = \sqrt{\text{SNR}^{1-p_i}} (\mathbf{u}_i - \tilde{\mathbf{u}}_i)$, we get

$$\begin{aligned}&\mathbb{E}_{\mathbf{h}_i}\{\mathbb{P}[\mathbf{u}_i \to \tilde{\mathbf{u}}_i]\} \\ &\quad \le \mathbb{E}_{\mathbf{h}_i}\left\{\exp\left[-\frac{\|h_{ii}(\Delta \mathbf{u}_i)\|^2 \text{SNR}^{p_i}}{4(1 + \text{SNR}^{-(1-p_j)+\alpha} |h_{ji}|^2)}\right]\right\}. \quad (101)\end{aligned}$$

Next, we use the fact that $\bar{\mathcal{O}}_{i1}(\Upsilon_{i1}(\mathbf{r}), \Psi_{i1}(\mathbf{s}^*), \mathbf{p}^*)$ entails $\frac{\text{SNR}^{p_i} |h_{ii}|^2}{1 + \text{SNR}^{-(1-p_j)+\alpha} |h_{ji}|^2} \ge \text{SNR}^{\psi_{i1}^i(\mathbf{s}^*)}$, where $i, j = 1, 2$ and $i \neq j$, and apply the union bound to upper-bound $\mathbb{P}\big[\mathcal{E}_{i1}^{HK} | \bar{\mathcal{O}}_{i1}(\Upsilon_{i1}(\mathbf{r}), \Psi_{i1}(\mathbf{s}^*), \mathbf{p}^*)\big]$ according to

$$\begin{aligned}&\mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_{i1}^{HK} | \bar{\mathcal{O}}_{i1}(\Upsilon_{i1}(\mathbf{r}), \Psi_{i1}(\mathbf{s}^*), \mathbf{p}^*)\big]\big\} \le \\ &\qquad \text{SNR}^{N s_i} \exp\left[-\frac{\text{SNR}^{\psi_{i1}^i(\mathbf{s}^*)} \|\Delta \mathbf{u}_i\|^2}{4}\right]. \quad (102)\end{aligned}$$

Since $\|\Delta \mathbf{u}_i\|^2 \dot{\ge} \text{SNR}^{-\psi_{i1}^i(\mathbf{s}^*)+\epsilon}$, with $\epsilon > 0$, by assumption, we further have

$$\begin{aligned}&\mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_{i1}^{HK}\big]\big\} \\ &\dot{\le} \mathbb{P}[\mathcal{O}_{i1}(\Upsilon_{i1}(\mathbf{r}), \Psi_{i1}(\mathbf{s}^*), \mathbf{p}^*)] + \text{SNR}^{N s_i} \exp\left[-\text{SNR}^\epsilon\right] \quad (103)\\ &\dot{\le} \mathbb{P}[\mathcal{O}_{i1}(\Upsilon_{i1}(\mathbf{r}), \Psi_{i1}(\mathbf{s}^*), \mathbf{p}^*)]. \quad (104)\end{aligned}$$

As the bounding of the remaining error events is similar to what was done above, we relegate the derivations to the Appendix. Next, we upper-bound $\mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_i^{HK}\big]\big\}$, $i = 1, 2$, as follows

$$\begin{aligned}\mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_i^{HK}\big]\big\} &\le \sum_{k=1}^{6} \mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_{ik}^{HK}\big]\big\} \quad (105)\\ &\dot{\le} \sum_{k=1}^{6} \mathbb{P}[\mathcal{O}_{ik}(\Upsilon_{ik}(\mathbf{r}), \Psi_{ik}(\mathbf{s}^*), \mathbf{p}^*)] \quad (106)\\ &\doteq \max_{k=1,2,\ldots,6} \mathbb{P}[\mathcal{O}_{ik}(\Upsilon_{ik}(\mathbf{r}), \Psi_{ik}(\mathbf{s}^*), \mathbf{p}^*)] \quad (107)\\ &\doteq \mathbb{P}\big[\mathcal{O}^{HK}(\mathbf{r})\big]. \quad (108)\end{aligned}$$

The error probability for the two-message, fixed-power-split-HK scheme is now given by

$$\begin{aligned}P\big(E^{HK}\big) &\doteq \max_{i=1,2} \mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_i^{HK}\big]\big\} \quad (109)\\ &\doteq \mathbb{P}\big[\mathcal{O}^{HK}(\mathbf{r})\big] \quad (110)\end{aligned}$$

where (110) follows from the definition of $\mathbb{P}\big[\mathcal{O}^{HK}(\mathbf{r})\big]$. From the outage lower bound (79), we now have that

$$\mathbb{P}\big[\mathcal{O}^{HK}(\mathbf{r})\big] \ \dot{\le} \ P\big(E^{HK}\big) \ \dot{\le} \ \mathbb{P}\big[\mathcal{O}^{HK}(\mathbf{r})\big] \quad (111)$$

and therefore,

$$P\big(E^{HK}\big) \doteq \mathbb{P}\big[\mathcal{O}^{HK}(\mathbf{r})\big]. \quad (112)$$

■

Discussion: It is unclear whether finite block-length codes satisfying the conditions of Theorem 2 exist. It is, however, interesting to note that a fixed power-split superposition HK-type scheme with infinite block-length codes and under an average power constraint, as put forward in [19], achieves exactly the DMT region characterized in Theorem 2.

Remark 1: It turns out that the total outage probability (79) can be described in a simpler fashion by recognizing that the constraints (73) and (76) are redundant. An inspection of (73) and (74) immediately yields that $d_{i3}(\mathbf{r}, \mathbf{s}, \mathbf{p}) \le d_{i2}(\mathbf{r}, \mathbf{s}, \mathbf{p})$ so that (73) can be eliminated. Finally, (76) can be eliminated by noting the following:

- whenever $p_j < 1 - \sum_{k=1}^{2} r_k + s_j$, we have
  $$d_{i6}(\mathbf{r}, \mathbf{s}, \mathbf{p}) \le d_{i5}(\mathbf{r}, \mathbf{s}, \mathbf{p}).$$
- whenever $p_j \ge 1 - \sum_{k=1}^{2} r_k + s_j$ and
  - ⋆ $p_j \ge 1 - \sum_{k=1}^{2} r_k + \sum_{l=1}^{2} s_l$, we have
    $$d_{i6}(\mathbf{r}, \mathbf{s}, \mathbf{p}) \le d_{i5}(\mathbf{r}, \mathbf{s}, \mathbf{p}).$$
  - ⋆ $p_j < 1 - \sum_{k=1}^{2} r_k + \sum_{l=1}^{2} s_l$, we have
    $$d_{j1}(\mathbf{r}, \mathbf{s}, \mathbf{p}) \le d_{i5}(\mathbf{r}, \mathbf{s}, \mathbf{p})$$

  with $i, j = 1, 2$ and $i \neq j$.

It is interesting to observe that analogues of the eliminations carried out above were reported in [17]. Specifically, we note that the elimination of (73) and (76) is equivalent (in terms of DMT) to eliminating conditions (67) and (70) in the characterization of the total outage event in (65). This, in turn, is equivalent (in terms of DMT) to eliminating (58) and (61) from the characterization of the achievable rate region $\mathcal{R}^*$. Now, it can be shown that the HK rate region described in [17] evaluates precisely to the rate region $\mathcal{R}^*$ in (64) after the elimination of (58) and (61) when the input distributions are taken to be i.i.d. Gaussian in [17] (which results in no loss of DMT-optimality).

IV. DMT of the Interference Channel

We start by showing that the *joint ML decoder* is a special case of the two-message, fixed-power-split HK scheme. With $p_i = -\infty$, $i = 1, 2$, the *joint ML decoder for the two-message, fixed-power-split HK scheme* will only attempt to decode the public messages at each $\mathcal{R}_i$. Hence, for $p_i = -\infty$, the *joint ML decoder* is equivalent to the *joint ML decoder for the two-message, fixed-power-split HK scheme* in terms of the DMT region. We will use the term *fixed-power-split HK scheme* to refer to both the *joint ML decoder* and the *joint ML decoder for the two-message, fixed-power-split HK scheme*.

We next put the pieces together to set the stage for the remainder of the paper. For a given rate tuple $\mathbf{r}$, obviously, either $d^{HK}(\mathbf{r})$ or $d^{JD}(\mathbf{r})$ dominates. Therefore, the maximum DMT of the fixed-power-split HK scheme is given by

$$d(\mathbf{r}) = \max\{d^{HK}(\mathbf{r}), d^{JD}(\mathbf{r})\}\,. \tag{113}$$

- If $d^{HK}(\mathbf{r}) \le d^{JD}(\mathbf{r})$, $d(\mathbf{r})$ can be achieved by satisfying the code design criteria in Theorem 1, in combination with the *joint ML decoder*. As already mentioned, it is shown in [14] that there exist codes satisfying these criteria for all $r_1 = r_2$ and $r_1 < 1.5$. Whether codes for the general case exist is an open problem.
- If $d^{HK}(\mathbf{r}) > d^{JD}(\mathbf{r})$, $d(\mathbf{r})$ is achievable using infinite block length codes [19]. Finding codes that satisfy the design criteria specified in Theorem 2, as already mentioned, remains an open problem.

In the following, we call ICs with $1 > \alpha \ge 2/3$, $2 > \alpha \ge 1$, and $\alpha \ge 2$ *moderate*, *strong*, and *very strong* ICs in the sense of [4], respectively. In the next section, we show that the fixed-power-split HK scheme is DMT-optimal under *strong* and *very strong interference* for all multiplexing rates, provided the constituent codes satisfy the corresponding code design criteria.

V. DMT-Optimality

We first derive an outer bound on the DMT region of the IC that is tighter than the outer bound derived in [9] *for some interference levels*. It turns out that for all multiplexing rate pairs, the fixed-power-split HK scheme achieves this outer bound for all $\alpha \ge 1$. Under the symmetric rate constraint, i.e., $r = r_1 = r_2$, and $1 > \alpha \ge 2/3$, we show that the outer bound can be achieved for all rates $r \le \alpha/2$ using finite block length codes that satisfy the design criteria in Theorem 1. As stated above, the existence of such codes was shown in [14]. Under the symmetric rate constraint with $r > \alpha/2$ and $1 > \alpha \ge 2/3$, it remains an open problem to construct codes that satify the criteria in Theorem 2.

