QR Decomposition of Laurent Polynomial Matrices Sampled on the Unit Circle

Davide Cescato, *Student Member, IEEE,* and Helmut Bölcskei, *Fellow, IEEE*

Abstract

We consider Laurent polynomial (LP) matrices defined on the unit circle of the complex plane. QR decomposition of an LP matrix $\mathbf{A}(s)$ yields QR factors $\mathbf{Q}(s)$ and $\mathbf{R}(s)$ that, in general, are neither LP nor rational matrices. In this paper, we present an invertible mapping that transforms $\mathbf{Q}(s)$ and $\mathbf{R}(s)$ into LP matrices. Furthermore, we show that, given QR factors of sufficiently many samples of $\mathbf{A}(s)$, it is possible to obtain QR factors of additional samples of $\mathbf{A}(s)$ through application of this mapping followed by interpolation and inversion of the mapping. The results of this paper find applications in the context of signal processing for multiple-input multiple-output (MIMO) wireless communication systems that employ orthogonal frequency-division multiplexing (OFDM).

Index Terms

Interpolation, Laurent polynomial matrices, QR decomposition, sampling.

I. Introduction

We consider Laurent polynomial (LP) matrices defined on the unit circle of the complex plane. QR decomposition of an LP matrix $\mathbf{A}(s)$ yields QR factors $\mathbf{Q}(s)$ and $\mathbf{R}(s)$ that, in general, are neither LP nor rational matrices. In this paper, we present an invertible mapping that transforms $\mathbf{Q}(s)$ and $\mathbf{R}(s)$ into LP matrices. The significance of this result lies in the fact that it enables interpolation-based computation of QR factors of samples of $\mathbf{A}(s)$. Specifically, given QR factors of sufficiently many samples of $\mathbf{A}(s)$, QR factors of additional samples of $\mathbf{A}(s)$ can be obtained through application of the mapping followed by interpolation and inversion of the mapping.

This work was supported in part by the Swiss National Science Foundation under grant No. 200021-100025/1. Parts of this paper were presented at the Sixth IEEE Workshop on Signal Processing Advances in Wireless Communications (SPAWC), New York, NY, June 2005. The authors are with the Communication Technology Laboratory, ETH Zurich, 8092 Zurich, Switzerland. E-mail: {dcescato, boelcskei}@nari.ee.ethz.ch.

The results presented in this paper find applications in multiple-input multiple-output (MIMO) wireless communication systems [1] that employ orthogonal frequency-division multiplexing (OFDM) [2]. A number of MIMO-OFDM receive algorithms, including successive cancelation detection [3], [4] and sphere decoding [5], [6], as well as the MIMO-OFDM transmit precoding schemes presented in [7], require QR decomposition or regularized QR decomposition, defined formally in Section II-B, of a sequence of matrices obtained by oversampling a polynomial matrix on the unit circle. Specific algorithms for efficient interpolation-based QR decomposition and regularized QR decomposition in MIMO-OFDM systems that make use of the main result in this paper are reported in [8]. Moreover, it is shown in [8] that interpolation-based QR decomposition can yield significant computational complexity savings over per-sample QR decomposition.

Finally, we mention that the results presented in this paper are related to interpolation-based inversion of polynomial matrices, as proposed in [9], [10] and applied in the context of MIMO-OFDM systems in [11].

II. Preliminaries

A. Notation

$\mathbb{C}^{P\times M}$ denotes the set of complex-valued $P \times M$ matrices. $\mathcal{U} \triangleq \{s \in \mathbb{C} : |s| = 1\}$ indicates the unit circle. Throughout the paper, we use the following conventions. First, if $k_2 < k_1$, $\sum_{k=k_1}^{k_2} \alpha_k = 0$, regardless of α_k. Second, sequences of integers of the form $k_1, k_1 + 1, \ldots, k_2$ simplify to the sequence k_1, k_2 if $k_2 = k_1 + 1$, to the single value k_1 if $k_2 = k_1$, and to the empty sequence if $k_2 < k_1$. $\mathbf{A}^*$, $\mathbf{A}^T$, $\mathbf{A}^H$, $\mathrm{rank}(\mathbf{A})$, and $\mathrm{range}(\mathbf{A})$ denote the entrywise conjugate, the transpose, the conjugate transpose, the rank, and the range space, respectively, of the matrix $\mathbf{A}$. $[\mathbf{A}]_{p,m}$ indicates the entry in the pth row and mth column of $\mathbf{A}$. $\mathbf{A}^{p_1,p_2}$ and $\mathbf{A}_{m_1,m_2}$ stand for the submatrix given by the rows $p_1, p_1 + 1, \ldots, p_2$ of $\mathbf{A}$ and the submatrix given by the columns $m_1, m_1 + 1, \ldots, m_2$ of $\mathbf{A}$, respectively. Furthermore, we set $\mathbf{A}_{m_1,m_2}^{p_1,p_2} \triangleq (\mathbf{A}_{m_1,m_2})^{p_1,p_2}$ and $\mathbf{A}_{m_1,m_2}^{H} \triangleq (\mathbf{A}_{m_1,m_2})^{H}$. A $P\times M$ matrix $\mathbf{A}$ is said to be upper triangular if all entries below its main diagonal $\{[\mathbf{A}]_{k,k} : k = 1, 2, \ldots, \min(P, M)\}$ are equal to zero. $\det(\mathbf{A})$ and $\mathrm{adj}(\mathbf{A})$ denote the determinant and the adjoint of a square matrix $\mathbf{A}$, respectively. $\mathrm{diag}(a_1, a_2, \ldots, a_M)$ indicates the $M \times M$ diagonal matrix with the scalar a_m as its mth main diagonal element. $\mathbf{I}_M$ stands for the $M \times M$ identity matrix and $\mathbf{0}$ denotes the all-zeros matrix of appropriate dimensions. Column vectors and row vectors are represented by lower-case bold symbols and by lower-case bold underlined symbols, respectively. Finally, orthogonality and norm of complex-valued column vectors $\mathbf{a}_1, \mathbf{a}_2$ are induced by the inner product $\mathbf{a}_1^H \mathbf{a}_2$.

