OUTAGE PROPERTIES OF SPACE-TIME BLOCK CODES IN CORRELATED RAYLEIGH OR RICEAN FADING ENVIRONMENTS

Rohit U. Nabar[1)*], *Helmut Bölcskei*[2)†], *and Arogyaswami J. Paulraj*[1)]

[1)] Information Systems Laboratory, Stanford University
228 Packard, 350 Serra Mall, Stanford, CA 94305
Email: {nabar, apaulraj}@stanford.edu

[2)] Communication Technology Laboratory, ETH Zurich
ETH Zentrum, ETF E122, Sternwartstrasse 7, CH-8092 Zürich
Email: boelcskei@nari.ee.ethz.ch

ABSTRACT

The performance of space-time block codes is well understood from an average (over the random channel) error point of view. However, inherent to the idea of diversity gain is the issue of reliability, which is better captured through an outage analysis indicating the quality of performance guaranteed with a certain level of reliability. In this paper, we study the outage performance of a simple space-time block code, the Alamouti scheme, in the presence of correlated Rayleigh or Ricean fading. We derive expressions for the cumulative distribution function of the uncoded symbol error rate and verify the accuracy of our analytical expressions through comparison with numerical results. In addition, we introduce a quantitative measure to compare the diversity gain offered by two channels at a given outage rate.

1. INTRODUCTION AND OUTLINE

Wireless links are impaired by random fluctuations in signal level, known as fading. Multiple-antenna systems offer *spatial diversity*, which is an effective means of combating fading. The use of spatial diversity combined with proper signal processing and coding can significantly improve the quality and reliability of transmission. *Receive diversity* techniques [1] may be exploited in systems with multiple antennas at the receiver. In practice, however, deploying multiple antennas at the subscriber unit is often not possible, due to cost or space limitations. An attractive alternative is the use of *transmit diversity* [2]-[4] where multiple antennas are employed at the transmitter. *Space-time coding* [3, 4] is a popular transmit diversity technique that is capable of extracting full diversity gain, without requiring channel knowledge in the transmitter. In this paper, we consider a popular space-time coding scheme, the Alamouti scheme [4], that is applicable to systems with two antennas at the transmitter and one or more antennas at the receiver.

The performance of the Alamouti scheme depends heavily on channel characteristics such as Ricean K-factor and fading signal correlation which in turn depend on antenna height and spacing and the scattering environment. The Alamouti scheme, and more generally space-time block codes, are well understood from an average (over the random channel) error point of view [3, 5, 6]. However, the issue of link reliability which is inherent to the idea of diversity gain is better captured through an outage analysis which indicates the quality of performance guaranteed with a certain level of reliability.

Contributions. In this paper, we study the outage performance of the Alamouti scheme in the presence of correlated Rayleigh or Ricean fading. Our contributions are as follows.

- We *derive* expressions for the *cumulative distribution function (CDF)* of the *uncoded symbol error* rate for the Alamouti scheme as a function of the channel statistics. The *accuracy* of our analytical expressions is demonstrated through comparison with numerical results.
- We apply our results to *quantify the impact of fading signal correlation and Ricean K-factor* on the outage error performance of the Alamouti scheme.
- Taking into account fading signal correlation and Ricean K-factor, we provide a new *quantitative measure* of diversity at a given outage rate.

Organization of the paper. The rest of this paper is organized as follows. Section 2 introduces the channel model. In Section 3, we derive an expression for the CDF of the uncoded symbol error rate of the Alamouti scheme and describe how our techniques can be used to quantify the impact of varying channel characteristics on its outage error performance. We present our simulation results in Section 4, and conclude in Section 5.

2. CHANNEL MODEL

We consider a wireless system with two transmit antennas and N receive antennas, and restrict our analysis to the case of frequency-flat fading. The input-output relation of such a channel is characterized by the $N \times 2$ channel matrix $\boldsymbol{H}$, whose elements $H_{i,j}$ $(i = 1, 2, ..., N,\ j = 1, 2)$ are (possibly correlated) complex Gaussian random variables. We

*R. U. Nabar's work was supported by the Dr. T. J. Rodgers Stanford Graduate Fellowship.

†The work of H. Bölcskei was supported by FWF grant J1868-TEC and by NSF grants CCR 99-79381 and ITR 00-85929.

write $\boldsymbol{H}$ as the sum of a fixed (possibly line-of sight) component[1] $\overline{\boldsymbol{H}} = \mathcal{E}\{\boldsymbol{H}\}$ and a variable component $\widetilde{\boldsymbol{H}}$ consisting of zero-mean circularly symmetric complex Gaussian random variables [6]. In the case of pure Rayleigh fading, we have $\overline{\boldsymbol{H}} = \mathbf{0}$, while $\overline{\boldsymbol{H}} \neq \mathbf{0}$ in the presence of Ricean fading.