A. Outer bound on the DMT

The outer bound is based on providing $\mathcal{R}_2$ with the side information $\mathbf{x}_1$. As $\mathcal{R}_2$ knows the fading coefficient h_{12} perfectly (by assumption), it can cancel the interference term $h_{12}\mathbf{x}_1$ out, leaving a one-sided IC as depicted in Fig. 1. Further, we assume that a genie reveals the fading coefficient h_{21} to $\mathcal{T}_2$. It is shown in [4] that the capacity region of the IC is contained in the following region

$$\mathcal{R}^1_{ETW} \triangleq \begin{cases} \mathcal{D}_1, & \text{SNR}^\alpha |h_{21}|^2 < 1 \\ \mathcal{D}_2, & \text{SNR}^\alpha |h_{21}|^2 \ge 1 \end{cases} \tag{114}$$

where

$$\begin{aligned} \mathcal{D}_1 \triangleq{}& (S_1, T_1, S_2, T_2): \\ & S_1 + T_1 \le \log\left(1 + \text{SNR}|h_{11}|^2\right) + 1 \\ & S_2 + T_2 \le \log\left(1 + \text{SNR}|h_{22}|^2\right) + 1 \\ \mathcal{D}_2 \triangleq{}& (S_1, T_1, S_2, T_2): \\ & S_1 + T_1 \le \log\left(1 + \text{SNR}|h_{11}|^2\right) + 1 \\ & S_1 + T_1 + T_2 \le \log\left(1 + \text{SNR}|h_{11}|^2 + \text{SNR}^\alpha |h_{21}|^2\right) + 1 \\ & S_2 \le \log\left(1 + \text{SNR}^{1-\alpha}\frac{|h_{22}|^2}{|h_{21}|^2}\right) + 1 \\ & S_2 + T_2 \le \log\left(1 + \text{SNR}|h_{22}|^2\right) + 1. \end{aligned}$$

Recall that for a set $\mathcal{S}$ of quadruples $\{S_1, T_1, S_2, T_2\}$, we defined $\prod(\mathcal{S})$ to be the corresponding set of rate pairs $\{R_1, R_2\}$ such that $R_1 = S_1 + T_1$ and $R_2 = S_2 + T_2$. We also recall that $S_i = s_i \log \text{SNR}$, $T_i = t_i \log \text{SNR}$, and $R_i = r_i \log \text{SNR}$, for $i = 1, 2$. Then, the set

$$\mathcal{R}^*_{ETW} \triangleq \prod\left(\mathcal{R}^1_{ETW}\right) \tag{115}$$

is an outer bound on the achievable rate region for the IC, i.e., we have

$$\mathcal{R}^*_{ETW} \supseteq \mathcal{R}^\dagger \tag{116}$$

where $\mathcal{R}^\dagger$ is any achievable rate region of the IC. Next, we define the events

$$\begin{aligned} \mathcal{A} &\triangleq \left\{h_{21} : \text{SNR}^\alpha |h_{21}|^2 < 1\right\} \\ \bar{\mathcal{A}} &\triangleq \left\{h_{21} : \text{SNR}^\alpha |h_{21}|^2 \ge 1\right\} \end{aligned}$$

and

$$\begin{aligned} & \mathcal{O}^{ETW}_{1i}(\mathbf{r}, \mathbf{s}) \triangleq \\ & \left\{\mathbf{h}_i : \log\left(1 + \text{SNR}|h_{ii}|^2\right) + 1 < S_i + T_i\right\},\ \text{for } i = 1, 2 \\ & \mathcal{O}^{ETW}_{13}(\mathbf{r}, \mathbf{s}) \triangleq \\ & \left\{\mathbf{h}_1 : \log\left(1 + \text{SNR}|h_{11}|^2 + \text{SNR}^\alpha |h_{21}|^2\right) + 1 < S_1 + T_1 + T_2\right\} \\ & \mathcal{O}^{ETW}_{14}(\mathbf{r}, \mathbf{s}) \triangleq \\ & \left\{\mathbf{h}_2 : \log\left(1 + \text{SNR}^{1-\alpha}\frac{|h_{22}|^2}{|h_{21}|^2}\right) + 1 < S_2\right\}. \end{aligned}$$

Since any achievable rate region for the IC is contained in $\mathcal{R}^*_{ETW}$, it follows that the error probability of any scheme communicating over the IC is lower-bounded by the outage probability

$$\mathbb{P}\left[\mathcal{O}^{ETW}(\mathbf{r})\right] \triangleq \min_{\mathbf{s}} \mathbb{P}\left[\mathcal{O}^{ETW}_1(\mathbf{r}, \mathbf{s})\right] \tag{117}$$

where the minimization is carried out subject to

$$r_i = s_i + t_i \tag{118}$$
$$s_i, t_i \geq 0 \tag{119}$$
$$s_i, t_i \leq r_i, \tag{120}$$

for $i = 1, 2$, with

$$\mathcal{O}_1^{ETW}(\mathbf{r}, \mathbf{s}) \triangleq \mathcal{K}_1(\mathbf{r}, \mathbf{s}) \bigcup \mathcal{K}_2(\mathbf{r}, \mathbf{s}) \tag{121}$$

and

$$\mathcal{K}_1(\mathbf{r}, \mathbf{s}) \triangleq \left(\bigcup_{i=1,2} \mathcal{O}_{1i}^{ETW}(\mathbf{r}, \mathbf{s}) \right) \bigcap \mathcal{A} \tag{122}$$
$$\mathcal{K}_2(\mathbf{r}, \mathbf{s}) \triangleq \left(\bigcup_{i=1,2,3,4} \mathcal{O}_{1i}^{ETW}(\mathbf{r}, \mathbf{s}) \right) \bigcap \bar{\mathcal{A}}. \tag{123}$$

Next, we compute $\mathbb{P}\big[\mathcal{O}_1^{ETW}(\mathbf{r}, \mathbf{s})\big]$. We note that an equivalent characterization of $\mathcal{O}_1^{ETW}(\mathbf{r}, \mathbf{s})$ is:

$$\mathcal{O}_1^{ETW}(\mathbf{r}, \mathbf{s}) = \left(\bigcup_{i=1,2} \mathcal{O}_{1i}^{ETW}(\mathbf{r}, \mathbf{s}) \right) \bigcup \left(\bigcup_{i=3,4} \mathcal{O}_{1i}^{ETW}(\mathbf{r}, \mathbf{s}) \bigcap \bar{\mathcal{A}} \right).$$

It follows that we can upper-bound $\mathbb{P}\big[\mathcal{O}_1^{ETW}(\mathbf{r}, \mathbf{s})\big]$ according to

$$\mathbb{P}\big[\mathcal{O}_1^{ETW}(\mathbf{r}, \mathbf{s})\big] \leq \sum_{i=1}^{2} \mathbb{P}\big[\mathcal{O}_{1i}^{ETW}(\mathbf{r}, \mathbf{s})\big] + \sum_{i=3}^{4} \mathbb{P}\Big[\mathcal{O}_{1i}^{ETW}(\mathbf{r}, \mathbf{s}) \bigcap \bar{\mathcal{A}}\Big]. \tag{124}$$

We can also lower-bound $\mathbb{P}\big[\mathcal{O}_1^{ETW}(\mathbf{r}, \mathbf{s})\big]$ according to

$$\mathbb{P}\big[\mathcal{O}_{1i}^{ETW}(\mathbf{r}, \mathbf{s})\big] \leq \mathbb{P}\big[\mathcal{O}_1^{ETW}(\mathbf{r}, \mathbf{s})\big], \tag{125}$$

for $i = 1, 2$. Further, for $i = 3, 4$, we have

$$\mathbb{P}\Big[\mathcal{O}_{1i}^{ETW}(\mathbf{r}, \mathbf{s}) \bigcap \bar{\mathcal{A}}\Big] \leq \mathbb{P}\big[\mathcal{O}_1^{ETW}(\mathbf{r}, \mathbf{s})\big]. \tag{126}$$

We next compute the SNR exponents of the upper and lower bounds in (124) and (126) and show that these exponents match in order to obtain the SNR exponent of $\mathbb{P}\big[\mathcal{O}_1^{ETW}(\mathbf{r}, \mathbf{s})\big]$. It is shown in [9] that

$$\mathbb{P}\big[\mathcal{O}_{1i}^{ETW}(\mathbf{r}, \mathbf{s})\big] \doteq \text{SNR}^{-d_{1i}^{ETW}(\mathbf{r}, \mathbf{s})} \tag{127}$$

where $d_{1i}^{ETW}(\mathbf{r}, \mathbf{s}) = (1 - r_i)^+$, for $i = 1, 2$, and

$$\mathbb{P}\big[\mathcal{O}_{13}^{ETW}(\mathbf{r}, \mathbf{s})\big] \doteq \text{SNR}^{-d_{13}^{ETW}(\mathbf{r}, \mathbf{s})} \tag{128}$$
$$\mathbb{P}\big[\mathcal{O}_{14}^{ETW}(\mathbf{r}, \mathbf{s})\big] \doteq \text{SNR}^{-d_{14}^{ETW}(\mathbf{r}, \mathbf{s})} \tag{129}$$

with

$$d_{13}^{ETW}(\mathbf{r}, \mathbf{s}) = (1 - r_1 - r_2 + s_2)^+ + (\alpha - r_1 - r_2 + s_2)^+$$
$$d_{14}^{ETW}(\mathbf{r}, \mathbf{s}) = \begin{cases} (1 - \alpha - s_2)^+, & \text{if } s_2 > 0 \text{ and } \alpha < 1 \\ 1, & \text{if } s_2 = 0 \\ 0, & \text{if } s_2 > 0 \text{ and } \alpha \geq 1. \end{cases} \tag{130}$$

Combining (127)-(129) with (124), (125) and (126), it follows that

$$\mathbb{P}\big[\mathcal{O}_1^{ETW}(\mathbf{r}, \mathbf{s})\big] \doteq \text{SNR}^{-d_1^{ETW}(\mathbf{r}, \mathbf{s})} \tag{131}$$

where

$$d_1^{ETW} = \min_{i=1,2,3,4} d_{1i}^{ETW}(\mathbf{r}, \mathbf{s}). \tag{132}$$

The SNR exponent of $\mathbb{P}\big[\mathcal{O}^{ETW}(\mathbf{r})\big]$ is then obtained as

$$\mathbb{P}\big[\mathcal{O}^{ETW}(\mathbf{r})\big] = \min_{\mathbf{s}} \text{SNR}^{-d_1^{ETW}(\mathbf{r}, \mathbf{s})} \tag{133}$$
$$= \text{SNR}^{-\max_{\mathbf{s}} d_1^{ETW}(\mathbf{r}, \mathbf{s})} \tag{134}$$

where the optimization is carried out subject to

$$r_i = s_i + t_i \tag{135}$$
$$s_i, t_i \geq 0 \tag{136}$$
$$s_i, t_i \leq r_i. \tag{137}$$

The SNR exponent of the error probability lower bound in (134) is in general difficult to evaluate. In the next section, we identify cases where an analytic evaluation is possible.

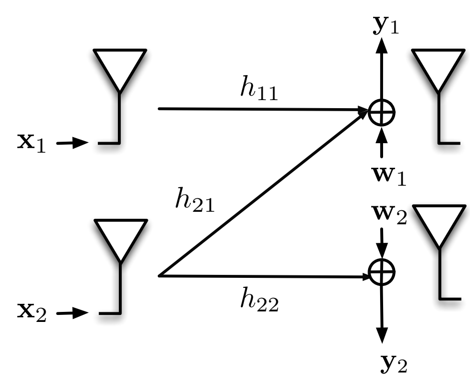

Fig. 1. One-sided interference channel.

B. *The case $\alpha \geq 1$*

We next show that the *joint ML decoder* in combination with codes that satisfy the criteria in Theorem 1 achieves the optimal DMT region of the IC for all interference levels $\alpha \geq 1$. We start by denoting the minimizing value of $\mathbf{s}$ in (134) by $\mathbf{s}^\dagger$ and noting that the DMT outer bound (134) in Section V-A can be simplified according to

$$d_1^{ETW}(\mathbf{r}, \mathbf{s}^\dagger) = d^{JD}(\mathbf{r}). \tag{138}$$

Upon inspection of (130), we see that choosing any $s_2 > 0$ results in $d_{i4}^{ETW}(\mathbf{r}, \mathbf{s}) = 0$ for $\alpha \geq 1$. Hence, for any $s_2 > 0$, we have $d_1^{ETW}(\mathbf{r}, \mathbf{s}) = 0$. For $s_2 = 0$, we get

$$d_{1i}^{ETW}(\mathbf{r}, \mathbf{s}) = (1 - r_i)^+, \ i = 1, 2 \tag{139}$$
$$d_{13}^{ETW}(\mathbf{r}, \mathbf{s}) = (1 - r_1 - r_2)^+ + (\alpha - r_1 - r_2)^+ \tag{140}$$
$$d_{14}^{ETW}(\mathbf{r}, \mathbf{s}) = 1. \tag{141}$$

Therefore, $d_1^{ETW}(\mathbf{r}, \mathbf{s}^\dagger)$ is equivalent to $d^{JD}(\mathbf{r})$ by inspection of (8) and (139)-(141).