B. QR Decomposition

Throughout this section, we consider a matrix $\mathbf{A} = [\mathbf{a}_1 \; \mathbf{a}_2 \; \cdots \; \mathbf{a}_M] \in \mathbb{C}^{P \times M}$ with $P \geq M$, where $\mathbf{a}_k$ denotes the kth column of $\mathbf{A}$ ($k = 1, 2, \ldots, M$). In the remainder of the paper, we use the following definition of QR decomposition, that allows us to state our results for general rank of $\mathbf{A}$ and independently of the QR decomposition algorithm used.

Definition 1: We call any factorization $\mathbf{A} = \mathbf{QR}$, for which the matrices $\mathbf{Q} \in \mathbb{C}^{P \times M}$ and $\mathbf{R} \in \mathbb{C}^{M \times M}$ satisfy the following conditions, a *QR decomposition* of $\mathbf{A}$ with *QR factors* $\mathbf{Q}$ and $\mathbf{R}$:

1) the nonzero columns of $\mathbf{Q}$ are orthonormal
2) $\mathbf{R}$ is upper triangular with real-valued nonnegative entries on its main diagonal
3) $\mathbf{R} = \mathbf{Q}^H \mathbf{A}$

Practical algorithms for QR decomposition are either based on Gram-Schmidt (GS) orthonormalization or on unitary transformations (UT). We next briefly review both classes of algorithms. GS-based QR decomposition is summarized as follows. For $k = 1, 2, \ldots, M$, the kth column of $\mathbf{Q}$, denoted by $\mathbf{q}_k$, is determined by

$$\mathbf{y}_k \triangleq \mathbf{a}_k - \sum_{i=1}^{k-1} \mathbf{q}_i^H \mathbf{a}_k \mathbf{q}_i \tag{1}$$

with

$$\mathbf{q}_k = \begin{cases} \frac{\mathbf{y}_k}{\sqrt{\mathbf{y}_k^H \mathbf{y}_k}}, & \mathbf{y}_k \neq \mathbf{0} \\ \mathbf{0}, & \mathbf{y}_k = \mathbf{0} \end{cases} \tag{2}$$

whereas the kth row of $\mathbf{R}$, denoted by $\underline{\mathbf{r}}_k$, is given by

$$\underline{\mathbf{r}}_k = \mathbf{q}_k^H \mathbf{A}. \tag{3}$$

UT-based QR decomposition of $\mathbf{A}$ is performed by left-multiplying $\mathbf{A}$ by the product $\mathbf{\Theta}_U \cdots \mathbf{\Theta}_2 \mathbf{\Theta}_1$ of $P \times P$ unitary matrices $\mathbf{\Theta}_u$, where the sequence of matrices $\mathbf{\Theta}_1, \mathbf{\Theta}_2, \ldots, \mathbf{\Theta}_U$ and the parameter U are not unique and are chosen such that the $P \times M$ matrix $\mathbf{\Theta}_U \cdots \mathbf{\Theta}_2 \mathbf{\Theta}_1 \mathbf{A}$ is upper triangular with real-valued nonnegative entries on its main diagonal. The matrices $\mathbf{\Theta}_u$ are typically either Givens rotation matrices [12] or Householder reflection matrices [12]. With $\mathbf{R} \triangleq (\mathbf{\Theta}_U \cdots \mathbf{\Theta}_2 \mathbf{\Theta}_1 \mathbf{A})^{1,M}$ and $\mathbf{Q} \triangleq ((\mathbf{\Theta}_U \cdots \mathbf{\Theta}_2 \mathbf{\Theta}_1)^H)_{1,M}$, we obtain that $\mathbf{Q}^H \mathbf{A} = \mathbf{R}$ and, since $\mathbf{\Theta}_U \cdots \mathbf{\Theta}_2 \mathbf{\Theta}_1$ is unitary, that $\mathbf{Q}^H \mathbf{Q} = \mathbf{I}_M$.

We note that since $\mathbf{y}_1 = \mathbf{0}$ is equivalent to $\mathbf{a}_1 = \mathbf{0}$ and $\mathbf{y}_k = \mathbf{0}$ is equivalent to $\mathrm{rank}(\mathbf{A}_{1,k-1}) = \mathrm{rank}(\mathbf{A}_{1,k})$ ($k = 2, 3, \ldots, M$) [13], GS-based QR decomposition sets $M - \mathrm{rank}(\mathbf{A})$ columns of $\mathbf{Q}$ and the corresponding $M - \mathrm{rank}(\mathbf{A})$ rows of $\mathbf{R}$ to zero. In contrast, UT-based QR decomposition yields a matrix $\mathbf{Q}$ such that $\mathbf{Q}^H\mathbf{Q} = \mathbf{I}_M$, regardless of the value of $\text{rank}(\mathbf{A})$, and sets $M - \text{rank}(\mathbf{A})$ entries on the main diagonal of $\mathbf{R}$ to zero [12]. Hence, for $\text{rank}(\mathbf{A}) < M$, different QR decomposition algorithms will, in general, produce different QR factors. The case $\text{rank}(\mathbf{A}) = M$ is considered next.

Proposition 1: If $\text{rank}(\mathbf{A}) = M$, Conditions 1 and 2 of Definition 1 simplify, respectively, to

1) $\mathbf{Q}^H\mathbf{Q} = \mathbf{I}_M$
2) $\mathbf{R}$ is upper triangular with $[\mathbf{R}]_{k,k} > 0, k = 1, 2, \ldots, M$

whereas Condition 3 is redundant. Moreover, $\mathbf{A}$ has unique QR factors.