In practice, as a result of insufficient antenna spacing and/or lack of scattering the elements of $\boldsymbol{H}$ will in general be correlated. This correlation may be concisely expressed through the covariance matrix[2]

$$\boldsymbol{R} = \mathcal{E}\left\{\tilde{\boldsymbol{h}}\tilde{\boldsymbol{h}}^H\right\}, \tag{1}$$

where $\tilde{\boldsymbol{h}} = \text{vec}\{\widetilde{\boldsymbol{H}}\}$. Furthermore, we set $\boldsymbol{R} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{U}^H$ with $\boldsymbol{\Sigma} = \text{diag}\{\sigma_i\}_{i=1}^{2N}$. For the sake of simplicity, we assume $\boldsymbol{R}$ to be non-singular in the presence of Ricean fading. Finally, we introduce vectorized representations of $\boldsymbol{H}$ and $\overline{\boldsymbol{H}}$ which we denote by $\boldsymbol{h}$ and $\overline{\boldsymbol{h}}$, respectively. Note that $\boldsymbol{R}$ and $\overline{\boldsymbol{H}}$ completely characterize the statistical behavior of the channel.

3. OUTAGE PROPERTIES OF THE ALAMOUTI SCHEME

In the Alamouti scheme during the first symbol period, the symbols s_1 and s_2 are transmitted from antennas 1 and 2, respectively. During the following symbol period $-s_2^*$ and s_1^* are sent from antennas 1 and 2, respectively. In the following, we assume that the data symbols are drawn from a constellation with unit average energy. Employing the receiver structure described in [4], we obtain the equivalent input-output relation

$$\hat{s}_i = \sqrt{E_s}\|\boldsymbol{H}\|_F^2 s_i + n_i, \qquad i = 1, 2,$$

where $\hat{s}_i$ denotes the reconstructed data symbols, $\|\boldsymbol{H}\|_F^2 = \boldsymbol{h}^H\boldsymbol{h}$ is the squared Frobenius norm of the channel matrix, E_s is an energy normalization factor, and n_i denotes additive white complex Gaussian noise with $\mathcal{E}\{|n_i|^2\} = \|\boldsymbol{H}\|_F^2\sigma_n^2$. The probability of symbol error for a given channel realization $\boldsymbol{H}$ satisfies [4, 7]

$$P_e \approx \overline{N}_e\, Q\left(\sqrt{d_{min}^2 \frac{E_s\|\boldsymbol{H}\|_F^2}{2\sigma_n^2}}\right), \tag{2}$$

where $\overline{N}_e$ and d_{min} are the average number of nearest neighbors and minimum distance of separation of the underlying constellation, respectively. Note that since the channel is random, P_e will be random as well. The operational meaning of a random error probability as defined above is as follows. We assume that the channel is block fading with sufficiently large block length. For a given channel realization $\boldsymbol{H}$ we then get a certain error probability according to (2). Now, we consider $\boldsymbol{H}$ random which renders the symbol error probability random. The $a\%$ outage error rate is defined as the symbol error rate that is guaranteed in $(100 - a)\%$ of the cases. From (2) it is clear that the statistics of P_e and hence the outage error performance of the Alamouti scheme depend on the statistics of $\|\boldsymbol{H}\|_F^2$. Furthermore, noting that $Q(x)$ decreases monotonically for $x \geq 0$, it is easy to verify that the CDF of P_e, $F_e(y) = P(P_e \leq y)$, satisfies[3]

$$F_e(y) \approx 1 - F\left(\frac{2\sigma_n^2}{d_{min}^2 E_s}\left\{Q^{-1}\left(\frac{y}{\overline{N}_e}\right)\right\}^2\right), \tag{3}$$

where $F(y)$ stands for the CDF of $\|\boldsymbol{H}\|_F^2$. Since $\|\boldsymbol{H}\|_F^2$ is a quadratic form in complex Gaussian random variables, the Laplace transform of the probability density function (PDF) of $\|\boldsymbol{H}\|_F^2$ is given by [8]