C. The case $1 > \alpha \geq 2/3$

For the case $1 > \alpha \geq 2/3$ and for general multiplexing rate pairs for the two transmitters, proving optimality of the fixed-power-split HK scheme remains elusive. However, we can show that the fixed-power-split HK scheme achieves the outer bound for $r_1 = r_2 = r$ and for all $r < \alpha/2$. The DMT outer bound is achieved for $1 > \alpha \geq 2/3$ and $r < \alpha/2$ by using the *joint ML decoder* in combination with codes that satify the criteria in Theorem 1. We recall that in the case of symmetric multiplexing rates ($r_1 = r_2 = r$), we have that $s_1 = s_2 = s$. It turns out that the DMT outer bound in (134) can be achieved for all $r < \alpha/2$ by setting $s = 0$. With these choices of optimizing values, an inspection of the DMT outer bound in (134) and the achievable region (113) reveals that the two DMT regions are equal for $r < \alpha/2$.

The problem with rates $r \geq \alpha/2$ is that the existence of finite block length codes satisfying the code design criteria of Theorem 2 under a peak power constraint, the case we consider, remains an open problem. As mentioned earlier, it is shown in [19] that a fixed-power-split superposition HK scheme with infinite-block length codes and under an average power constraint achieves the DMT outer bound for rates $r \geq \alpha/2$.

VI. Very Strong Interference

We recall that channels with $\alpha \geq 2$ are called *very strong interference channels* in the sense of [4]. We shall see that the condition $\alpha \geq 2$ enables each transmitter-receiver pair to communicate as if no interference were present. Specifically, we show, in this section, that a *stripping decoder*, which decodes interference while treating the intended signal as noise, subtracts the result out, and then decodes the desired signal, is DMT-optimal for the IC under very strong interference. In this section, we take $N = 1$ throughout; we will see later that DMT-optimal performance can be achieved for $N \geq 1$, in contrast to the fixed-power-split HK scheme which needs $N \geq 2$. In the following, we use the short-hand x_i for the first element of the transmit signal vector $\mathbf{x}_i$, y_i for the first element of the receive signal vector $\mathbf{y}_i$, and $\mathcal{X}_i$ for $\mathcal{C}_i(\text{SNR}, r_i)$.

We write $\mathbb{P}[E_{ij}|\mathbf{h}_j]$, for $i, j = 1, 2$, and $i \neq j$, for the ML error probability corresponding to decoding of $\mathcal{T}_i$ at $\mathcal{R}_j$ under the assumption of the signal from $\mathcal{T}_j$ being treated as noise. We define the respective average ML decoding error probability as $P(E_{ij}) = \mathbb{E}_{\mathbf{h}_j}\{\mathbb{P}[E_{ij}|\mathbf{h}_j]\}$. It is assumed throughout that the transmit symbols are equally likely for both transmitters, and hence $\mathbb{P}[x_i] = \frac{1}{|\mathcal{X}_i|}$, for $i = 1, 2$.

In the following, we show that a stripping decoder achieves the DMT outer bound in [9] given by

$$d(\mathbf{r}) \leq \min\{(1 - r_1)^+, (1 - r_2)^+\}. \tag{142}$$

Theorem 3: For the fading IC with I/O relation (3)-(4), the average error probability of the stripping decoder obeys

$$P(E) \doteq \text{SNR}^{-\min\{(1-r_1)^+,(1-r_2)^+\}} \tag{143}$$

provided that $\Delta x_i = x_i^j - x_i^k$ satisfies $|\Delta x_i|^2 \dot{\geq} \text{SNR}^{-r_i+\epsilon}$ for every pair x_i^j, x_i^k in each codebook $\mathcal{X}_i$, $i = 1, 2$, and for some[9] $\epsilon > 0$.

[9] ϵ is allowed to be different for $i = 1, 2$ throughout.

Proof: We start by decoding $\mathcal{T}_2$ at $\mathcal{R}_1$ while treating $\mathcal{T}_1$ as noise with known codebook, i.e., we have the effective I/O relation

$$y_1 = \sqrt{\text{SNR}^{\alpha}} h_{21} x_2 + \tilde{z} \tag{144}$$

where $\tilde{z}$ is the effective noise term with variance $1 + \text{SNR}|h_{11}|^2$. We next note that the worst case (in terms of mutual information and hence outage probability) uncorrelated (with the transmit signal) additive noise under a variance constraint is Gaussian [18, Theorem 1]. In the following, we use the corresponding worst case outage probability in combination with the design criteria in Theorem 3 to exponentially upper-bound $P(E_{21})$, i.e., we set $\tilde{z} \sim \mathcal{CN}(0, 1 + \text{SNR}|h_{11}|^2)$. We start by normalizing the received signal according to

$$\frac{y_1}{\sqrt{1 + \text{SNR}|h_{11}|^2}} = \sqrt{\frac{\text{SNR}^{\alpha}}{1 + \text{SNR}|h_{11}|^2}} h_{21} x_2 + z \tag{145}$$

where $z \sim \mathcal{CN}(0, 1)$. We can now upper-bound $\mathbb{P}[E_{21}|\mathbf{h}_1]$ as

$$\begin{aligned}
\mathbb{P}[E_{21}|\mathbf{h}_1] &= \sum_{x_2 \in \mathcal{X}_2} \mathbb{P}[x_2]\, \mathbb{P}[E_{21}|\mathbf{h}_1, x_2] && (146)\\
&= \frac{1}{|\mathcal{X}_2|} \sum_{i=1}^{|\mathcal{X}_2|} \mathbb{P}\left[\bigcup_{\substack{j=1\\ j\neq i}}^{|\mathcal{X}_2|} x_2^i \to x_2^j \mid \mathbf{h}_1 \right] && (147)\\
&\leq |\mathcal{X}_2| \mathbb{P}\left[x_2^{\tilde{i}} \to x_2^{\tilde{j}} \mid \mathbf{h}_1 \right] && (148)\\
&\leq |\mathcal{X}_2| \mathrm{Q}\left(\sqrt{\frac{\text{SNR}^{\alpha}|h_{21}|^2 |\Delta x_2|^2}{2(1 + \text{SNR}|h_{11}|^2)}} \right), && (149)
\end{aligned}$$

where $\left\{x_2^{\tilde{i}}, x_2^{\tilde{j}}\right\}$ denotes the (or "a" in the case of multiple pairs with the same distance) pair of symbols with minimum Euclidean distance among all possible pairs of different symbols. We now define the outage event $\mathcal{O}_{ii}$ associated with decoding $\mathcal{T}_i$ at $\mathcal{R}_i$ ($i = 1, 2$) in the absence of interference and its complementary event $\bar{\mathcal{O}}_{ii}$ as follows

$$\mathcal{O}_{ii} \triangleq \left\{ h_{ii} : \log\left(1 + \text{SNR}|h_{ii}|^2\right) < R_i \right\} \tag{150}$$

$$\bar{\mathcal{O}}_{ii} \triangleq \left\{ h_{ii} : \log\left(1 + \text{SNR}|h_{ii}|^2\right) \geq R_i \right\}. \tag{151}$$

We note that this definition is consistent with the definition of $P(E_{ii})$. Similarly, we define the event $\mathcal{O}_{ij}$ associated with decoding $\mathcal{T}_i$ at $\mathcal{R}_j$ while treating $\mathcal{T}_j$ as noise ($i, j = 1, 2$ and $i \neq j$) and its complementary event $\bar{\mathcal{O}}_{ij}$ as follows

$$\mathcal{O}_{ij} \triangleq \left\{ \mathbf{h}_j : \log\left(1 + \frac{\text{SNR}^{\alpha}|h_{ij}|^2}{1 + \text{SNR}|h_{jj}|^2}\right) < R_i \right\}$$

$$\bar{\mathcal{O}}_{ij} \triangleq \left\{ \mathbf{h}_j : \log\left(1 + \frac{\text{SNR}^{\alpha}|h_{ij}|^2}{1 + \text{SNR}|h_{jj}|^2}\right) \geq R_i \right\}.$$

Next, we upper-bound $P(E_{21})$ according to

$$\begin{aligned}
&P(E_{21}) = \mathbb{E}_{\mathbf{h}_1}\{\mathbb{P}[E_{21}|\mathbf{h}_1]\} = \\
&\mathbb{E}_{\mathbf{h}_1}\left\{\mathbb{P}[\mathcal{O}_{21}]\, \mathbb{P}[E_{21}|\mathbf{h}_1, \mathcal{O}_{21}] + \mathbb{P}\left[\bar{\mathcal{O}}_{21}\right] \mathbb{P}\left[E_{21}|\mathbf{h}_1, \bar{\mathcal{O}}_{21}\right]\right\} && (152)\\
&\leq \mathbb{P}[\mathcal{O}_{21}] + \mathbb{E}_{\mathbf{h}_1}\left\{\mathbb{P}\left[E_{21}|\mathbf{h}_1, \bar{\mathcal{O}}_{21}\right]\right\} && (153)\\
&\leq \mathbb{P}[\mathcal{O}_{21}] + \text{SNR}^{r_2} \mathrm{Q}\left(\sqrt{\frac{\text{SNR}^{r_2}|\Delta x_2|^2}{2}} \right) && (154)
\end{aligned}$$

where (152) follows from Bayes's rule and (153) is obtained by upper-bounding $\mathbb{P}[E_{21}|\mathbf{h}_1, \mathcal{O}_{21}]$ and $\mathbb{P}\big[\bar{\mathcal{O}}_{21}\big]$ by 1. Finally, (154) follows by using the fact that $\bar{\mathcal{O}}_{21}$ entails $\frac{\text{SNR}^{\alpha}|h_{21}|^2}{1+\text{SNR}|h_{11}|^2} \geq 2^{R_2}-1$, and invoking $R_2 = r_2 \log \text{SNR}$, $|\mathcal{X}_2| = \text{SNR}^{r_2}$, and $\text{SNR} \gg 1$ in (149). It can be shown that $\mathbb{P}[\mathcal{O}_{21}] \doteq \text{SNR}^{-(\alpha-1-r_2)^+}$ for $\alpha \geq 2$ [9]. Since $|\Delta x_2|^2 \dot{\geq} \text{SNR}^{-r_2+\epsilon}$, for $\epsilon > 0$, by assumption, we can further simplify the above as the second term in (154) decays exponentially in SNR whereas the first term decays polynomially, i.e., we get

$$\mathbb{E}_{\mathbf{h}_1}\{\mathbb{P}[E_{21}|\mathbf{h}_1]\} \ \dot{\leq} \ \mathbb{P}[\mathcal{O}_{21}] \doteq \text{SNR}^{-(\alpha-1-r_2)^+}. \tag{155}$$