Proof: Since $\mathbf{A} = \mathbf{QR}$ implies $\text{rank}(\mathbf{A}) \leq \min\{\text{rank}(\mathbf{Q}), \text{rank}(\mathbf{R})\}$, it follows from $\text{rank}(\mathbf{A}) = M$ that $\text{rank}(\mathbf{Q}) = \text{rank}(\mathbf{R}) = M$. Now, $\text{rank}(\mathbf{Q}) = M$ implies that the $P \times M$ matrix $\mathbf{Q}$ can not contain all-zero columns, and hence Condition 1 is equivalent to $\mathbf{Q}^H\mathbf{Q} = \mathbf{I}_M$. Moreover, $\text{rank}(\mathbf{R}) = M$ implies $\det(\mathbf{R}) \neq 0$ and, since $\mathbf{R}$ is upper triangular, we have $\det(\mathbf{R}) = \prod_{k=1}^{M}[\mathbf{R}]_{k,k}$. Hence, Condition 2 becomes $[\mathbf{R}]_{k,k} > 0, k = 1, 2, \ldots, M$. Condition 3 is redundant since $\mathbf{A} = \mathbf{QR}$, together with $\mathbf{Q}^H\mathbf{Q} = \mathbf{I}_M$, implies $\mathbf{Q}^H\mathbf{A} = \mathbf{R}$. The uniqueness of $\mathbf{Q}$ and $\mathbf{R}$ is proven in [13, Sec. 2.6]. ■

We note that for full-rank $\mathbf{A}$, the uniqueness of $\mathbf{Q}$ and $\mathbf{R}$ implies that $\mathbf{A} = \mathbf{QR}$ can be called *the* QR decomposition of $\mathbf{A}$ with *the* QR factors $\mathbf{Q}$ and $\mathbf{R}$. Throughout the paper, whenever $\mathbf{A}$ is not guaranteed to have full rank, we simply speak of *a* QR decomposition of $\mathbf{A}$ with QR factors $\mathbf{Q}$ and $\mathbf{R}$.

Finally, we introduce the concept of regularized QR decomposition, relevant, e.g., in MIMO-OFDM detectors with minimum mean-square error (MMSE) regularization [1]. We note that our main results will be derived for QR decomposition only, whereas the straightforward extensions to regularized QR decomposition will be briefly discussed in Section III-D.

Definition 2: The *regularized QR decomposition* of $\mathbf{A}$ with the real-valued *regularization parameter* $\alpha > 0$, is the unique factorization $\mathbf{A} = \mathbf{QR}$, where the *regularized QR factors* $\mathbf{Q} \in \mathbb{C}^{P \times M}$ and $\mathbf{R} \in \mathbb{C}^{M \times M}$ are obtained as follows: $\bar{\mathbf{A}} = \bar{\mathbf{Q}}\mathbf{R}$ is the unique QR decomposition of the full-rank $(P + M) \times M$ augmented matrix $\bar{\mathbf{A}} \triangleq [\,\mathbf{A}^T \quad \alpha\mathbf{I}_M\,]^T$, and $\mathbf{Q} \triangleq \bar{\mathbf{Q}}^{1,P}$.

C. Laurent Polynomials and Interpolation

Definition 3: Given a matrix-valued function $\mathbf{A} : \mathcal{U} \to \mathbb{C}^{P \times M}$ and integers $V_1, V_2 \geq 0$, the notation $\mathbf{A}(s) \sim (V_1, V_2)$ indicates that there exist coefficient matrices $\mathbf{A}_v \in \mathbb{C}^{P \times M}, v = -V_1, -V_1 + 1, \ldots, V_2$, such that

$$\mathbf{A}(s) = \sum_{v=-V_1}^{V_2} \mathbf{A}_v s^{-v}, \qquad s \in \mathcal{U}. \tag{4}$$

If $\mathbf{A}(s) \sim (V_1, V_2)$, then $\mathbf{A}(s)$ is a *Laurent polynomial* (LP) matrix with *maximum degree* $V_1 + V_2$. In the following, we briefly list the following statements which follow directly from Definition 3. First, $\mathbf{A}(s) \sim (V_1, V_2)$ implies $\mathbf{A}(s) \sim (V_1', V_2')$ for any $V_1' \geq V_1, V_2' \geq V_2$. Moreover, since for $s \in \mathcal{U}$ we have $s^* = s^{-1}$, $\mathbf{A}(s) \sim (V_1, V_2)$ implies $\mathbf{A}^H(s) \sim (V_2, V_1)$. Finally, given LP matrices $\mathbf{A}_1(s) \sim (V_{11}, V_{12})$ and $\mathbf{A}_2(s) \sim (V_{21}, V_{22})$, if $\mathbf{A}_1(s)$ and $\mathbf{A}_2(s)$ have the same dimensions, then $(\mathbf{A}_1(s) + \mathbf{A}_2(s)) \sim (\max(V_{11}, V_{21}), \max(V_{12}, V_{22}))$, whereas if the dimensions of $\mathbf{A}_1(s)$ and $\mathbf{A}_2(s)$ are such that the product $\mathbf{A}_1(s)\mathbf{A}_2(s)$ is defined, then $\mathbf{A}_1(s)\mathbf{A}_2(s) \sim (V_{11} + V_{21}, V_{12} + V_{22})$.

In the following, we consider an LP $a(s) \sim (V_1, V_2)$ of maximum degree $V \triangleq V_1 + V_2$. Borrowing terminology from signal analysis, we call the value of $a(s)$ at a given point $s_0 \in \mathcal{U}$ the *sample* $a(s_0)$. If more than $V + 1$ samples of $a(s)$ are known, we say that $a(s)$ is *oversampled*. The LP $a(s)$ can be interpolated from its samples at $V + 1$ distinct points in $\mathcal{U}$. Interpolation of LP matrices is performed entrywise.

III. QR Decomposition of LP Matrices

A. Additional Properties of QR Decomposition

We next set the stage for the formulation of our main result by presenting additional properties of QR decomposition of a matrix $\mathbf{A} \in \mathbb{C}^{P \times M}$, with $P \geq M$, that are directly implied by Definition 1.

Proposition 2: Let $\mathbf{A} = \mathbf{QR}$ be a QR decomposition of $\mathbf{A}$. Then, for a given $k \in \{1, 2, \ldots, M\}$, $\mathbf{A}_{1,k} = \mathbf{Q}_{1,k}\mathbf{R}_{1,k}^{1,k}$ is a QR decomposition of $\mathbf{A}_{1,k}$.

