Equivalence of Two Methods for Constructing Tight Gabor Frames

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Abstract—**Recently, in the context of Orthogonal Frequency Division Multiplexing (OFDM), a new method (FAB-method) [11] for constructing tight Gabor frames (with redundancy 2) from a (nontight) Gaussian g was proposed. In this letter, we prove that the FAB-method yields the tight window function canonically associated to the Gaussian. We furthermore provide a necessary and sufficient condition on the initial window function g in the Zak transform domain for the FAB-method to yield a tight Gabor frame. This yields a characterization of all initial window functions g for which the FAB-method works.**

Index Terms—**Signal representation, transforms.**

I. Introduction

RECENTLY, in the context of orthogonal frequency division multiplexing (OFDM), LeFloch *et al.* [11] proposed a new method for constructing tight Gabor frames for oversampling factor 2. In the following, this method will be referred to as the FAB-method. The essence of the FAB method is to start from a (nontight) Gaussian g and to "tighten" it using a two-step orthogonalization procedure.

In this letter, we show that the FAB-method is equivalent to applying the positive definite inverse square root of the Gabor frame operator to the initial window function. As a consequence, the FAB-method yields the tight Gabor frame canonically associated to g. We provide a necessary and sufficient condition on the initial window function g in the Zak transform domain for the FAB-method to yield a tight Gabor frame. This condition characterizes all functions for which the FAB-method works (i.e. yields a tight Gabor frame).

The letter is organized as follows. In Section II, we provide a brief review of Gabor theory and Zak transforms, and we describe the FAB-method. In Section III, we consider the FAB-method in the Zak transform domain, and in Section IV, we show that the FAB-method yields the canonical tight frame for Gaussian g.

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II. Gabor Frames, Zak Transforms, and the FAB-Method

A. Gabor Frames

Let $a > 0$, $b > 0$ and denote for $x, y \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$

$$f_{x,y}(t) = e^{2\pi i y t} f(t-x), \qquad t \in \mathbb{R}. \tag{1}$$

We say that g generates a Gabor frame for $L^2(\mathbb{R})$ for the shift parameters a, b when there exist constants $A > 0$ and $B < \infty$ such that

$$A\|f\|^2 \le \sum_{n,m} |(f, g_{na,mb})|^2 \le B\|f\|^2, \qquad f \in L^2(\mathbb{R}). \tag{2}$$

The constants A and B are called a lower and upper frame bound for g, respectively. When (2) holds for $A = B$, we say that g generates a tight frame. It is well known that for g to generate a Gabor frame for $L^2(\mathbb{R})$, it is necessary that $ab \le 1$ [3, Section IV-A], [10], [7]. The cases $ab = 1$ and $ab < 1$ are referred to as critical sampling and oversampling, respectively. The frame condition (2) can be equivalently written as

$$A\boldsymbol{I} \le \boldsymbol{S} \le B\boldsymbol{I} \tag{3}$$

where $\boldsymbol{I}$ is the identity operator of $L^2(\mathbb{R})$, and $\boldsymbol{S}$ is the frame operator defined by

$$\boldsymbol{S}f = \sum_{n,m} (f, g_{na,mb}) g_{na,mb} \qquad f \in L^2(\mathbb{R}). \tag{4}$$

When g generates a frame and $f \in L^2(\mathbb{R})$, f can be represented as

$$f = \sum_{n,m} (f, \gamma^0_{na,mb}) g_{na,mb} \tag{5}$$

where $\gamma^0 = \boldsymbol{S}^{-1} g$ is the minimal (or Wexler–Raz) dual of g. The importance of tight Gabor frames is derived from the fact that $\gamma^0 = (1/A)\, g$ [3].

To every $g \in L^2(\mathbb{R})$ generating a Gabor frame for $L^2(\mathbb{R})$, we can associate a tight frame generated by [3]

$$\begin{aligned} h &= \boldsymbol{S}^{-1/2} g \\ &= \left(\frac{2}{B+A}\right)^{1/2} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \\ &\quad \cdot \left(\boldsymbol{I} - \frac{2}{B+A}\boldsymbol{S}\right)^n g \end{aligned} \tag{6}$$

where $\boldsymbol{S}^{-1/2}$ denotes the positive definite operator square root of the inverse Gabor frame operator. We say that h is canonically associated to g. For further generalities about Gabor frames, the interested reader is referred to [4], [3], [5], and [1].