We proceed to analyze decoding of $\mathcal{T}_1$ at $\mathcal{R}_1$ and start by defining $\bar{x}_2$ as the result of decoding $\mathcal{T}_2$ at $\mathcal{R}_1$. Note that we do not need to assume that $\mathcal{T}_2$ was decoded correctly at $\mathcal{R}_1$. We begin by upper-bounding $\mathbb{P}[E_{11}|\mathbf{h}_1]$ given $\bar{x}_2$:

$$\begin{aligned}
&\mathbb{P}[E_{11}|\mathbf{h}_1, \bar{x}_2] \\
&= \sum_{x_1 \in \mathcal{X}_1} \sum_{x_2 \in \mathcal{X}_2} \mathbb{P}[x_1]\,\mathbb{P}[x_2]\,\mathbb{P}[E_1|\mathbf{h}_1, x_1, x_2, \bar{x}_2] && (156)\\
&= \frac{1}{|\mathcal{X}_1||\mathcal{X}_2|} \sum_{i=1}^{|\mathcal{X}_1|} \sum_{k=1}^{|\mathcal{X}_2|} \mathbb{P}\left[\bigcup_{\substack{j=1\\ j\neq i}}^{|\mathcal{X}_1|} x_1^i \to x_1^j \mid \mathbf{h}_1, x_2^k, \bar{x}_2\right] && (157)\\
&\leq \frac{|\mathcal{X}_1|}{|\mathcal{X}_2|} \sum_{k=1}^{|\mathcal{X}_2|} \mathbb{P}\Big[x_1^{\tilde{i}} \to x_1^{\tilde{j}} \mid \mathbf{h}_1, x_2^k, \bar{x}_2\Big], && (158)
\end{aligned}$$

where $\Big\{x_1^{\tilde{i}}, x_1^{\tilde{j}}\Big\}$ denotes the (or "a" in the case of multiple pairs with the same distance) pair of symbols with minimum Euclidean distance among all possible pairs of different symbols. Next, we further upper-bound $\mathbb{P}[E_{11}|\mathbf{h}_1, \bar{x}_2]$ by considering two events; namely, when $\mathcal{R}_1$ decodes the message from $\mathcal{T}_2$ correctly and when it does not:

$$\begin{aligned}
&\mathbb{P}[E_{11}|\mathbf{h}_1, \bar{x}_2] \leq \\
&\frac{|\mathcal{X}_1|}{|\mathcal{X}_2|} \sum_{k=1}^{|\mathcal{X}_2|} \Big(\mathbb{P}\big[\bar{x}_2 = x_2^k|\mathbf{h}_1, x_2^k\big]\mathbb{P}\Big[x_1^{\tilde{i}} \to x_1^{\tilde{j}}\,|\mathbf{h}_1, x_2^k, \bar{x}_2, \bar{x}_2 = x_2^k\Big] \\
&+\mathbb{P}\big[\bar{x}_2 \neq x_2^k|\mathbf{h}_1, x_2^k\big]\mathbb{P}\Big[x_1^{\tilde{i}} \to x_1^{\tilde{j}}\,|\mathbf{h}_1, x_2^k, \bar{x}_2, \bar{x}_2 \neq x_2^k\Big]\Big),
\end{aligned} \tag{159}$$

where $\mathbb{P}\Big[x_1^{\tilde{i}} \to x_1^{\tilde{j}}\,|\mathbf{h}_1, x_2^k, \bar{x}_2, \bar{x}_2 = x_2^k\Big]$ is the probability of mistakenly decoding $x_1^{\tilde{i}}$ for $x_1^{\tilde{j}}$ given that $\mathcal{T}_2$ transmitted x_2^k and $\mathcal{R}_1$ decoded $\mathcal{T}_2$ correctly, i.e., $\bar{x}_2 = x_2^k$. The quantity $\mathbb{P}\big[\bar{x}_2 = x_2^k|\mathbf{h}_1, x_2^k\big]$ is the probability of decoding $\mathcal{T}_2$ correctly given that x_2^k was transmitted. By upper-bounding $\mathbb{P}\big[\bar{x}_2 = x_2^k|\mathbf{h}_1, x_2^k\big]$ and $\mathbb{P}\Big[x_1^{\tilde{i}} \to x_1^{\tilde{j}}\,|\mathbf{h}_1, x_2^k, \bar{x}_2, \bar{x}_2 \neq x_2^k\Big]$ in (159) by 1, we arrive at

$$\begin{aligned}
\mathbb{P}[E_{11}|\mathbf{h}_1, \bar{x}_2] \leq \frac{|\mathcal{X}_1|}{|\mathcal{X}_2|} \sum_{k=1}^{|\mathcal{X}_2|} \mathbb{P}\Big[x_1^{\tilde{i}} \to x_1^{\tilde{j}} \mid \mathbf{h}_1, x_2^k, \bar{x}_2, \bar{x}_2 = x_2^k\Big] + \\
\frac{|\mathcal{X}_1|}{|\mathcal{X}_2|} \sum_{k=1}^{|\mathcal{X}_2|} \mathbb{P}\big[\bar{x}_2 \neq x_2^k|\mathbf{h}_1, x_2^k\big]\,.
\end{aligned} \tag{160}$$

Next, noting that $\frac{1}{|\mathcal{X}_2|} \sum_{k=1}^{|\mathcal{X}_2|} \mathbb{P}\big[\bar{x}_2 \neq x_2^k|\mathbf{h}_1, x_2^k\big] \leq \mathbb{P}[E_{21}|\mathbf{h}_1]$ and invoking the corresponding upper bound (149) in (160), we get

$$\begin{aligned}
\mathbb{P}[E_1|\mathbf{h}_1, \bar{x}_2] \leq |\mathcal{X}_1|\text{Q}\left(\sqrt{\frac{\text{SNR}|h_{11}|^2|\Delta x_1|^2}{2}}\right) + \\
|\mathcal{X}_1||\mathcal{X}_2|\text{Q}\left(\sqrt{\frac{\text{SNR}^{\alpha}|h_{21}|^2|\Delta x_2|^2}{2(1+\text{SNR}|h_{11}|^2)}}\right).
\end{aligned} \tag{161}$$

The first term on the RHS of (161) follows from the first term on the RHS of (160), since given $\bar{x}_2 = x_2^k$, the interference can be subtracted out, leaving an effective SISO channel without interference. We are now in a position to upper-bound $P(E_{11})$:

$$\begin{aligned}
&P(E_{11}) = \mathbb{E}_{\mathbf{h}_1}\{\mathbb{P}[E_{11}|\mathbf{h}_1]\} \leq \mathbb{E}_{\mathbf{h}_1}\{\mathbb{P}[E_{11}|\mathbf{h}_1, \bar{x}_2]\} && (162)\\
&\leq \mathbb{E}_{\mathbf{h}_1}\left\{|\mathcal{X}_1|\text{Q}\left(\sqrt{\frac{\text{SNR}|h_{11}|^2|\Delta x_1|^2}{2}}\right)\right\} + \\
&\mathbb{E}_{\mathbf{h}_1}\left\{|\mathcal{X}_1||\mathcal{X}_2|\text{Q}\left(\sqrt{\frac{\text{SNR}^{\alpha}|h_{21}|^2|\Delta x_2|^2}{2(1+\text{SNR}|h_{11}|^2)}}\right)\right\}. && (163)
\end{aligned}$$

Here, (162) follows since the error probability incurred by using the stripping decoder constitutes a natural upper bound on $\mathbb{E}_{\mathbf{h}_1}\{\mathbb{P}[E_{11}|\mathbf{h}_1]\}$. We upper-bound (163) by splitting each of the two terms into outage and no-outage sets using Bayes's rule to arrive at

$$\begin{aligned}
&P(E_{11}) = \mathbb{E}_{\mathbf{h}_1}\{\mathbb{P}[E_{11}|\mathbf{h}_1]\} \leq \\
&\mathbb{P}[\mathcal{O}_{11}] + \text{SNR}^{r_1}\text{Q}\bigg(\frac{\text{SNR}^{r_1}|\Delta x_1|^2}{2}\bigg) + \mathbb{P}[\mathcal{O}_{21}] + \\
&\text{SNR}^{r_1+r_2}\text{Q}\bigg(\frac{\text{SNR}^{r_2}|\Delta x_2|^2}{2}\bigg).
\end{aligned} \tag{164}$$

The second and fourth terms on the RHS of (164) follow from (163) since $\bar{\mathcal{O}}_{11}$ and $\bar{\mathcal{O}}_{21}$ entail $\text{SNR}|h_{11}|^2 \geq 2^{R_1}-1$ and $\frac{\text{SNR}^{\alpha}|h_{21}|^2}{1+\text{SNR}|h_{11}|^2} \geq 2^{R_2}-1$, respectively, and since $R_i = r_i \log \text{SNR}$, $|\mathcal{X}_i| = \text{SNR}^{r_i}$, for $i = 1, 2$, and $\text{SNR} \gg 1$. Given that the minimum Euclidean distances in each codebook, $|\Delta x_1|^2$ and $|\Delta x_2|^2$, obey $|\Delta x_1|^2 \dot{\geq} \text{SNR}^{-r_1+\epsilon}$ and $|\Delta x_2|^2 \dot{\geq} \text{SNR}^{-r_2+\epsilon}$, for some $\epsilon > 0$, by assumption, we get

$$\begin{aligned}
P(E_{11}) = \mathbb{E}_{\mathbf{h}_1}\{\mathbb{P}[E_{11}|\mathbf{h}_1]\} \ &\dot{\leq} \ \mathbb{P}[\mathcal{O}_{11}] + \mathbb{P}[\mathcal{O}_{21}] && (165)\\
&\doteq \ \text{SNR}^{-(1-r_1)^+} + \text{SNR}^{-(\alpha-1-r_2)^+} && (166)\\
&\doteq \ \text{SNR}^{-\min\{(1-r_1)^+, (\alpha-1-r_2)^+\}}. && (167)
\end{aligned}$$

Similar derivations for decoding at $\mathcal{R}_2$ lead to

$$P(E_{22}) \ \dot{\leq} \ \text{SNR}^{-\min\{(1-r_2)^+, (\alpha-1-r_1)^+\}}. \tag{168}$$

We note that the error probability of decoding $\mathcal{T}_i$ at $\mathcal{R}_i$ is exponentially lower-bounded by $\mathbb{P}[\mathcal{O}_{ii}]$, for $i = 1, 2$ [13]. Hence, $P(E_{ii})$ is sandwiched according to

$$\text{SNR}^{-(1-r_i)^+} \dot{\leq} P(E_{ii}) \ \dot{\leq} \ \text{SNR}^{-\min\{(1-r_i)^+, (\alpha-1-r_j)^+\}}, \tag{169}$$

for $i, j = 1, 2$, and $i \neq j$. The proof is concluded by first upper-bounding

$$P(E) = \max\{P(E_{11}), P(E_{22})\} \tag{170}$$

as

$$P(E) \dot{\leq} \max\Big\{\text{SNR}^{-\min\{(1-r_1)^+,(\alpha-1-r_2)^+\}},$$
$$\text{SNR}^{-\min\{(1-r_2)^+,(\alpha-1-r_1)^+\}}\Big\}$$
$$\doteq \text{SNR}^{-\min\{(1-r_1)^+,(1-r_2)^+\}} \quad (171)$$

where (171) is a consequence of the assumption $\alpha \geq 2$. Second, $P(E)$ can be lower-bounded using the outage bounds on the individual error probabilities:

$$\text{SNR}^{-\min\{(1-r_1)^+,(1-r_2)^+\}} \dot{\leq} P(E). \quad (172)$$

Since the SNR exponents in the upper bound (171) and the lower bound (172) match, we can conclude that

$$P(E) \doteq \text{SNR}^{-\min\{(1-r_1)^+,(1-r_2)^+\}} \quad (173)$$

which establishes the desired result. ■

Remark 2: We can immediately conclude from Theorem 3 that using a sequence of codebooks that is DMT-optimal for the SISO channel for both users results in DMT-optimality for the IC under very strong interference.

Remark 3: If $r_1 = r_2 = r$ and we use sequences of codebooks $\mathcal{C}(\text{SNR}, r)$ satisfying the conditions of Theorem 3 for both users, then we have

$$P(E_{11}) \doteq P(E_{22}) \doteq \text{SNR}^{-(1-r)^+} \quad (174)$$

as a simple consequence of (169). This means that in the special case, where $\mathcal{T}_1$ and $\mathcal{T}_2$ transmit at the same multiplexing rate $r = r_1 = r_2$, we have the stronger result that the single user DMT, i.e., the DMT that is achievable for a SISO channel in the absence of any interferers, is achievable for both users. In effect, under very strong interference and when the two users operate at the same multiplexing rate, the interference channel effectively gets *decoupled*. For a stripping decoder and $r_1 \neq r_2$, we can, in general, not arrive at the same conclusion as the SNR exponents in (169) do not necessarily match.

VII. Suboptimal Strategies

In the following, we investigate the DMT performance obtained i) by treating the IC as a combination of two multiple-access channels (MACs) and ii) by sharing transmission time between the two transmitters. These strategies are suboptimal; in fact, it can be shown that the fixed-power-split HK scheme always outperforms these schemes. Nevertheless, we analyze these two schemes as they are of some practical importance.

A. Achievable DMT for treating the IC as a combination of two MACs

A simple achievable rate region for the IC is obtained by treating the IC as a MAC at each receiver $\mathcal{R}_j$, $j = 1, 2$. Next, we formally define the corresponding strategy.