Proof: From $\mathbf{A} = \mathbf{QR}$ we obtain $\mathbf{A}_{1,k} = (\mathbf{QR})_{1,k} = \mathbf{Q}_{1,k}\mathbf{R}_{1,k}^{1,k} + \mathbf{Q}_{k+1,M}\mathbf{R}_{1,k}^{k+1,M}$, which simplifies to $\mathbf{A}_{1,k} = \mathbf{Q}_{1,k}\mathbf{R}_{1,k}^{1,k}$, since the upper triangularity of $\mathbf{R}$ implies that $\mathbf{R}_{1,k}^{k+1,M} = \mathbf{0}$. $\mathbf{Q}_{1,k}$ and $\mathbf{R}_{1,k}^{1,k}$ satisfy Conditions 1 and 2 of Definition 1 since all columns of $\mathbf{Q}_{1,k}$ are also columns of $\mathbf{Q}$ and since $\mathbf{R}_{1,k}^{1,k}$ is a principal submatrix of $\mathbf{R}$, respectively. Finally, $\mathbf{R} = \mathbf{Q}^H\mathbf{A}$ implies $\mathbf{R}_{1,k}^{1,k} = (\mathbf{Q}^H\mathbf{A})_{1,k}^{1,k} = \mathbf{Q}_{1,k}^H\mathbf{A}_{1,k}$ and hence Condition 3 of Definition 1 is satisfied. ■

Proposition 3: Let $\mathbf{A} = \mathbf{QR}$ be a QR decomposition of $\mathbf{A}$. Then, for $M > 1$ and for a given $k \in \{2, 3, \ldots, M\}$, $\mathbf{A}_{k,M} - \mathbf{Q}_{1,k-1}\mathbf{R}_{k,M}^{1,k-1} = \mathbf{Q}_{k,M}\mathbf{R}_{k,M}^{k,M}$ is a QR decomposition of $\mathbf{A}_{k,M} - \mathbf{Q}_{1,k-1}\mathbf{R}_{k,M}^{1,k-1}$.

Proof: From $\mathbf{A} = \mathbf{Q}_{1,k-1}\mathbf{R}^{1,k-1} + \mathbf{Q}_{k,M}\mathbf{R}^{k,M}$ we obtain $\mathbf{A}_{k,M} = \mathbf{Q}_{1,k-1}\mathbf{R}_{k,M}^{1,k-1} + \mathbf{Q}_{k,M}\mathbf{R}_{k,M}^{k,M}$ and hence $\mathbf{A}_{k,M} - \mathbf{Q}_{1,k-1}\mathbf{R}_{k,M}^{1,k-1} = \mathbf{Q}_{k,M}\mathbf{R}_{k,M}^{k,M}$. $\mathbf{Q}_{k,M}$ and $\mathbf{R}_{k,M}^{k,M}$ satisfy Conditions 1 and 2 of Definition 1 since all columns of $\mathbf{Q}_{k,M}$ are also columns of $\mathbf{Q}$ and since $\mathbf{R}_{k,M}^{k,M}$ is a principal submatrix of $\mathbf{R}$, respectively. Moreover, $\mathbf{R} = \mathbf{Q}^H\mathbf{A}$ implies $\mathbf{R}_{k,M}^{k,M} = (\mathbf{Q}^H\mathbf{A})_{k,M}^{k,M} = \mathbf{Q}_{k,M}^H\mathbf{A}_{k,M}$. Using $\mathbf{Q}_{k,M}^H\mathbf{Q}_{1,k-1} = \mathbf{0}$, which follows from the fact that the nonzero columns of $\mathbf{Q}$ are orthonormal, we can write $\mathbf{R}_{k,M}^{k,M} =$

$\mathbf{Q}_{k,M}^H \mathbf{A}_{k,M} - \mathbf{Q}_{k,M}^H \mathbf{Q}_{1,k-1} \mathbf{R}_{k,M}^{1,k-1} = \mathbf{Q}_{k,M}^H (\mathbf{A}_{k,M} - \mathbf{Q}_{1,k-1} \mathbf{R}_{k,M}^{1,k-1})$. Hence, Condition 3 of Definition 1 is satisfied. ■

In order to characterize QR factors of $\mathbf{A}$ in the general case $\mathrm{rank}(\mathbf{A}) \leq M$, we introduce the following concept.

Definition 4: The *ordered column rank* of $\mathbf{A}$ is the number

$$K \triangleq \begin{cases} 0, & \mathrm{rank}(\mathbf{A}_{1,1}) = 0 \\ \max\{k \in \{1, 2, \ldots, M\} : \mathrm{rank}(\mathbf{A}_{1,k}) = k\}, & \text{else.} \end{cases}$$

We note that $K = 0$ is equivalent to $\mathbf{a}_1 = \mathbf{0}$, and that $K < M$ is equivalent to $\mathbf{A}$ being rank-deficient. With the terminology and the results presented so far, we can now characterize QR factors of $\mathbf{A}$ as follows:

Proposition 4: QR factors $\mathbf{Q}$ and $\mathbf{R}$ of a matrix $\mathbf{A}$ of ordered column rank $K > 0$ satisfy the following properties:

1) $\mathbf{Q}_{1,K}^H \mathbf{Q}_{1,K} = \mathbf{I}_K$
2) $[\mathbf{R}]_{k,k} > 0$ for $k = 1, 2, \ldots, K$
3) $\mathbf{Q}_{1,K}$ and $\mathbf{R}^{1,K}$ are unique
4) $\mathrm{range}(\mathbf{Q}_{1,k}) = \mathrm{range}(\mathbf{A}_{1,k})$ for $k = 1, 2, \ldots, K$
5) if $K < M$, $[\mathbf{R}]_{K+1,K+1} = 0$