$$\psi(s) = \frac{\exp\left(-\sum_{j=1}^{2N}|b_j|^2 + \sum_{j=1}^{2N}\frac{|b_j|^2}{1+s\sigma_j}\right)}{\prod_{j=1}^{2N}(1+s\sigma_j)}, \tag{4}$$

where b_j represents the j-th element of the $2N \times 1$ vector $\boldsymbol{b} = \boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{U}^H\overline{\boldsymbol{h}}$ (recall that $\boldsymbol{R}$ was assumed non-singular). We first consider the case of Rayleigh fading, i.e., $\overline{\boldsymbol{h}} = \mathbf{0}$. Expressing distinct non-zero eigenvalues of $\boldsymbol{R}$ by $\widetilde{\sigma}_j (j = 1, 2, ..., L, 1 \leq L \leq 2N)$ and denoting their respective multiplicities by $m_j (j = 1, 2, ..., L)$, we can express $\psi(s)$ in (4) via partial fraction expansion as

$$\psi(s) = \sum_{j=1}^{L}\sum_{k=1}^{m_j}\frac{A_{jk}}{(1+s\widetilde{\sigma}_j)^k}, \tag{5}$$

where the $A_{jk} (j = 1, 2, ..., L, k = 1, 2, ..., m_j)$ are determined by solving a system of linear equations. Note that $\psi(0) = 1$ implies that $\sum_{j=1}^{L}\sum_{k=1}^{m_j} A_{jk} = 1$. The PDF of $\|\boldsymbol{H}\|_F^2$ which is simply the inverse Laplace transform of (5) can then be expressed as[4] [8]

$$f(x) = \sum_{j=1}^{L}\sum_{k=1}^{m_j} A_{jk}\frac{x^{k-1}}{(k-1)!\,\widetilde{\sigma}_j^k}e^{-\frac{x}{\widetilde{\sigma}_j}}u(x). \tag{6}$$

From (6), the CDF of $\|\boldsymbol{H}\|_F^2$ can be derived as [8]

$$F(y) = \left(1 - \sum_{j=1}^{L}\sum_{k=1}^{m_j} A_{jk}\sum_{i=0}^{k-1}\frac{\left(\frac{y}{\widetilde{\sigma}_j}\right)^{k-1-i}}{(k-1-i)!}e^{-\frac{y}{\widetilde{\sigma}_j}}\right)u(y). \tag{7}$$

In the case of Ricean fading ($\overline{\boldsymbol{h}} \neq \mathbf{0}$) the PDF of $\|\boldsymbol{H}\|_F^2$ can be derived through more complicated Laplace transform inversion techniques involving series expansion of $\psi(s)$ [8, 9] to give

$$f(x) = \sum_{k=0}^{\infty} c_k \frac{x^{2N+k-1}}{(2N+k-1)!}u(x), \tag{8}$$

[1] $\mathcal{E}$ stands for the expectation operator.

[2] The superscript ^{H} stands for conjugate transpose.

[3] Q^{-1} stands for the inverse Q-function.

[4] $u(x)$ is the unit step function defined as $u(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$.

where c_0 is given by the equation

$$c_0 = \prod_{j=1}^{2N} \sigma_j^{-1} \exp\left(-\sum_{j=1}^{2N} |b_j|^2 \right), \tag{9}$$

and c_k $(k \geq 1)$ can be calculated recursively using the following relations

$$c_k = \frac{1}{k} \sum_{r=0}^{k-1} d_{k-r} c_r, \; k \geq 1 \tag{10}$$

$$d_k = (-1)^k \sum_{j=1}^{2N} (1 - k|b_j|^2) \sigma_j^{-k}, \; k \geq 1. \tag{11}$$

From (8) it is straightforward to show that $F(y)$ for the Ricean case is given by

$$F(y) = \sum_{k=0}^{\infty} c_k \frac{y^{2N+k}}{(2N+k)!} u(y). \tag{12}$$

Combining (3), (7), and (12) we can compute the CDF of the uncoded symbol error rate of the Alamouti scheme corresponding to a given outage requirement. Simulations in Section 4 reveal an almost identical match between the so obtained CDF of the uncoded symbol error rate and the numerically computed CDF. We furthermore note that the power series expression for $F(y)$ in the case of Ricean fading simplifies to the closed-form expression for Rayleigh fading when $\overline{\boldsymbol{h}} = \mathbf{0}$.