Definition 4: A *joint MAC-ML decoder* at $\mathcal{R}_j$ ($j = 1, 2$) carries out joint ML detection on the messages from both transmitters. An error is declared if any one or both messages are decoded in error. The ML error probability and the average ML error probability of this receiver are denoted by $\mathbb{P}\big[\mathcal{E}_j^{MAC}\big]$ and $P\big(E_j^{MAC}\big) \triangleq \mathbb{E}_{\mathbf{h}_j}\big\{\mathbb{P}\big[\mathcal{E}_j^{MAC}\big]\big\}$, respectively. The total error probability of this scheme is defined to be $P\big(E^{MAC}\big) \triangleq \max_{j=1,2} P\big(E_j^{MAC}\big)$.

The following theorem provides the achievable DMT region for the strategy of treating the IC as a combination of two MACs.

Theorem 4: The DMT corresponding to treating the IC as a MAC at each receiver is given by

$$d^{MAC}(\mathbf{r}) = \min_{\substack{i=1,2\\k=1,2,3}} d_{ik}^{MAC}(\mathbf{r}) \quad (175)$$

where

$$d_{i1}^{MAC}(\mathbf{r}) = (1 - r_i)^+$$
$$d_{i2}^{MAC}(\mathbf{r}) = (\alpha - r_j)^+, \ i, j = 1, 2, \ i \neq j$$
$$d_{i3}^{MAC}(\mathbf{r}) = (1 - r_1 - r_2)^+ + (\alpha - r_1 - r_2)^+ .$$

Denote

$$[i^* \ k^*] = \arg \min_{\substack{i=1,2\\k=1,2,3}} d_{ik}^{MAC}(\mathbf{r}). \quad (176)$$

Let $\Xi_{ik}(\mathbf{r}) = [\xi_{ik}^1(\mathbf{r}) \ \xi_{ik}^2(\mathbf{r})]^T$ be functions[10] such that

$$d_{i^*k^*}^{MAC}(\mathbf{r}) = d_{ik}^{MAC}(\Xi_{ik}(\mathbf{r})), \quad (177)$$

for $i = 1, 2$, $k = 1, 2, 3$. If a sequence (in SNR) of codebooks with block length $N \geq 2$ satisfies

$$\|\Delta\mathbf{x}_i\|^2 \dot{\geq} \text{SNR}^{-\min\{\xi_{i1}^i(\mathbf{r}),\xi_{j2}^i(\mathbf{r})\}+\epsilon} \quad (178)$$
$$\lambda_{\min}\big(\Delta\mathbf{X}_{ij}(\Delta\mathbf{X}_{ij})^H\big) \dot{\geq} \text{SNR}^{-\xi_{i3}^1(\mathbf{r})-\xi_{i3}^2(\mathbf{r})+\epsilon}, \quad (179)$$

for all pairs of codewords $\mathbf{x}_i^{n_i}, \mathbf{x}_i^{\tilde{n}_i} \in \mathcal{C}_i(\text{SNR}, r_i)$ s.t. $\mathbf{x}_i^{n_i} \neq \mathbf{x}_i^{\tilde{n}_i}$, $\mathbf{x}_j^{n_j}, \mathbf{x}_j^{\tilde{n}_j} \in \mathcal{C}_j(\text{SNR}, r_j)$ s.t. $\mathbf{x}_j^{n_j} \neq \mathbf{x}_j^{\tilde{n}_j}$, for $i, j = 1, 2$, and $i \neq j$, where $\Delta\mathbf{x}_i = \mathbf{x}_i^{n_i} - \mathbf{x}_i^{\tilde{n}_i}$, $\Delta\mathbf{x}_j = \mathbf{x}_j^{n_j} - \mathbf{x}_j^{\tilde{n}_j}$, and $\Delta\mathbf{X}_{ij} = [\Delta\mathbf{x}_i \ \Delta\mathbf{x}_j]$, and $\lambda_{\min}(\Delta\mathbf{X}_{ij}(\Delta\mathbf{X}_{ij})^H)$ denotes the smallest nonzero eigenvalue of $\Delta\mathbf{X}_{ij}(\Delta\mathbf{X}_{ij})^H$, for some[11] $\epsilon > 0$, then $P(E^{MAC})$ obeys

$$P(E^{MAC}) \doteq \text{SNR}^{-d^{MAC}(\mathbf{r})}. \quad (180)$$

Proof: We first identify an outer bound on the DMT region and then show, using an appropriate inner bound, that this DMT region is, indeed, achievable. We define the outage events corresponding to decoding of $\mathcal{T}_i$, decoding of $\mathcal{T}_j$, and joint decoding of $\mathcal{T}_i$ *and* $\mathcal{T}_j$ at $\mathcal{R}_i$, for $i, j = 1, 2$, and $i \neq j$, by

$$\mathcal{O}_{i1}^{MAC} \triangleq \{\mathbf{h}_i : I(\mathbf{x}_i; \mathbf{y}_i|\mathbf{x}_j, \mathbf{h}_i) < R_i\} \quad (181)$$
$$\mathcal{O}_{i2}^{MAC} \triangleq \{\mathbf{h}_i : I(\mathbf{x}_j; \mathbf{y}_i|\mathbf{x}_i, \mathbf{h}_i) < R_j\} \quad (182)$$
$$\mathcal{O}_{i3}^{MAC} \triangleq \{\mathbf{h}_i : I(\mathbf{x}_i, \mathbf{x}_j; \mathbf{y}_i|\mathbf{h}_i) < R_1 + R_2\}. \quad (183)$$

We define an outage event for the MAC at $\mathcal{R}_i$ as

$$\mathcal{O}_i^{MAC} \triangleq \bigcup_{k=1}^{3} \mathcal{O}_{ik}^{MAC}. \quad (184)$$

We define the total outage probability for treating the IC as a combination of MACs as

$$\mathbb{P}\big[\mathcal{O}^{MAC}\big] \triangleq \max\big\{\mathbb{P}\big[\mathcal{O}_1^{MAC}\big], \mathbb{P}\big[\mathcal{O}_2^{MAC}\big]\big\}. \quad (185)$$

[10] We note that the functions $\Xi_{ik}(\mathbf{r})$ might not be unique.

[11] We note that ϵ is allowed to be different in (178) and (179).

Using a standard argument along the lines of [10], [12], we can see that taking both transmitters to employ i.i.d. Gaussian codebooks does not result in a loss of DMT-optimality. We can therefore evaluate (181)-(183) as

$$\mathcal{O}_{i1}^{MAC}(\mathbf{r}) \triangleq \left\{\mathbf{h}_i : \log\left(1+\text{SNR}|h_{ii}|^2\right) < R_i\right\}$$
$$\mathcal{O}_{i2}^{MAC}(\mathbf{r}) \triangleq \left\{\mathbf{h}_i : \log\left(1+\text{SNR}^{\alpha}|h_{ji}|^2\right) < R_j\right\}$$
$$\mathcal{O}_{i3}^{MAC}(\mathbf{r}) \triangleq$$
$$\left\{\mathbf{h}_i : \log\left(1+\text{SNR}^{\alpha}|h_{ji}|^2+\text{SNR}|h_{ii}|^2\right) < R_1+R_2\right\}$$

with $i,j=1,2$ and $i \neq j$. We can now establish the asymptotic behavior of $\mathcal{O}_i^{MAC}$. By the union bound, we have

$$\mathbb{P}\left[\mathcal{O}_i^{MAC}\right] \leq \sum_{k=1}^{3}\mathbb{P}\left[\mathcal{O}_{ik}^{MAC}(\mathbf{r})\right] \quad (186)$$
$$\doteq \max_{k=1,2,3}\mathbb{P}\left[\mathcal{O}_{ik}^{MAC}(\mathbf{r})\right]. \quad (187)$$

It is shown in [13] and [9] that

$$\mathbb{P}\left[\mathcal{O}_{i1}^{MAC}(\mathbf{r})\right] \doteq \text{SNR}^{-d_{i1}^{MAC}(\mathbf{r})} \quad (188)$$
$$\mathbb{P}\left[\mathcal{O}_{i2}^{MAC}(\mathbf{r})\right] \doteq \text{SNR}^{-d_{i2}^{MAC}(\mathbf{r})} \quad (189)$$
$$\mathbb{P}\left[\mathcal{O}_{i3}^{MAC}(\mathbf{r})\right] \doteq \text{SNR}^{-d_{i3}^{MAC}(\mathbf{r})} \quad (190)$$

with

$$d_{i1}^{MAC}(\mathbf{r}) = (1-r_i)^+ \quad (191)$$
$$d_{i2}^{MAC}(\mathbf{r}) = (\alpha-r_j)^+ \quad (192)$$
$$d_{i3}^{MAC}(\mathbf{r}) = (1-r_1-r_2)^+ + (\alpha-r_1-r_2)^+, \quad (193)$$

for $i,j=1,2$, and $i \neq j$. We point out that (191) and (192) define six SNR exponents $d_{ik}^{MAC}(\mathbf{r})$, $i=1,2$, $k=1,2,3$. The DMT-characterization of the outage event corresponding to joint decoding of the signals from both transmitters at $\mathcal{R}_1$ is identical to the DMT-characterization of the outage event corresponding to joint decoding of the signals from both transmitters at $\mathcal{R}_2$. Hence, the corresponding SNR exponents of the outage probabilities of these events, namely, $d_{13}^{MAC}(\mathbf{r})$ and $d_{23}^{MAC}(\mathbf{r})$, are equal. The total outage probability corresponding to treating the IC as a combination of two MACs then satisfies

$$\mathbb{P}\left[\mathcal{O}^{MAC}\right] = \max\left\{\mathbb{P}\left[\mathcal{O}_1^{MAC}\right], \mathbb{P}\left[\mathcal{O}_2^{MAC}\right]\right\}. \quad (194)$$

From (187), it follows that

$$\mathbb{P}\left[\mathcal{O}_i^{MAC}\right] \doteq \max_{k=1,2,3}\mathbb{P}\left[\mathcal{O}_{ik}^{MAC}(\mathbf{r})\right]$$
$$\doteq \text{SNR}^{-\min\limits_{k=1,2,3} d_{ik}^{MAC}(\mathbf{r})}. \quad (195)$$

Hence, combining (194) and (195), we get

$$\mathbb{P}\left[\mathcal{O}^{MAC}\right] \doteq \max_{i=1,2}\text{SNR}^{-\min\limits_{k=1,2,3} d_{ik}^{MAC}(\mathbf{r})} \quad (196)$$
$$\doteq \text{SNR}^{-d^{MAC}(\mathbf{r})} \quad (197)$$

where

$$d^{MAC}(\mathbf{r}) = \min_{\substack{i=1,2\\k=1,2,3}} d_{ik}^{MAC}(\mathbf{r}). \quad (198)$$

We note that (196) can be simplified by eliminating either $d_{13}^{MAC}(\mathbf{r})$ or $d_{23}^{MAC}(\mathbf{r})$ as explained earlier.