Proof: Since $\mathbf{Q}_{1,K}$ and $\mathbf{R}_{1,K}^{1,K}$ are QR factors of $\mathbf{A}_{1,K}$, as stated in Proposition 2, and since $\mathrm{rank}(\mathbf{A}_{1,K}) = K$, Properties 1 and 2, as well as the uniqueness of $\mathbf{Q}_{1,K}$ stated in Property 3, are obtained directly by applying Proposition 1 to the full-rank matrix $\mathbf{A}_{1,K}$. The uniqueness of $\mathbf{R}^{1,K}$ stated in Property 3 is implied by the uniqueness of $\mathbf{Q}_{1,K}$ and by $\mathbf{R}^{1,K} = \mathbf{Q}_{1,K}^H \mathbf{A}$, which follows from Condition 3 of Definition 1. For $k = 1, 2, \ldots, K$, $\mathrm{range}(\mathbf{Q}_{1,k}) = \mathrm{range}(\mathbf{A}_{1,k})$ is a trivial consequence of $\mathbf{A}_{1,k} = \mathbf{Q}_{1,k} \mathbf{R}_{1,k}^{1,k}$ and of $\mathrm{rank}(\mathbf{R}_{1,k}^{1,k}) = k$, which follows from the fact that $\mathbf{R}_{1,k}^{1,k}$ is upper triangular with nonzero entries on its main diagonal. This proves Property 4. If $K < M$, Condition 3 of Definition 1 implies $[\mathbf{R}]_{K+1,K+1} = \mathbf{q}_{K+1}^H \mathbf{a}_{K+1}$. If $\mathbf{q}_{K+1} = \mathbf{0}$, $[\mathbf{R}]_{K+1,K+1} = 0$ follows trivially. If $\mathbf{q}_{K+1} \neq \mathbf{0}$, Condition 1 of Definition 1 implies that $\mathbf{q}_{K+1}$ is orthogonal to $\mathrm{range}(\mathbf{Q}_{1,K})$, whereas the definition of K implies that $\mathbf{a}_{K+1} \in \mathrm{range}(\mathbf{A}_{1,K})$. Since $\mathrm{range}(\mathbf{Q}_{1,K}) = \mathrm{range}(\mathbf{A}_{1,K})$, we obtain $\mathbf{q}_{K+1}^H \mathbf{a}_{K+1} = [\mathbf{R}]_{K+1,K+1} = 0$, which proves Property 5. ■

We emphasize that for $K > 0$, the uniqueness of $\mathbf{Q}_{1,K}$ and $\mathbf{R}^{1,K}$ has two significant consequences. First, the GS orthonormalization procedure (1)–(3), evaluated for $k = 1, 2, \ldots, K$, determines the submatrices $\mathbf{Q}_{1,K}$ and $\mathbf{R}^{1,K}$ of the matrices $\mathbf{Q}$ and $\mathbf{R}$ produced by *any* QR decomposition algorithm. Second, the nonuniqueness of $\mathbf{Q}$ and $\mathbf{R}$ in the case of rank-deficient $\mathbf{A}$, demonstrated in Section II-B, is restricted to the submatrices $\mathbf{Q}_{K+1,M}$ and $\mathbf{R}^{K+1,M}$.

Finally, we note that Property 5 of Proposition 4 is valid for the case $K = 0$ as well. In fact, Condition 3 of Definition 1 implies $[\mathbf{R}]_{1,1} = \mathbf{q}_1^H \mathbf{a}_1$. Since $K = 0$ implies $\mathbf{a}_1 = \mathbf{0}$, we immediately obtain $[\mathbf{R}]_{1,1} = 0$.

B. Mapping of $\mathbf{Q}(s)$ and $\mathbf{R}(s)$ to LP Matrices

In the remainder of Section III, we consider a $P \times M$ LP matrix $\mathbf{A}(s) \sim (V_1, V_2)$, $s \in \mathcal{U}$, with $P \geq M$. Despite $\mathbf{A}(s)$ being an LP matrix, QR factors $\mathbf{Q}(s)$ and $\mathbf{R}(s)$ of $\mathbf{A}(s)$ will, in general, neither be LP nor rational matrices. To see this, consider the case where $\mathrm{rank}(\mathbf{A}(s)) = M$ for all $s \in \mathcal{U}$. It follows from the results in Sections II-B and III-A that $\mathbf{Q}(s)$ and $\mathbf{R}(s)$ are unique and determined through (1)–(3). The division and the square root operation in (2), in general, prevent $\mathbf{Q}(s)$, and hence also $\mathbf{R}(s) = \mathbf{Q}^H(s)\mathbf{A}(s)$, from being LP or rational matrices. As shown next, there exists a simple mapping that transforms $\mathbf{Q}(s)$ and $\mathbf{R}(s)$ into corresponding LP matrices $\tilde{\mathbf{Q}}(s)$ and $\tilde{\mathbf{R}}(s)$ by essentially eliminating, through a series of multiplications, the divisions and square root operations incurred when expressing $\mathbf{Q}(s)$ and $\mathbf{R}(s)$ as functions of $\mathbf{A}(s)$ according to (1)–(3). In Section IV, we will describe how this mapping enables interpolation-based QR decomposition of samples of $\mathbf{A}(s)$.

For a given $s_0 \in \mathcal{U}$, the mapping $\mathcal{M} : \mathbb{C}^{P\times M} \times \mathbb{C}^{M\times M} \to \mathbb{C}^{P\times M} \times \mathbb{C}^{M\times M}$ produces the matrices $(\tilde{\mathbf{Q}}(s_0), \tilde{\mathbf{R}}(s_0))$ from the matrices $(\mathbf{Q}(s_0), \mathbf{R}(s_0))$, according to the definition below. The functions $\tilde{\mathbf{Q}}(s)$ and $\tilde{\mathbf{R}}(s)$ are formally obtained by applying $\mathcal{M}$ to $(\mathbf{Q}(s), \mathbf{R}(s))$ pointwise for all $s \in \mathcal{U}$. We note that in order to keep the notation compact, in the following we omit the dependence of all involved quantities on s_0. We start by defining the auxiliary variables Δ_k as

$$\Delta_k \triangleq \Delta_{k-1}[\mathbf{R}]^2_{k,k}, \qquad k = 1, 2, \ldots, M \tag{5}$$

with $\Delta_0 \triangleq 1$. Next, we introduce the vectors

$$\tilde{\mathbf{q}}_k \triangleq \Delta_{k-1}\,[\mathbf{R}]_{k,k}\,\mathbf{q}_k, \qquad k = 1, 2, \ldots, M \tag{6}$$