An outage rate dependent measure of diversity. We shall next employ the analysis developed thus far to compare the diversity gain offered by two channels with different characteristics at a given outage requirement. Assume that $F_1(y)$ and $F_2(y)$ are the CDFs of $\|\boldsymbol{H}\|_F^2$ for two different channels with different statistics and $F_1(y1) = F_2(y2) = a\%$. Noting again that $Q(x)$ decreases monotonically for $x \geq 0$, it is clear that the channel with higher $y_i (i = 1, 2)$ will have a lower $a\%$ outage symbol error rate. We can quantify the gain (or loss) in diversity at the $a\%$ outage level offered by channel 1 with channel 2 as reference as follows

$$\eta(a\%) = 10 \log_{10} \left(\frac{y_1}{y_2} \right) (\text{dB}). \tag{13}$$

Note that (13) can be used to quantify the loss/gain in diversity due to fading signal correlation and a Ricean component by using the classical i.i.d. Rayleigh fading MIMO channel as a reference.

4. SIMULATIONS

We simulated the Alamouti scheme for a system with 2 transmit and 2 receive antennas. The SNR in our simulations was defined as $10 \log_{10} \left(\frac{2E_s}{\sigma_n^2} \right)$(dB). The data symbols were drawn from a 16-QAM constellation. Though our techniques are generally applicable, for the sake of simplicity we consider the following simplified channel model. We chose $\overline{\boldsymbol{H}}$ and $\widetilde{\boldsymbol{H}}$ to be of the form

$$\overline{\boldsymbol{H}} = \sqrt{\frac{K}{1+K}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{14}$$

$$\widetilde{\boldsymbol{H}} = \sqrt{\frac{1}{1+K}} \begin{bmatrix} \widetilde{G}_{1,1} & \widetilde{G}_{1,2} \\ \widetilde{G}_{2,1} & \widetilde{G}_{2,2} \end{bmatrix}, \tag{15}$$

where $\widetilde{G}_{i,j} (i, j = 1, 2)$ are (possibly correlated) zero-mean complex Gaussian random variables with unit variance and K is the Ricean K-factor of the system. In the case of pure Rayleigh fading $K = 0$. Furthermore, we define the following correlations[5]

$$t = \mathcal{E}\{\widetilde{G}_{1,1} \widetilde{G}_{1,2}^*\} = \mathcal{E}\{\widetilde{G}_{2,1} \widetilde{G}_{2,2}^*\} \tag{16}$$

$$r = \mathcal{E}\{\widetilde{G}_{1,1} \widetilde{G}_{2,1}^*\} = \mathcal{E}\{\widetilde{G}_{1,2} \widetilde{G}_{2,2}^*\}, \tag{17}$$

where t is referred to as the transmit correlation and r is the receive correlation coefficient. For the sake of simplicity, we assume $\mathcal{E}\{\widetilde{G}_{1,1}\widetilde{G}_{2,2}^*\} = \mathcal{E}\{\widetilde{G}_{1,2}\widetilde{G}_{2,1}^*\} = 0$.

Simulation Example 1. The first simulation example serves to assess the accuracy of our analytical expressions in determining the CDF of the uncoded symbol error rate of the Alamouti scheme. We simulated the Alamouti scheme for a channel with $K = 2$, $r = 0.1$, and $t = 0.2$. Fig. 1 shows the empirical (using Monte Carlo methods) CDF of the uncoded symbol error rate and the CDF calculated using (3) and (12) for an SNR of 10 dB. The first 150 terms in the power series expansion of $F(y)$ were considered in calculating the analytical CDF for the uncoded symbol error rate. The plot indicates a close match between the estimated and analytical CDF, verifying the accuracy of our analysis.

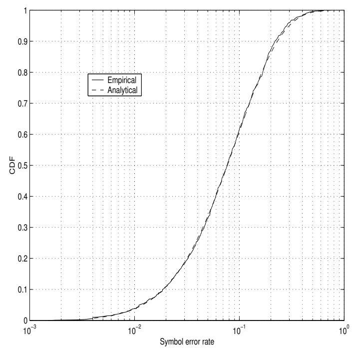

Fig. 1. *Comparison of the empirical and analytical CDF of the symbol error rate for the Alamouti scheme.*

Simulation Example 2. In this simulation example, we study the impact of receive correlation on the outage properties of the Alamouti scheme in the presence of Rayleigh fading. In Figs. 2a) and b), we present the empirical and the analytical CDF, respectively, of $\|\boldsymbol{H}\|_F^2$ for $t = 0$ and varying degrees of receive correlation r. The simulation results reveal (as expected) that receive correlation is detrimental to the outage performance of the Alamouti scheme for low outage requirements. Comparing the channels with $r = 0$ (i.i.d. Rayleigh fading) and $r = 1$ we find that the

[5]The superscript * stands for complex conjugate.

loss in performance due to receive correlation at the 10% outage level is given by $\eta(10\%) = -2$ dB and similarly $\eta(30\%) = -1$ dB. We therefore conclude that the loss in diversity gain depends strongly on the outage level.