B. The Zak Transform

In the cases of rational oversampling (i.e., $ab = p/q, p \in \mathbb{Z}, q \in \mathbb{Z}, p < q, (p, q) = 1$) and critical sampling, the Gabor frame operator can conveniently be expressed using the Zak transform (ZT) [6], [2], [14], [15]. For $\lambda > 0$, the ZT of a signal $f \in L^2\mathbb{R}$ is defined as

$$(Z_\lambda f)(t, \nu) = \lambda^{1/2} \sum_{k=-\infty}^{\infty} f(\lambda(t+k))e^{-2\pi ik\nu}. \tag{7}$$

In this letter, we will be only concerned with the case $a = 1/2$, $b = 1$ and $\lambda = 1/b = 1$. The Zibulski–Zeevi representation [14], [15], [9] of $\boldsymbol{S}$ reads

$$(Z\boldsymbol{S}f)(t, \nu) = (|(Zg)(t, \nu)|^2 + |(Zg)(t+1/2, \nu)|^2)(Zf)(t, \nu) \tag{8}$$

where (Zf) is the ZT of f with $\lambda = 1$. A g generates a tight frame if and only if

$$|(Zg)(t, \nu)|^2 + |(Zg)(t+1/2, \nu)|^2 = c \tag{9}$$

for some constant c. Also, the canonical tight frame generating h associated to g is given in the ZT domain by

$$(Zh)(t, \nu) = \frac{(Zg)(t, \nu)}{(|(Zg)(t, \nu)|^2 + |(Zg)(t+1/2, \nu)|^2)^{1/2}}. \tag{10}$$

We shall need the following properties of the ZT. When $f \in L^2(\mathbb{R})$, H is 1-periodic and Ψ is any function defined on $[0, 1]$, we have under mild smoothness and decay properties of f, H and Ψ that

$$f(t) = \int_0^1 (Zf)(t, \nu) d\nu, \ t \in \mathbb{R} \tag{11}$$

$$(Z\boldsymbol{F}f)(\nu, -t) = e^{-2\pi it\nu}(Zf)(t, \nu), \ t, \nu \in \mathbb{R} \tag{12}$$

$$(Zf)(t-n, \nu) = e^{-2\pi in\nu}(Zf)(t, \nu), \ t, \nu \in \mathbb{R}, n \in \mathbb{Z} \tag{13}$$

$$(Zf)(t, \nu - m) = (Zf)(t, \nu), \quad t, \nu \in \mathbb{R}, m \in \mathbb{Z} \tag{14}$$

$$|(Zf)(t, \nu)|^2 = \sum_{n, m} (f, f_{n,m}) e^{2\pi imt - 2\pi in\nu}, \ t, \nu \in \mathbb{R} \tag{15}$$

$$Z(Hf)(t, \nu) = H(t)(Zf)(t, \nu), \quad t, \nu \in \mathbb{R} \tag{16}$$

$$Z\left[\int_0^1 \Psi(\mu)(Zf)(\cdot, \mu)\, d\mu\right](t, \nu) = \Psi(\nu)(Zf)(t, \nu), \quad t, \nu \in \mathbb{R} \tag{17}$$

$$\int_0^1 |(Zf)(t, \nu)|^2\, dt = \sum_k |(\boldsymbol{F}f)(\nu - k)|^2, \quad \nu \in \mathbb{R}. \tag{18}$$

In (12), we have written $\boldsymbol{F}f$ for the Fourier transform of f. The properties (11)–(16) are well known in ZT theory. Property (17) follows from (11) and the fact that Z is an isometry of $L^2(\mathbb{R})$ onto $L^2([0, 1]^2)$. Indeed, either side of (17) is a ZT of an L^2-function. These functions are the same, as can be seen from integrating either side of (17) over $\nu \in [0, 1]$ and using (11). Hence, their ZT's are the same, and this is (17). Finally, property (18) follows from (12) and Parseval's theorem for Fourier series.

C. The FAB-Method

In the FAB-method for computing a tight Gabor frame for oversampling factor 2, one starts with $g_\alpha(t) = (2\alpha)^{1/4}e^{-\pi\alpha t^2}$ and computes

$$h^{(\mathrm{FAB})}(t) = \boldsymbol{O}_{1/2}\boldsymbol{F}^{-1}\boldsymbol{O}_1\boldsymbol{F}g_\alpha \tag{19}$$

where

$$(\boldsymbol{O}_\lambda f)(t) = \frac{f(t)}{\left(\lambda \sum_{k=-\infty}^{\infty} |f(t-k\lambda)|^2\right)^{1/2}} \tag{20}$$

and $\boldsymbol{F}$ denotes the Fourier transform operator, as above. This approach for computing tight Gabor frames with redundancy 2 recently occurred in certain work on OFDM [11]–[13]. That the FAB-method works is a nontrivial fact for which we could not find a proof in the literature.

III. FAB-Method in the Zak Transform Domain

In this section, we consider the FAB-method in the ZT domain. More specifically, we shall show that the FAB-method, starting from a general well-behaved g, indeed yields a tight frame if and only if we have a factorization

$$|(Zg)(t, \nu)|^2 + |(Zg)(t+1/2, \nu)|^2 = \Psi(\nu)K(t) \tag{21}$$

where Ψ is 1-periodic in ν and K is 1/2-periodic in t, and in that case, we have that $h^{(\mathrm{FAB})}$ in (19) equals the canonical tight frame generating h of (6). The following lemma is basic.