$$\underline{\tilde{\mathbf{r}}}_k \triangleq \Delta_{k-1}\,[\mathbf{R}]_{k,k}\,\underline{\mathbf{r}}_k, \qquad k = 1, 2, \ldots, M \tag{7}$$

and define the mapping $\mathcal{M} : (\mathbf{Q}, \mathbf{R}) \mapsto (\tilde{\mathbf{Q}}, \tilde{\mathbf{R}})$ by $\tilde{\mathbf{Q}} \triangleq [\tilde{\mathbf{q}}_1\ \tilde{\mathbf{q}}_2\ \cdots\ \tilde{\mathbf{q}}_M]$ and $\tilde{\mathbf{R}} \triangleq [\underline{\tilde{\mathbf{r}}}_1^T\ \underline{\tilde{\mathbf{r}}}_2^T\ \cdots\ \underline{\tilde{\mathbf{r}}}_M^T]^T$. We emphasize that the factors $\Delta_{k-1}\,[\mathbf{R}]_{k,k}$, $k = 1, 2, \ldots, M$, in (6) and (7), are functions of the entries $[\mathbf{R}]_{k',k'}$, $k' = 1, 2, \ldots, k$, on the main diagonal of the matrix $\mathbf{R}$, which in turn is a function of s_0.

Now, we consider the ordered column rank K of $\mathbf{A}$, and note that Property 2 in Proposition 4 implies that, if $K > 0$, $\Delta_{k-1}\,[\mathbf{R}]_{k,k} > 0$ for $k = 1, 2, \ldots, K$, as seen by unfolding the recursion in (5). Hence, for $K > 0$ and $k = 1, 2, \ldots, K$, we can compute $\mathbf{q}_k$ and $\underline{\mathbf{r}}_k$ from $\tilde{\mathbf{q}}_k$ and $\underline{\tilde{\mathbf{r}}}_k$, respectively, according to

$$\mathbf{q}_k = \left(\Delta_{k-1}\left[\mathbf{R}\right]_{k,k}\right)^{-1}\tilde{\mathbf{q}}_k \tag{8}$$

$$\underline{\mathbf{r}}_k = \left(\Delta_{k-1}\left[\mathbf{R}\right]_{k,k}\right)^{-1}\underline{\tilde{\mathbf{r}}}_k \tag{9}$$

where $\Delta_{k-1}\left[\mathbf{R}\right]_{k,k}$ is obtained from the entries on the main diagonal of $\tilde{\mathbf{R}}$ as

$$\Delta_{k-1}\left[\mathbf{R}\right]_{k,k} = \begin{cases} \sqrt{[\tilde{\mathbf{R}}]_{k,k}}, & k = 1 \\ \sqrt{[\tilde{\mathbf{R}}]_{k-1,k-1}[\tilde{\mathbf{R}}]_{k,k}}, & k = 2, 3, \ldots, K. \end{cases} \tag{10}$$

If $K = M$, i.e., for full-rank $\mathbf{A}$, we have $\Delta_{k-1}\left[\mathbf{R}\right]_{k,k} \neq 0$ for all $k = 1, 2, \ldots, M$, and the mapping $\mathcal{M}$ is invertible. In the case $K < M$, Property 5 in Proposition 4 states that $[\mathbf{R}]_{K+1,K+1} = 0$, which combined with (5)–(7) implies that $\Delta_k = 0$, $\tilde{\mathbf{q}}_k = \mathbf{0}$, and $\underline{\tilde{\mathbf{r}}}_k = \mathbf{0}$ for $k = K+1, K+2, \ldots, M$. Hence, the mapping $\mathcal{M}$ is not invertible for $K < M$, since the information contained in $\mathbf{Q}_{K+1,M}$ and $\mathbf{R}^{K+1,M}$ can not be extracted from $\tilde{\mathbf{Q}}_{K+1,M} = \mathbf{0}$ and $\tilde{\mathbf{R}}^{K+1,M} = \mathbf{0}$. Nevertheless, we can recover $\mathbf{Q}_{K+1,M}$ and $\mathbf{R}^{K+1,M}$ as follows. For $0 < K < M$, setting $k = K+1$ in Proposition 3 shows that $\mathbf{Q}_{K+1,M}$ and $\mathbf{R}^{K+1,M}_{K+1,M}$ can be obtained by QR decomposition of $\mathbf{A}_{K+1,M} - \mathbf{Q}_{1,K}\mathbf{R}^{1,K}_{K+1,M}$. Then, $\mathbf{R}^{K+1,M}$ is obtained as $\mathbf{R}^{K+1,M} = [\mathbf{R}^{K+1,M}_{1,K} \;\; \mathbf{R}^{K+1,M}_{K+1,M}]$ with $\mathbf{R}^{K+1,M}_{1,K} = \mathbf{0}$ because of the upper triangularity of $\mathbf{R}$. For $K = 0$, since $\tilde{\mathbf{Q}}$ and $\tilde{\mathbf{R}}$ are all-zero matrices, $\mathbf{Q}_{K+1,M} = \mathbf{Q}$ and $\mathbf{R}^{K+1,M}_{K+1,M} = \mathbf{R}$ must be obtained by performing QR decomposition on $\mathbf{A}$. In the remainder of the paper, we denote by *inverse mapping* $\mathcal{M}^{-1} : (\tilde{\mathbf{Q}}, \tilde{\mathbf{R}}) \mapsto (\mathbf{Q}, \mathbf{R})$ the procedure[1] formulated in the following steps:

1) If $K > 0$, for $k = 1, 2, \ldots, K$, compute the scaling factor $\left(\Delta_{k-1}\left[\mathbf{R}\right]_{k,k}\right)^{-1}$ using (10) and scale $\tilde{\mathbf{q}}_k$ and $\underline{\tilde{\mathbf{r}}}_k$ according to (8) and (9), respectively.
2) If $0 < K < M$, compute $\mathbf{Q}_{K+1,M}$ and $\mathbf{R}^{K+1,M}_{K+1,M}$ by performing QR decomposition on $\mathbf{A}_{K+1,M} - \mathbf{Q}_{1,K}\mathbf{R}^{1,K}_{K+1,M}$, and construct $\mathbf{R}^{K+1,M} = [\mathbf{0} \;\; \mathbf{R}^{K+1,M}_{K+1,M}]$.
3) If $K = 0$, compute $\mathbf{Q}$ and $\mathbf{R}$ by performing QR decomposition on $\mathbf{A}$.