With (195) we arrived at a lower bound on the error probability of the *joint MAC-ML decoder* at $\mathcal{R}_i$. This lower bound, by definition, gives an outer bound on the DMT region. We next try to find an upper bound on the error probability $P(E^{MAC})$ that has the same exponential behavior as this lower bound. To this end, consider next the error probability corresponding to the *joint MAC-ML decoder*. We first define the relevant error events. Let $\mathbf{x}_i^{n_i}$ and $\mathbf{x}_j^{n_j}$ with $n_i \in \{1,2,\ldots,2^{NR_i}\}$, $n_j \in \{1,2,\ldots,2^{NR_j}\}$ ($i,j=1,2$ and $i \neq j$) be the codewords transmitted by $\mathcal{T}_i$ and $\mathcal{T}_j$, respectively. The results of joint ML decoding of $\mathcal{T}_i$ and $\mathcal{T}_j$ at $\mathcal{R}_i$ are denoted by $\mathbf{x}_i^{\tilde{n}_i}$ and $\mathbf{x}_j^{\tilde{n}_j}$, respectively, with $\tilde{n}_i \in \{1,2,\ldots,2^{NR_i}\}$, $\tilde{n}_j \in \{1,2,\ldots,2^{NR_j}\}$, for $i,j=1,2$, and $i \neq j$. We have the error events corresponding to $\mathcal{T}_i$ only, $\mathcal{T}_j$ only, and $\mathcal{T}_i$ and $\mathcal{T}_j$ being decoded in error at $\mathcal{R}_i$, respectively, as

$$\mathcal{E}_{i1}^{MAC} \triangleq \{\tilde{n}_i \neq n_i,\ \tilde{n}_j = n_j\} \quad (199)$$
$$\mathcal{E}_{i2}^{MAC} \triangleq \{\tilde{n}_i = n_i,\ \tilde{n}_j \neq n_j\} \quad (200)$$
$$\mathcal{E}_{i3}^{MAC} \triangleq \{\tilde{n}_i \neq n_i,\ \tilde{n}_j \neq n_j\}, \quad (201)$$

for $i,j=1,2$, and $i \neq j$. We will also need the total error event defined as

$$\mathcal{E}_i^{MAC} \triangleq \bigcup_{k=1,2,3}\mathcal{E}_{ik}^{MAC}. \quad (202)$$

We denote

$$[i^*\ k^*] = \arg\min_{\substack{i=1,2\\k=1,2,3}} d_{ik}^{MAC}(\mathbf{r}). \quad (203)$$

Let $\Xi_{ik}(\mathbf{r}) = [\xi_{ik}^1(\mathbf{r})\ \xi_{ik}^2(\mathbf{r})]^T$ be functions[12] such that

$$d_{i^*k^*}^{MAC}(\mathbf{r}) = d_{ik}^{MAC}(\Xi_{ik}(\mathbf{r})), \quad (204)$$

for $i=1,2$, $k=1,2,3$.

We next find an upper bound on the probability of the events $\mathcal{E}_{ik}^{MAC}$ as follows:

$$\begin{aligned}&\mathbb{P}\left[\mathcal{E}_{ik}^{MAC}\right]\\ &= \mathbb{P}\left[\mathcal{E}_{ik}^{MAC}, \mathcal{O}_{ik}^{MAC}(\Xi_{ik}(\mathbf{r}))\right] + \mathbb{P}\left[\mathcal{E}_{ik}^{MAC}, \bar{\mathcal{O}}_{ik}^{MAC}(\Xi_{ik}(\mathbf{r}))\right]\\ &\leq \mathbb{P}\left[\mathcal{O}_{ik}^{MAC}(\Xi_i(\mathbf{r}))\right] + \mathbb{P}\left[\mathcal{E}_{ik}^{MAC}|\bar{\mathcal{O}}_{ik}^{MAC}(\Xi_i(\mathbf{r}))\right]. \quad (205)\end{aligned}$$

In order to obtain an upper bound on $\mathbb{P}\left[\mathcal{E}_{ik}^{MAC}\right]$, we derive an upper bound on the average pairwise error probability (PEP) of each error event $\mathcal{E}_{ik}^{MAC}$ for $i=1,2$ and $k=1,2,3$. Let us first consider an $\mathcal{E}_{i3}^{MAC}$-type error event corresponding to the case where the messages from both transmitters are decoded in error. The probability of the ML decoder mistakenly deciding in favor of the codeword $\mathbf{X}_{ij}^{\tilde{n}_i\tilde{n}_j} = [\mathbf{x}_i^{\tilde{n}_i}\ \mathbf{x}_j^{\tilde{n}_j}]$ when $\mathbf{X}_{ij}^{n_in_j} = [\mathbf{x}_i^{n_i}\ \mathbf{x}_j^{n_j}]$ (with $\mathbf{x}_i^{n_i}, \mathbf{x}_i^{\tilde{n}_i} \in \mathcal{C}_i(\text{SNR}, r_i)$ and $\mathbf{x}_j^{n_j}, \mathbf{x}_j^{\tilde{n}_j} \in \mathcal{C}_j(\text{SNR}, r_j)$, $i,j=1,2$ and $i \neq j$) was actually transmitted, can be upper-bounded according to (36):

$$\mathbb{E}_{\mathbf{h}_i}\left\{\mathbb{P}\left[\mathbf{X}_{ij}^{n_in_j} \to \mathbf{X}_{ij}^{\tilde{n}_i\tilde{n}_j}\right]\right\} \leq \quad (206)$$
$$\mathbb{E}_{\mathbf{h}_i}\left\{\exp\left[-\lambda_{\min}\frac{\text{SNR}|h_{ii}|^2+\text{SNR}^{\alpha}|h_{ji}|^2}{4}\right]\right\} \quad (207)$$

where $\tilde{\mathbf{h}}_i = [\sqrt{\text{SNR}}h_{ii}\ \sqrt{\text{SNR}^{\alpha}}h_{ji}]^T$, for $i,j=1,2$, and $i \neq j$, and $\lambda_{\min}$ is the smallest nonzero eigenvalue of $\Delta\mathbf{X}_{ij}(\Delta\mathbf{X}_{ij})^H$.

[12] We note that the functions $\Xi_{ik}(\mathbf{r})$ might not be unique.

Noting that the no-outage event $\bar{\mathcal{O}}_{i3}^{MAC}(\Xi_{i3}(\mathbf{r}))$ entails $\text{SNR}|h_{ii}|^2 + \text{SNR}^{\alpha}|h_{ji}|^2 \geq \text{SNR}^{\xi_{i3}^1(\mathbf{r})+\xi_{i3}^2(\mathbf{r})} - 1$, Eq. (205) implies an upper bound on $\mathbb{P}\big[\mathcal{E}_{i3}^{MAC}\big]$ according to:

$$\mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_{i3}^{MAC}\big]\big\} \dot{\leq} \mathbb{P}\big[\mathcal{O}_{i3}^{MAC}(\Xi_3(\mathbf{r}))\big] + \text{SNR}^{N(r_1+r_2)} \exp\left[-\frac{\lambda_{\min}\text{SNR}^{\xi_{i3}^1(\mathbf{r})+\xi_{i3}^2(\mathbf{r})}}{4}\right]. \tag{208}$$

Here, we used the definitions $R_i = r_i \log \text{SNR}$, $i = 1, 2$, and $\exp[-\frac{\lambda_{\min}}{4}(\text{SNR}^{\xi_{i3}^1(\mathbf{r})+\xi_{i3}^2(\mathbf{r})} - 1)] \doteq \exp[-\frac{\lambda_{\min}}{4}\text{SNR}^{\xi_{i3}^1(\mathbf{r})+\xi_{i3}^2(\mathbf{r})}]$. Since we have, by assumption, that $\lambda_{\min} \dot{\geq} \text{SNR}^{-\xi_{i3}^1(\mathbf{r})-\xi_{i3}^2(\mathbf{r})+\epsilon}$ with $\epsilon > 0$, we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_{i3}^{MAC}\big]\big\} \\
&\dot{\leq} \mathbb{P}\big[\mathcal{O}_{i3}^{MAC}(\Xi_{i3}(\mathbf{r}))\big] + \text{SNR}^{N(r_1+r_2)} \exp\left[-\frac{\text{SNR}^{\epsilon}}{4}\right] \quad (209)\\
&\doteq \mathbb{P}\big[\mathcal{O}_{i3}^{MAC}(\Xi_{i3}(\mathbf{r}))\big] \\
&\doteq \text{SNR}^{-d_{i^*k^*}^{MAC}(\mathbf{r})} \quad (210)
\end{aligned}$$

as the second term on the RHS of (209) decays exponentially in SNR whereas the first term decays polynomially. Eq. (210) is a consequence of the definition of the function $\Xi_{i3}(\mathbf{r})$.

A similar analysis for the $\mathcal{E}_{i1}^{MAC}$-type error event results in

$$\mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathbf{x}_i^{n_i} \to \mathbf{x}_i^{\tilde{n}_i}\big]\big\} \leq \mathbb{E}_{\mathbf{h}_i}\left\{\exp\left[-\frac{\text{SNR}|h_{ii}|^2\|\Delta\mathbf{x}_i\|^2}{4}\right]\right\} \tag{211}$$

which, upon invoking

$$\|\Delta\mathbf{x}_i\|^2 \dot{\geq} \text{SNR}^{-\min\{\xi_{i1}^i(\mathbf{r}),\xi_{j2}^i(\mathbf{r})\}+\epsilon},$$

and using the fact that $\bar{\mathcal{O}}_{i1}^{MAC}(\Xi_{i1}(\mathbf{r}))$ entails $\text{SNR}|h_{ii}|^2 \geq \text{SNR}^{\xi_{i1}^i(\mathbf{r})} - 1$, yields

$$\begin{aligned}
\mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_{i1}^{MAC}\big]\big\} &\dot{\leq} \mathbb{P}\big[\mathcal{O}_{i1}^{MAC}(\Xi_{i1}(\mathbf{r}))\big] + \\
&\text{SNR}^{Nr_i} \exp\left[-\frac{\text{SNR}^{\xi_{i1}^i(\mathbf{r})-\min\{\xi_{i1}^i(\mathbf{r}),\xi_{j2}^i(\mathbf{r})\}+\epsilon}}{4}\right] \quad (212)\\
&\doteq \mathbb{P}\big[\mathcal{O}_{i1}^{MAC}(\Xi_{i1}(\mathbf{r}))\big] \doteq \text{SNR}^{-d_{i^*k^*}^{MAC}(\mathbf{r})}, \quad (213)
\end{aligned}$$

for $i = 1, 2$.

A similar analysis for the $\mathcal{E}_{i2}^{MAC}$-type error event results in

$$\mathbb{E}_{\mathbf{h}_i}\left\{\mathbb{P}\left[\mathbf{x}_j^{n_j} \to \mathbf{x}_j^{\tilde{n}_j}\right]\right\} \leq \mathbb{E}_{\mathbf{h}_i}\left\{\exp\left[-\frac{\text{SNR}^{\alpha}|h_{ji}|^2\|\Delta\mathbf{x}_j\|^2}{4}\right]\right\} \tag{214}$$

which, upon invoking

$$\|\Delta\mathbf{x}_j\|^2 \dot{\geq} \text{SNR}^{-\min\{\xi_{j1}^j(\mathbf{r}),\xi_{i2}^j(\mathbf{r})\}+\epsilon},$$

and using the fact that $\bar{\mathcal{O}}_{i2}^{MAC}(\Xi_{i2}(\mathbf{r}))$ entails $\text{SNR}^{\alpha}|h_{ji}|^2 \geq \text{SNR}^{\xi_{i2}^j(\mathbf{r})} - 1$, yields

$$\begin{aligned}
\mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_{i2}^{MAC}\big]\big\} &\dot{\leq} \mathbb{P}\big[\mathcal{O}_{i2}^{MAC}(\Xi_{i2}(\mathbf{r}))\big] + \\
&\text{SNR}^{Nr_j} \exp\left[-\frac{\text{SNR}^{\xi_{i2}^j(\mathbf{r})-\min\{\xi_{j1}^j(\mathbf{r}),\xi_{i2}^j(\mathbf{r})\}+\epsilon}}{4}\right] \quad (215)\\
&\doteq \mathbb{P}\big[\mathcal{O}_{i2}^{MAC}(\Xi_{i2}(\mathbf{r}))\big] \doteq \text{SNR}^{-d_{i^*k^*}^{MAC}(\mathbf{r})}, \quad (216)
\end{aligned}$$

for $i, j = 1, 2$, and $i \neq j$. To complete the proof, we note that

$$\begin{aligned}
\mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_i^{MAC}\big]\big\} &\leq \sum_{k=1}^{3} \mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_{ik}^{MAC}\big]\big\} \quad (217)\\
&\dot{\leq} \sum_{k=1}^{3} \mathbb{P}\big[\mathcal{O}_{ik}^{MAC}(\Xi_{ik}(\mathbf{r}))\big] \quad (218)\\
&= 3\text{SNR}^{-d_{i^*k^*}^{MAC}(\mathbf{r})} \doteq \text{SNR}^{-d^{MAC}(\mathbf{r})}.
\end{aligned}$$