Lemma 1: For general well-behaved g, we have

$$Z(\boldsymbol{O}_{1/2}\boldsymbol{F}^{-1}\boldsymbol{O}_1\boldsymbol{F}g)(t, \nu) = \frac{(Zg)(t, \nu)\Phi^{-1/2}(\nu)}{\left(\frac{1}{2}\int_0^1 (|(Zg)(t, \mu)|^2 + |(Zg)(t+1/2, \mu)|^2)\Phi^{-1}(\mu)\, d\mu\right)^{1/2}} \tag{22}$$

where $\Phi(\nu) = \sum_{k=-\infty}^{\infty} |(\boldsymbol{F}g)(\nu - k)|^2$ is assumed to be strictly positive.

Proof: We compute using (12)

$$\begin{aligned}(\boldsymbol{F}^{-1}\boldsymbol{O}_1\boldsymbol{F}g)(t) &= \int_{-\infty}^{\infty} e^{2\pi i\nu t}(\boldsymbol{O}_1\boldsymbol{F}g)(\nu)\, d\nu \\ &= \int_{-\infty}^{\infty} e^{2\pi i\nu t} \cdot \frac{(\boldsymbol{F}g)(\nu)}{\left(\sum_{k=-\infty}^{\infty} |(\boldsymbol{F}g)(\nu-k)|^2\right)^{1/2}} d\nu \\ &= \int_0^1 \Phi^{-1/2}(\nu) \sum_{k=-\infty}^{\infty} (\boldsymbol{F}g)(\nu+k) \cdot e^{2\pi i(\nu+k)t}\, d\nu \\ &= \int_0^1 \Phi^{-1/2}(\nu)e^{2\pi i\nu t}(Z\boldsymbol{F}g)(\nu, -t)\, d\nu \\ &= \int_0^1 \Phi^{-1/2}(\nu)(Zg)(t, \nu)\, d\nu. \end{aligned} \tag{23}$$

Then from (13) and Parseval's theorem

$$\begin{aligned}\sum_{k=-\infty}^{\infty} &|(\boldsymbol{F}^{-1}\boldsymbol{O}_1\boldsymbol{F}g)(t-k/2)|^2 \\ &= \sum_{n=-\infty}^{\infty} \left|\int_0^1 \Phi^{-1/2}(\nu)(Zg)(t-n,\,\nu)\,d\nu\right|^2 \\ &\quad + \sum_{n=-\infty}^{\infty} \left|\int_0^1 \Phi^{-1/2}(\nu)(Zg)(t+1/2-n,\,\nu)\,d\nu\right|^2 \\ &= \int_0^1 \Phi^{-1}(\nu)(|(Zg)(t,\,\nu)|^2 + |(Zg)(t+1/2,\nu)|^2)\,d\nu \\ &=: H(t). \end{aligned} \tag{24}$$

By $1/2$- (and thus 1-) periodicity of H and (16), we then get

$$\begin{aligned} Z(&\boldsymbol{O}_{1/2}\boldsymbol{F}^{-1}\boldsymbol{O}_1\boldsymbol{F}g)(t,\,\nu) \\ &= Z\left[\frac{(\boldsymbol{F}^{-1}\boldsymbol{O}_1\boldsymbol{F}g)}{(\frac{1}{2}\,H)^{1/2}}\right](t,\,\nu) \\ &= (\tfrac{1}{2}\,H(t))^{-1/2} Z(\boldsymbol{F}^{-1}\boldsymbol{O}_1\boldsymbol{F}g)(t,\,\nu). \end{aligned} \tag{25}$$

Finally, an application of (17) using (23) yields the desired result. □

We now show the results stated in the beginning of this section. Assume that we have a g such that $h^{(\mathrm{FAB})}$ generates a tight frame for $a = 1/2$, $b = 1$. Then we have that

$$|(Zh^{(\mathrm{FAB})})(t,\,\nu)|^2 + |(Zh^{(\mathrm{FAB})})(t+1/2,\nu)|^2 = c \tag{26}$$

for some constant c. From the Lemma we thus see that

$$\begin{aligned} |(Zg)&(t,\,\nu)|^2 + |(Zg)(t+1/2,\nu)|^2 \\ &= \frac{c}{2}\,\Phi(\nu)\int_0^1 (|(Zg)(t,\mu)|^2 + |(Zg)(t+1/2,\mu)|^2) \\ &\quad \cdot \Phi^{-1}(\mu)\,d\mu \end{aligned} \tag{27}$$

showing that the left-hand side of (27) has a factorization as in (21).