We note that the nonuniqueness of QR decomposition in the case $K < M$ has the following consequence. Given QR factors $\mathbf{Q}_1$ and $\mathbf{R}_1$ of $\mathbf{A}$, the application of the mapping $\mathcal{M}$ to $(\mathbf{Q}_1, \mathbf{R}_1)$ followed by application of the inverse mapping $\mathcal{M}^{-1}$ yields matrices $\mathbf{Q}_2$ and $\mathbf{R}_2$ that may not be equal to $\mathbf{Q}_1$ and $\mathbf{R}_1$, respectively. However, $\mathbf{Q}_2$ and $\mathbf{R}_2$ are QR factors of $\mathbf{A}$ in the sense of Definition 1.

We are now ready to present the main result of this paper.

[1] Note that for $K < M$, the inverse mapping $\mathcal{M}^{-1}$ requires explicit knowledge of $\mathbf{A}_{K+1,M}$.

Theorem 1: Given $\mathbf{A} : \mathcal{U} \rightarrow \mathbb{C}^{P\times M}$ with $P \geq M$, such that $\mathbf{A}(s) \sim (V_1, V_2)$ with maximum degree $V = V_1 + V_2$. The functions $\Delta_k(s)$, $\tilde{\mathbf{q}}_k(s)$, and $\underline{\tilde{\mathbf{r}}}_k(s)$, obtained by applying the mapping $\mathcal{M}$ as in (5)–(7) to QR factors $\mathbf{Q}(s)$ and $\mathbf{R}(s)$ of $\mathbf{A}(s)$ for all $s \in \mathcal{U}$, satisfy the following properties:

1) $\Delta_k(s) \sim (kV, kV)$
2) $\tilde{\mathbf{q}}_k(s) \sim ((k-1)V + V_1, (k-1)V + V_2)$
3) $\underline{\tilde{\mathbf{r}}}_k(s) \sim (kV, kV)$.

The results stated in Theorem 1 show that despite $\mathbf{q}_k(s)$ and $\underline{\mathbf{r}}_k(s)$ being neither LP nor rational, the multiplication with $\Delta_{k-1}(s)[\mathbf{R}(s)]_{k,k}$ (which is also neither LP nor rational) in (6) and (7) yields functions $\tilde{\mathbf{q}}_k(s)$ and $\underline{\tilde{\mathbf{r}}}_k(s)$, respectively, that are LP ($k = 1, 2, \ldots, M$). Finally, we emphasize that Theorem 1 applies to any QR factors satisfying Definition 1 and is therefore not affected by the nonuniqueness of QR decomposition arising in the rank-deficient case.

C. *Proof of Theorem 1*

The proof consists of three steps, summarized as follows. In Step 1, we focus on a given $s_0 \in \mathcal{U}$ and aim at writing $\Delta_k(s_0)$, $\tilde{\mathbf{q}}_k(s_0)$, and $\underline{\tilde{\mathbf{r}}}_k(s_0)$ as functions of $\mathbf{A}(s_0)$ for all $(K(s_0), k) \in \mathcal{K} \triangleq \{0, 1, \ldots, M\} \times \{1, 2, \ldots, M\}$, where $K(s_0)$ denotes the ordered column rank of $\mathbf{A}(s_0)$. Step 1 is split into Steps 1a and 1b, in which the two disjoint subsets $\mathcal{K}_1 \triangleq \{(K', k') \in \mathcal{K} : 0 < K' \leq M, 1 \leq k' \leq K'\}$ and $\mathcal{K}_2 \triangleq \{(K', k') \in \mathcal{K} : 0 \leq K' < M, K' + 1 \leq k' \leq M\}$ (with $\mathcal{K}_1 \cup \mathcal{K}_2 = \mathcal{K}$) are considered, respectively. In Step 1a, we note that for $(K(s_0), k) \in \mathcal{K}_1$, $\mathbf{Q}_{1,K(s_0)}(s_0)$ and $\mathbf{R}^{1,K(s_0)}(s_0)$ are unique and can be obtained by evaluating (1)–(3) for $k = 1, 2, \ldots, K(s_0)$. By unfolding the recursions in (1)–(3) and in (5)–(7), we write $\Delta_k(s_0)$, $\tilde{\mathbf{q}}_k(s_0)$, and $\underline{\tilde{\mathbf{r}}}_k(s_0)$ as functions of $\mathbf{A}(s_0)$ for $(K(s_0), k) \in \mathcal{K}_1$. In Step 1b, we show that the expressions for $\Delta_k(s_0)$, $\tilde{\mathbf{q}}_k(s_0)$, and $\underline{\tilde{\mathbf{r}}}_k(s_0)$, derived in Step 1a for $(K(s_0), k) \in \mathcal{K}_1$, are also valid for $(K(s_0), k) \in \mathcal{K}_2$ and hence, as a consequence of $\mathcal{K}_1 \cup \mathcal{K}_2 = \mathcal{K}$, for all $(K(s_0), k) \in \mathcal{K}$. In Step 2, we note that the derivations in Step 1 carry over to all $s_0 \in \mathcal{U}$, and generalize the expressions obtained in Step 1 to expressions for $\Delta_k(s)$, $\tilde{\mathbf{q}}_k(s)$, and $\underline{\tilde{\mathbf{r}}}_k(s)$ that hold for $k = 1, 2, \ldots, M$ and for all $s \in \mathcal{U}$. Making use of $\mathbf{A}(s) \sim (V_1, V_2)$, in Step 3 it is finally shown that $\Delta_k(s)$, $\tilde{\mathbf{q}}_k(s)$, and $\underline{\tilde{\mathbf{r}}}_k(s)$ satisfy Properties 1–3 in the statement of Theorem 1.

Step 1a: Throughout Steps 1a and 1b, in order to simplify the notation, we drop the dependence of all quantities on s_0. In Step 1a, we assume that $(K, k) \in \mathcal{K}_1$ and, unless stated otherwise, all equations and statements involving k are valid for all $k = 1, 2, \ldots, K$.