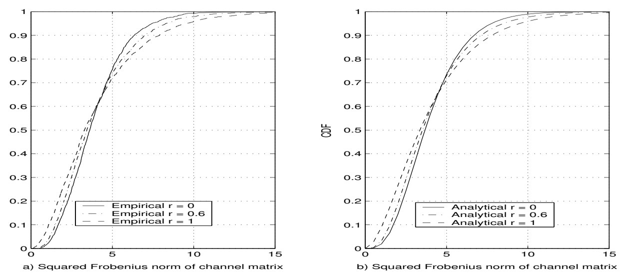

Fig. 2. *Effect of receive correlation on outage error performance of the Alamouti scheme.*

Simulation Example 3. In the last simulation example, we study gain/loss in diversity as a function of outage rate for varying channel conditions using the i.i.d. Rayleigh fading channel model as a reference. Fig. 3 shows the gain/loss in diversity according to (13) for three channels with different r, t, K as a function of outage rate (5% to 30%). We observe that a channel with moderately high transmit and receive correlation offers more diversity gain than a channel with severely high receive correlation and no (very little) transmit correlation. We can also see that the channel experiencing Ricean fading offers consistently higher diversity gain than an i.i.d. Rayleigh fading channel. This can be accounted for by the presence of an invariant component in the channel that stabilizes the link.

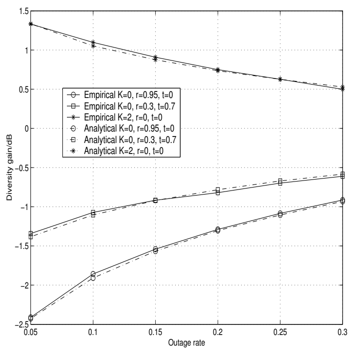

Fig. 3. *Diversity gain as a function of outage rate for varying channel conditions.*

5. CONCLUSIONS

We studied the outage error performance of the Alamouti scheme in the presence of correlated Rayleigh or Ricean fading. We derived expressions for the CDF of the uncoded symbol error rate for the Alamouti scheme and defined an outage dependent diversity measure which allows to compare the outage error performance of two channels experiencing different fading conditions. The accuracy of our analytical expressions was verified through comparison with empirically obtained results. Our techniques provide a tool to predict the reliability of a multi-antenna link employing the Alamouti scheme accounting for real-world propagation conditions, without having to resort to time-consuming computer simulations. Finally, we note that the analysis devised in this paper applies to arbitrary space-time block codes.

6. REFERENCES

[1] W. C. Jakes, *Microwave Mobile Communications*, Wiley, New York, NY, 1974.

[2] N. Seshadri and J. Winters, "Two signaling schemes for improving the error performance of frequency-division-duplex (FDD) transmission systems using transmitter antenna diversity," *Int. J. Wireless Information Networks*, vol. 1, pp. 49–60, 1994.

[3] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communication: Performance criterion and code construction," *IEEE Trans. Inf. Theory*, vol. 44, no. 2, pp. 744–765, March 1998.

[4] S. M. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE J. Sel. Areas Comm.*, vol. 16, pp. 1451–1468, Oct. 1998.

[5] H. Bölcskei and A. J. Paulraj, "Performance analysis of space-time codes in correlated Rayleigh fading environments," in *Proc. of Asilomar Conf. Signals, Systems, and Computers*, Pacific Grove, CA, Nov. 2000, pp. 687–693.

[6] R. U. Nabar, H. Bölcskei, V. Erceg, D. Gesbert, and A. J. Paulraj, "Performance of multi-antenna signaling techniques in the presence of polarization diversity," *IEEE Trans. Sig. Proc.*, submitted, Aug. 2001.

[7] J. M. Cioffi, *Class Reader for EE379a – Digital Communication: Signal Processing*, Stanford University, Stanford, CA. Available online at http://www.stanford.edu/class/ee379a.

[8] R. U. Nabar, H. Bölcskei, and A. J. Paulraj, "Outage performance of space-time block codes for generalized MIMO channels," in preparation.

[9] A. M. Mathai and S. B. Provost, *Quadratic Forms in Random Variables*, Marcel Dekker, New York, NY, 1992.