We finally get

$$\begin{aligned}
P\big(E^{MAC}\big) &= \max_{i=1,2} \mathbb{E}_{\mathbf{h}_i}\big\{\mathbb{P}\big[\mathcal{E}_i^{MAC}\big]\big\} \quad (219)\\
&\dot{\leq} \text{SNR}^{-d^{MAC}(\mathbf{r})}. \quad (220)
\end{aligned}$$

Since the SNR exponent in (220) gives an upper bound on the error probability that matches the SNR exponent of the lower bound in (197), the proof is complete. ∎

B. Time sharing between transmitters

We assume that the transmitters are orthogonalized in time or frequency such that $\mathcal{T}_i$ $(i = 1, 2)$ uses a fraction θ_i of the channel resources with $\theta_1 + \theta_2 = 1$ and $0 \leq \theta_i \leq 1$. Then, $\mathcal{T}_i$ enjoys an effective SISO channel θ_i fractions of time or frequency. Let $P\big(E_i^{TS}\big)$ be the average ML error probability for decoding $\mathcal{T}_i$ at $\mathcal{R}_i$ for the time sharing system. It follows from results in [13] that

$$P\big(E_i^{TS}\big) \doteq \begin{cases} \text{SNR}^{-(1-r_i/\theta_i)^+}, & \text{if } \theta_i > 0 \\ 1, & \text{if } \theta_i = 0 \end{cases} \tag{221}$$

for $i = 1, 2$. The achievable DMT of this strategy is then

$$P\big(E^{TS}\big) = \max\big\{P\big(E_1^{TS}\big), P\big(E_2^{TS}\big)\big\}.$$

We can optimize over the parameters θ_i to get the best possible DMT of this strategy according to

$$P\big(E^{OTS}\big) \triangleq \min_{\theta_1,\theta_2} \max\big\{P\big(E_1^{TS}\big), P\big(E_2^{TS}\big)\big\} \tag{222}$$

subject to

$$\begin{aligned} \theta_1 + \theta_2 &= 1 \\ 0 \leq \theta_i &\leq 1, \end{aligned}$$

for $i = 1, 2$.

Numerical results

Figs. 2-5 show the DMT achieved by the fixed-power-split HK scheme (HK) in comparison to the outer bound we derived in (134) (ETW), the outer bound in [9] (AL08), to treating interference as noise (TIAN), and to time-sharing (TS), all for symmetric rates $r_1 = r_2 = r$ and for $\alpha = 1/2$, $\alpha = 2/3$, $\alpha = 1$, and $\alpha = 1.5$, respectively.

From Fig. 2, we can see that the fixed-power-split HK scheme is only DMT-optimal for multiplexing rates $r < 1/4$, and falls short of achieving the outer bound (134) (ETW) and the outer bound in [9] (AL08) for multiplexing rates $r \geq 1/4$. It is interesting to note that the outer bound (134) is tighter than the outer bound in [9] for multiplexing rates $r \leq 0.45$, whereas for $r > 0.45$ the opposite is true, i.e., the outer bound [9] is tighter than the outer bound (134).

From Figs. 3-4, we can conclude that the fixed-power-split HK scheme is DMT-optimal for $\alpha = 1$ and achieves the DMT outer bound in (134). We also observe that the outer bound (134) is tighter than the outer bound in [9] for all (symmetric) multiplexing rates.

Fig. 5 shows that the two-message, fixed-power-split HK scheme achieves the DMT outer bound (134), and therefore, is DMT-optimal for $\alpha = 1.5$. We note that for $\alpha = 1.5$, the outer bound (134) and the outer bound in [9] are identical.

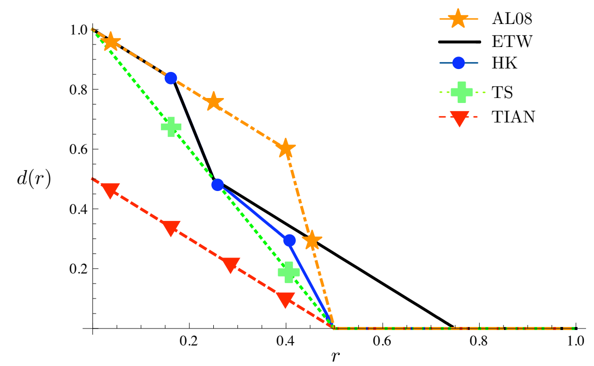

Fig. 2. Symmetric rate DMT for $\alpha = 1/2$ and for various schemes.

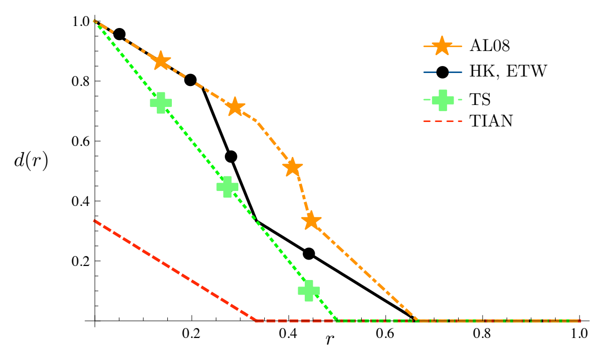

Fig. 3. Symmetric rate DMT for $\alpha = 2/3$ and for various schemes.

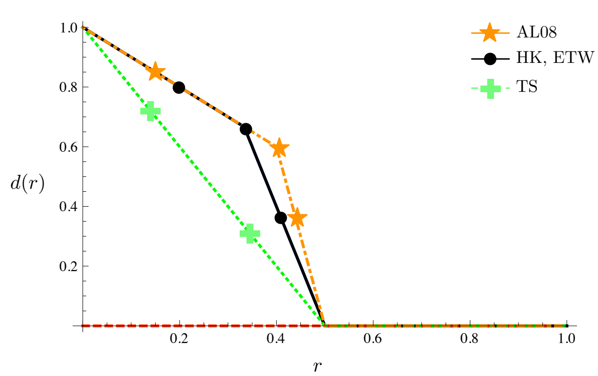

Fig. 4. Symmetric rate DMT for $\alpha = 1$ and for various schemes.

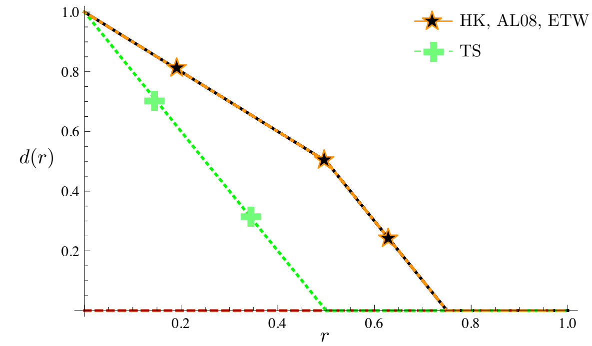

Fig. 5. Symmetric rate DMT for $\alpha = 1.5$ and for various schemes.

VIII. Conclusions

We characterized the optimal DMT of the two-user fading IC for the cases of *strong* and *very strong* interference. Further, we proved that a fixed-power-split HK superposition coding scheme achieves the optimal DMT of the two-user fading IC under *strong*, and *very strong* interference. We provided code design criteria for the corresponding superposition codes. A complete characterization of the optimal DMT of the two-user fading IC under *weak* interference remains an open problem.

IX. Appendix

We upper-bound the probabilities of the events $\mathcal{E}_{ij}^{HK}$ for $j = 2, 3, ..., 6$ in the proof of Theorem 2.

For the event $\mathcal{E}_{i2}^{HK}$, the receiver can cancel out the contributions of the correctly decoded messages $\mathbf{u}_i$ and $\mathbf{w}_j$. Following steps similar to those leading to (101), we obtain

$$\mathbb{E}_{\mathbf{h}_i}\{\mathbb{P}[\mathbf{w}_i \to \tilde{\mathbf{w}}_i]\} \leq \mathbb{E}_{\mathbf{h}_i}\left\{\exp\left[-\frac{\|h_{ii}\Delta\mathbf{w}_i\|^2 \mathsf{SNR}}{4(1+\mathsf{SNR}^{-(1-p_j)+\alpha}|h_{ji}|^2)}\right]\right\}.$$

Next, an application of the union bound to $\mathbb{P}[\mathcal{E}_{i2}^{HK}|\bar{\mathcal{O}}_{i2}(\Upsilon_{i2}(\mathbf{r}), \Psi_{i2}(\mathbf{s}^*), \mathbf{p}^*)]$ yields

$$\mathbb{E}_{\mathbf{h}_i}\{\mathbb{P}[\mathcal{E}_{i2}^{HK}|\bar{\mathcal{O}}_{i2}(\Upsilon_{i2}(\mathbf{r}), \Psi_{i2}(\mathbf{s}^*), \mathbf{p}^*)]\} \leq \mathsf{SNR}^{Nt_i}\exp\left[-\frac{\mathsf{SNR}^{\upsilon_{i2}^i(\mathbf{r})-\psi_{i2}^i(\mathbf{s}^*)}\|\Delta\mathbf{w}_i\|^2}{4}\right] \quad (223)$$

as the event $\bar{\mathcal{O}}_{i2}(\Upsilon_{i2}(\mathbf{r}), \Psi_{i2}(\mathbf{s}^*), \mathbf{p}^*)$ entails

$$\frac{\mathsf{SNR}|h_{ii}|^2}{1+\mathsf{SNR}^{\alpha+p_j-1}|h_{ji}|^2} \geq \mathsf{SNR}^{\upsilon_{i2}^i(\mathbf{r})-\psi_{i2}^i(\mathbf{s}^*)}. \quad (224)$$

Since $\|\Delta\mathbf{w}_i\|^2 \dot{\geq} \mathsf{SNR}^{-\upsilon_{i2}^i(\mathbf{r})+\psi_{i2}^i(\mathbf{s}^*)+\epsilon}$, with $\epsilon > 0$, by assumption, we further have

$$\mathbb{E}_{\mathbf{h}_i}\{\mathbb{P}[\mathcal{E}_{i2}^{HK}]\} \dot{\leq} \mathbb{P}[\mathcal{O}_{i2}(\Upsilon_{i2}(\mathbf{r}), \Psi_{i2}(\mathbf{s}^*), \mathbf{p}^*)] + \mathsf{SNR}^{Nt_i}\exp[-\mathsf{SNR}^{\epsilon}] \quad (225)$$

$$\dot{\leq} \mathbb{P}[\mathcal{O}_{i2}(\Upsilon_{i2}(\mathbf{r}), \Psi_{i2}(\mathbf{s}^*), \mathbf{p}^*)]. \quad (226)$$

For the event $\mathcal{E}_{i3}^{HK}$, the receiver can cancel out the contribution of the correctly decoded message $\mathbf{w}_j$. We define $\mathbf{x}_i^{\hat{i}_i^x} = \mathbf{u}_i^{\hat{i}_i^u} + \mathbf{w}_i^{\hat{i}_i^w}$, and recall that $\mathbf{x}_i^{i_i^x} = \mathbf{u}_i^{i_i^u} + \mathbf{w}_i^{i_i^w}$. The PEP of deciding in favor of $\mathbf{x}_i^{\hat{\imath}_i^x}$ when $\mathbf{x}_i^{\imath_i^x}$ was actually transmitted can be upper-bounded as

$$\mathbb{E}_{\mathbf{h}_i}\{\mathbb{P}[\mathbf{x}_i \to \tilde{\mathbf{x}}_i]\} \le \mathbb{E}_{\mathbf{h}_i}\left\{\exp\left[-\frac{\|h_{ii}\Delta\mathbf{x}_i\|^2\text{SNR}}{4(1+\text{SNR}^{-(1-p_j)+\alpha}|h_{ji}|^2)}\right]\right\}$$

where $\Delta\mathbf{x}_i = \mathbf{x}_i^{\imath_i^x} - \mathbf{x}_i^{\hat{\imath}_i^x}$ (as defined before). Next, applying the union bound, we get

$$\mathbb{E}_{\mathbf{h}_i}\left\{\mathbb{P}\left[\mathcal{E}_{i3}^{HK}|\bar{\mathcal{O}}_{i3}(\Upsilon_{i3}(\mathbf{r}),\Psi_{i3}(\mathbf{s}^*),\mathbf{p}^*)\right]\right\} \le \text{SNR}^{Nr_i}\exp\left[-\frac{\text{SNR}^{\upsilon_{i3}^i(\mathbf{r})}\|\Delta\mathbf{x}_i\|^2}{4}\right]$$

since the event $\bar{\mathcal{O}}_{i3}(\Upsilon_{i3}(\mathbf{r}),\Psi_{i3}(\mathbf{s}^*),\mathbf{p}^*)$ entails

$$\frac{\text{SNR}|h_{ii}|^2}{1+\text{SNR}^{\alpha+p_j-1}|h_{ji}|^2} \ge \text{SNR}^{\upsilon_{i3}^i(\mathbf{r})}. \tag{227}$$