Conversely, assume that (21) holds with a 1-periodic Ψ and a $1/2$-periodic K. Then we have from (18) that

$$\begin{aligned}\Psi(\nu)\int_0^{1/2} K(t)\,dt &= \int_0^1 |(Zg)(t,\,\nu)|^2\,dt \\ &= \sum_{k=-\infty}^{\infty} |(\boldsymbol{F}g)(\nu-k)|^2 = \Phi(\nu) \end{aligned} \tag{28}$$

with Φ as in the Lemma. We thus get from the Lemma that

$$\begin{aligned} |(Zh^{(\mathrm{FAB})})&(t,\,\nu)|^2 + |(Zh^{(\mathrm{FAB})})(t+1/2,\nu)|^2 \\ &= \frac{\Psi(\nu)K(t)\Phi^{-1}(\nu)}{\frac{1}{2}\displaystyle\int_0^1 \Psi(\mu)K(t)\Phi^{-1}(\mu)\,d\mu} \\ &= 2. \end{aligned} \tag{29}$$

Hence, $h^{(\mathrm{FAB})}$ generates a tight frame for $a = 1/2$, $b = 1$.

Finally, when $h^{(\mathrm{FAB})}$ generates a tight frame, we see from (27) and the Lemma that

$$\begin{aligned} (Zh^{(\mathrm{FAB})})&(t,\,\nu) \\ &= \frac{(Zg)(t,\,\nu)\Phi^{-1/2}(\nu)}{\left(\dfrac{1}{c}\,\Phi^{-1}(\nu)(|(Zg)(t,\,\nu)|^2 + |(Zg)(t+1/2,\nu)|^2)\right)^{1/2}} \\ &= c^{1/2}\frac{(Zg)(t,\,\nu)}{(|(Zg)(t,\,\nu)|^2 + |(Zg)(t+1/2,\nu)|^2)^{1/2}}. \end{aligned} \tag{30}$$

This shows that $h^{(\mathrm{FAB})}$ coincides, within a constant [see (10)], with the canonical tight frame generating h.

We finally note that the formulation of the FAB-method, as well as the results of this section, admits straightforward generalization to the case that the oversampling factor a^{-1} is an integer ≥ 2.

IV. Application to Gaussian g

In this section, we show that the FAB-method applied to a Gaussian $g_\alpha(t) = (2\alpha)^{1/4}e^{-\pi\alpha t^2}$ indeed yields a tight frame generating $h^{(\mathrm{FAB})}$, which coincides (up to a factor) with the canonical tight frame generating h. For all this, it is enough to show that (21) holds with Ψ and K periodic functions of period 1 and $1/2$, respectively. It follows from (15) for general g that

$$\begin{aligned} |(Zg)(t,\,\nu)|^2 &+ |(Zg)(t+1/2,\,\nu)|^2 \\ &= 2\sum_{k,\,l=-\infty}^{\infty} (g,\,g_{k,\,2l})e^{4\pi ilt - 2\pi ik\nu}. \end{aligned} \tag{31}$$

For $g = g_\alpha$ and $x,\,y \in \mathbb{R}$, we have

$$(g,\,g_{x,\,y}) = e^{-(1/2)\pi\alpha x^2 - (1/2)\,\pi\alpha^{-1}y^2}e^{-\pi ixy}. \tag{32}$$

Hence, when $x = k$, $y = 2l$ with $k,\,l \in \mathbb{Z}$, the phase factor $e^{-\pi ixy}$ in the right-hand side expression in (32) equals unity, and we get

$$\begin{aligned}\sum_{k,\,l=-\infty}^{\infty} &(g,\,g_{k,\,2l})e^{4\pi ilt-2\pi ik\nu} \\ &= \sum_{k=-\infty}^{\infty} e^{-(1/2)\pi\alpha k^2 - 2\pi ik\nu} \sum_{l=-\infty}^{\infty} e^{-2\pi\alpha^{-1}l^2 + 4\pi ilt} \end{aligned} \tag{33}$$

which is the desired factorization.

We finally note that Gaussians g are among the very few functions for which a factorization as in (21) holds (see [8] for some more examples).

V. Conclusion

We proved that the tight Gabor frame generated by the FAB-method applied to the Gaussian is equal to the canonical tight Gabor frame associated to the Gaussian. Hence, the FAB-method is equivalent to applying the positive definite inverse square root of the Gabor frame operator to the initial window function. We furthermore showed that for general windows g, the FAB-method yields a tight Gabor frame (and indeed, the canonical tight frame) if and only if $|(Zg)(t, \nu)|^2 + |(Zg)(t+1/2, \nu)|^2$ is separable, and we thereby characterized all window functions for which the FAB-method works (i.e., yields a tight Gabor frame). It appears that the Zak transform-based method is more general since it does not impose any restrictions on the initial window function.

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