As $\|\Delta\mathbf{x}_i\|^2 \dot{\ge} \text{SNR}^{-\upsilon_{i3}^i(\mathbf{r})+\epsilon}$, for $\epsilon > 0$, by assumption, we further have

$$\begin{aligned}&\mathbb{E}_{\mathbf{h}_i}\left\{\mathbb{P}\left[\mathcal{E}_{i3}^{HK}\right]\right\}\\ &\dot{\le}\, \mathbb{P}[\mathcal{O}_{i3}(\Upsilon_{i3}(\mathbf{r}),\Psi_{i3}(\mathbf{s}^*),\mathbf{p}^*)] + \text{SNR}^{Nr_i}\exp\left[-\text{SNR}^{\epsilon}\right] \quad (228)\\ &\dot{\le}\, \mathbb{P}[\mathcal{O}_{i3}(\Upsilon_{i3}(\mathbf{r}),\Psi_{i3}(\mathbf{s}^*),\mathbf{p}^*)]\,. \quad (229)\end{aligned}$$

For the event $\mathcal{E}_{i4}^{HK}$, the receiver can cancel out the contribution of the correctly decoded message $\mathbf{w}_i$. Denoting $\mathbf{A}_{ij} = [\sqrt{\text{SNR}^{1-p_i}}\mathbf{u}_i^{\imath_i^u} \ \ \mathbf{w}_j^{\imath_j^w}]$, $\tilde{\mathbf{A}}_{ij} = [\sqrt{\text{SNR}^{1-p_i}}\mathbf{u}_i^{\hat{\imath}_i^u} \ \ \mathbf{w}_j^{\hat{\imath}_j^w}]$, $\tilde{\mathbf{h}} = [\sqrt{\text{SNR}^{p_i}}h_{ii} \ \ \sqrt{\text{SNR}^{\alpha}}h_{ji}]^T$, and recalling that $\Delta\mathbf{A}_{ij} = \mathbf{A}_{ij} - \tilde{\mathbf{A}}_{ij}$, the PEP that corresponds to deciding in favor of $\tilde{\mathbf{A}}_{ij}$ when $\mathbf{A}_{ij}$ was actually transmitted can be upper-bounded according to

$$\begin{aligned}&\mathbb{E}_{\mathbf{h}_i}\left\{\mathbb{P}\left[\mathbf{A}_{ij} \to \tilde{\mathbf{A}}_{ij}\right]\right\}\\ &\le \mathbb{E}_{\mathbf{h}_i}\left\{\exp\left[-\frac{\|\Delta\mathbf{A}_{ij}\tilde{\mathbf{h}}\|^2}{4(1+\text{SNR}^{\alpha-(1-p_j)}|h_{ji}|^2)}\right]\right\}\\ &\le \mathbb{E}_{\mathbf{h}_i}\left\{\exp\left[-\lambda_{\min}\frac{\text{SNR}^{p_i}|h_{ii}|^2+\text{SNR}^{\alpha}|h_{ji}|^2}{4(1+\text{SNR}^{\alpha-(1-p_j)}|h_{ji}|^2)}\right]\right\}\\ &\le \exp\left[-\lambda_{\min}\text{SNR}^{\psi_{i4}^i(\mathbf{s}^*)+\upsilon_{j4}^j(\mathbf{r})-\psi_{j4}^j(\mathbf{s}^*)}\right]\end{aligned}$$

where $\lambda_{\min}$ is the smallest nonzero eigenvalue of $\Delta\mathbf{A}_{ij}(\Delta\mathbf{A}_{ij})^H$. As

$$\lambda_{\min} \dot{\ge} \text{SNR}^{-\psi_{i4}^i(\mathbf{s}^*)-\upsilon_{j4}^j(\mathbf{r})+\psi_{j4}^j(\mathbf{s}^*)+\epsilon} \tag{230}$$

with some $\epsilon > 0$, by assumption, we have

$$\begin{aligned}&\mathbb{E}_{\mathbf{h}_i}\left\{\mathbb{P}\left[\mathcal{E}_{i4}^{HK}\right]\right\}\\ &\dot{\le}\, \mathbb{P}[\mathcal{O}_{i4}(\Upsilon_{i4}(\mathbf{r}),\Psi_{i4}(\mathbf{s}^*),\mathbf{p}^*)] + \text{SNR}^{N(s_i+t_j)}\exp\left[-\text{SNR}^{\epsilon}\right]\\ &\dot{\le}\, \mathbb{P}[\mathcal{O}_{i4}(\Upsilon_{i4}(\mathbf{r}),\Psi_{i4}(\mathbf{s}^*),\mathbf{p}^*)]\,.\end{aligned}$$

For the event $\mathcal{E}_{i5}^{HK}$, the receiver cancels out the contributions of the correctly decoded $\mathbf{u}_i$. Denoting $\mathbf{B}_{ij} = [\mathbf{w}_i^{\imath_i^w} \ \ \mathbf{w}_j^{\imath_j^w}]$, $\tilde{\mathbf{B}}_{ij} = [\mathbf{w}_i^{\hat{\imath}_i^w} \ \ \mathbf{w}_j^{\hat{\imath}_j^w}]$, $\tilde{\mathbf{h}} = [\sqrt{\text{SNR}}h_{ii} \ \ \sqrt{\text{SNR}^{\alpha}}h_{ji}]^T$, and recalling that $\Delta\mathbf{B}_{ij} = \mathbf{B}_{ij} - \tilde{\mathbf{B}}_{ij}$, we have

$$\begin{aligned}&\mathbb{E}_{\mathbf{h}_i}\left\{\mathbb{P}\left[\mathbf{B}_{ij} \to \tilde{\mathbf{B}}_{ij}\right]\right\}\\ &\le \mathbb{E}_{\mathbf{h}_i}\left\{\exp\left[-\frac{\|\Delta\mathbf{B}_{ij}\tilde{\mathbf{h}}\|^2}{4(1+\text{SNR}^{-(1-p_j)+\alpha}|h_{ji}|^2)}\right]\right\}\\ &\le \mathbb{E}_{\mathbf{h}_i}\left\{\exp\left[-\lambda_{\min}\frac{\text{SNR}|h_{ii}|^2+\text{SNR}^{\alpha}|h_{ji}|^2}{4(1+\text{SNR}^{-(1-p_j)+\alpha}|h_{ji}|^2)}\right]\right\}\\ &\le \exp\left[-\lambda_{\min}\text{SNR}^{\sum_{k=1}^{2}\upsilon_{k5}^k(\mathbf{r})-\sum_{j=1}^{2}\psi_{j5}^j(\mathbf{s}^*)}\right]\end{aligned}$$

where $\lambda_{\min}$ is the smallest nonzero eigenvalue of $\Delta\mathbf{B}_{ij}(\Delta\mathbf{B}_{ij})^H$. As

$$\lambda_{\min} \dot{\ge} \text{SNR}^{-\sum_{k=1}^{2}\upsilon_{k5}^k(\mathbf{r})+\sum_{j=1}^{2}\psi_{j5}^j(\mathbf{s}^*)+\epsilon} \tag{231}$$

with some $\epsilon > 0$, by assumption, we have

$$\begin{aligned}&\mathbb{E}_{\mathbf{h}_i}\left\{\mathbb{P}\left[\mathcal{E}_{i5}^{HK}\right]\right\}\\ &\dot{\le}\, \mathbb{P}[\mathcal{O}_{i5}(\Upsilon_{i5}(\mathbf{r}),\Psi_{i5}(\mathbf{s}^*),\mathbf{p}^*)] + \text{SNR}^{N(t_1+t_2)}\exp\left[-\text{SNR}^{\epsilon}\right]\\ &\dot{\le}\, \mathbb{P}[\mathcal{O}_{i5}(\Upsilon_{i5}(\mathbf{r}),\Psi_{i5}(\mathbf{s}^*),\mathbf{p}^*)]\,.\end{aligned}$$

Finally, for the event $\mathcal{E}_{i6}^{HK}$, all codewords are in error, so that there is nothing to cancel out. Denoting $\mathbf{C}_{ij} = [\mathbf{x}_i^{\imath_i^x} \ \ \mathbf{w}_j^{\imath_j^w}]$, $\tilde{\mathbf{C}}_{ij} = [\mathbf{x}_i^{\hat{\imath}_i^x} \ \ \mathbf{w}_j^{\hat{\imath}_j^w}]$, $\tilde{\mathbf{h}} = [\sqrt{\text{SNR}}h_{ii} \ \sqrt{\text{SNR}^{\alpha}}h_{ji}]^T$, and recalling that $\Delta\mathbf{C}_{ij} = \mathbf{C}_{ij} - \tilde{\mathbf{C}}_{ij}$, we obtain

$$\begin{aligned}&\mathbb{E}_{\mathbf{h}_i}\left\{\mathbb{P}\left[\mathbf{C}_{ij} \to \tilde{\mathbf{C}}_{ij}\right]\right\}\\ &\le \mathbb{E}_{\mathbf{h}_i}\left\{\exp\left[-\frac{\|\Delta\mathbf{C}_{ij}\tilde{\mathbf{h}}\|^2}{4(1+\text{SNR}^{-(1-p_j)+\alpha}|h_{ji}|^2)}\right]\right\}\\ &\le \mathbb{E}_{\mathbf{h}_i}\left\{\exp\left[-\lambda_{\min}\frac{\text{SNR}|h_{ii}|^2+\text{SNR}^{\alpha}|h_{ji}|^2}{4(1+\text{SNR}^{-(1-p_j)+\alpha}|h_{ji}|^2)}\right]\right\}\\ &\le \exp\left[-\lambda_{\min}\text{SNR}^{\upsilon_{i6}^i(\mathbf{r})+\upsilon_{j6}^j(\mathbf{r})-\psi_{j6}^j(\mathbf{s}^*)}\right]\end{aligned}$$

where $\lambda_{\min}$ is the smallest nonzero eigenvalue of $\Delta\mathbf{C}_{ij}(\Delta\mathbf{C}_{ij})^H$. As

$$\lambda_{\min} \dot{\ge} \text{SNR}^{-\upsilon_{i6}^i(\mathbf{r})-\upsilon_{j6}^j(\mathbf{r})+\psi_{j6}^j(\mathbf{s}^*)+\epsilon} \tag{232}$$

with some $\epsilon > 0$, by assumption, we have

$$\begin{aligned}&\mathbb{E}_{\mathbf{h}_i}\left\{\mathbb{P}\left[\mathcal{E}_{i6}^{HK}\right]\right\}\\ &\dot{\le}\, \mathbb{P}[\mathcal{O}_{i6}(\Upsilon_{i6}(\mathbf{r}),\Psi_{i6}(\mathbf{s}^*),\mathbf{p}^*)] + \text{SNR}^{N(r_i+t_j)}\exp\left[-\text{SNR}^{\epsilon}\right]\\ &\dot{\le}\, \mathbb{P}[\mathcal{O}_{i6}(\Upsilon_{i6}(\mathbf{r}),\Psi_{i6}(\mathbf{s}^*),\mathbf{p}^*)]\,.\end{aligned}$$

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