We start by listing preparatory results. We recall from Section III-A that the submatrices $\mathbf{Q}_{1,K}$ and $\mathbf{R}^{1,K}$ are unique and that, consequently, $\mathbf{q}_k$ and $\underline{\mathbf{r}}_k$ are determined by (1)–(3). From $\mathbf{q}_k \neq \mathbf{0}$, implied by

Property 1 in Proposition 4, and from (2) we deduce that $\mathbf{y}_k \neq \mathbf{0}$. Then, from (1) and (2) we obtain

$$\mathbf{y}_k^H \mathbf{y}_k = \mathbf{y}_k^H \mathbf{a}_k - \sum_{i=1}^{k-1} \mathbf{q}_i^H \mathbf{a}_k \sqrt{\mathbf{y}_k^H \mathbf{y}_k} \mathbf{q}_k^H \mathbf{q}_i = \mathbf{y}_k^H \mathbf{a}_k \tag{11}$$

as $\mathbf{q}_k^H \mathbf{q}_i = 0$ for $i = 1, 2, \ldots, k-1$. Consequently, we can write $[\mathbf{R}]_{k,k}$, using (2) and (3), as

$$[\mathbf{R}]_{k,k} = \mathbf{q}_k^H \mathbf{a}_k = \frac{\mathbf{y}_k^H \mathbf{a}_k}{\sqrt{\mathbf{y}_k^H \mathbf{y}_k}} = \sqrt{\mathbf{y}_k^H \mathbf{y}_k} \tag{12}$$

thus implying $[\mathbf{R}]_{k,k} \mathbf{q}_k = \mathbf{y}_k$ and hence, by (6),

$$\tilde{\mathbf{q}}_k = \Delta_{k-1} \mathbf{y}_k. \tag{13}$$

Furthermore, using (5) and (12), we can write $\Delta_k = \Delta_{k-1} \mathbf{y}_k^H \mathbf{y}_k$ or alternatively, in recursion-free form,

$$\Delta_k = \prod_{i=1}^{k} \mathbf{y}_i^H \mathbf{y}_i. \tag{14}$$

Next, we note that (1) implies

$$\mathbf{y}_k = \mathbf{a}_k + \sum_{i=1}^{k-1} \alpha_i^{(k)} \mathbf{a}_i \tag{15}$$

with unique coefficients $\alpha_i^{(k)}, i = 1, 2, \ldots, k-1$, since $\mathbf{y}_1 = \mathbf{a}_1$ and since for $k > 1$, we have $\operatorname{rank}(\mathbf{A}_{1,k-1}) = k-1$ and, as stated in Property 4 of Proposition 4, $\operatorname{range}(\mathbf{Q}_{1,k-1}) = \operatorname{range}(\mathbf{A}_{1,k-1})$. Next, we consider the relation between $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\}$. Inserting (2) into (1) yields

$$\mathbf{y}_k = \mathbf{a}_k - \sum_{i=1}^{k-1} \frac{\mathbf{y}_i^H \mathbf{a}_k}{\mathbf{y}_i^H \mathbf{y}_i} \mathbf{y}_i.$$

Hence, using (11), we obtain

$$\begin{aligned}\mathbf{a}_{k'} &= \mathbf{y}_{k'} + \sum_{i=1}^{k'-1} \frac{\mathbf{y}_i^H \mathbf{a}_{k'}}{\mathbf{y}_i^H \mathbf{y}_i} \mathbf{y}_i \\ &= \sum_{i=1}^{k'} \frac{\mathbf{y}_i^H \mathbf{a}_{k'}}{\mathbf{y}_i^H \mathbf{y}_i} \mathbf{y}_i, \qquad k' = 1, 2, \ldots, k.\end{aligned} \tag{16}$$

We next note that (16) can be rewritten, for $k' = 1, 2, \ldots, k$, in vector-matrix form as

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_k \end{bmatrix} = \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_k \end{bmatrix} \mathbf{V}_k \tag{17}$$

with the $k \times k$ matrix

$$\mathbf{V}_k \triangleq \begin{bmatrix} \frac{\mathbf{y}_1^H \mathbf{a}_1}{\mathbf{y}_1^H \mathbf{y}_1} & \frac{\mathbf{y}_1^H \mathbf{a}_2}{\mathbf{y}_1^H \mathbf{y}_1} & \cdots & \frac{\mathbf{y}_1^H \mathbf{a}_k}{\mathbf{y}_1^H \mathbf{y}_1} \\ 0 & \frac{\mathbf{y}_2^H \mathbf{a}_2}{\mathbf{y}_2^H \mathbf{y}_2} & \cdots & \frac{\mathbf{y}_2^H \mathbf{a}_k}{\mathbf{y}_2^H \mathbf{y}_2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\mathbf{y}_k^H \mathbf{a}_k}{\mathbf{y}_k^H \mathbf{y}_k} \end{bmatrix}$$

satisfying $\det(\mathbf{V}_k) = 1$ because of $\mathbf{y}_k \neq \mathbf{0}$ and of (11). Next, we can write $\mathbf{V}_k$ as $\mathbf{V}_k = \mathbf{D}_k^{-1}\mathbf{U}_k$ with the $k \times k$ nonsingular matrices $\mathbf{D}_k \triangleq \mathrm{diag}(\mathbf{y}_1^H\mathbf{y}_1, \mathbf{y}_2^H\mathbf{y}_2, \ldots, \mathbf{y}_k^H\mathbf{y}_k)$ and

$$\mathbf{U}_k \triangleq \begin{bmatrix} \mathbf{y}_1^H\mathbf{a}_1 & \mathbf{y}_1^H\mathbf{a}_2 & \cdots & \mathbf{y}_1^H\mathbf{a}_k \\ 0 & \mathbf{y}_2^H\mathbf{a}_2 & \cdots & \mathbf{y}_2^H\mathbf{a}_k \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{y}_k^H\mathbf{a}_k \end{bmatrix}. \tag{18}$$

We next express Δ_k as a function of $\mathbf{A}_{1,k}$. From (11), (14), and (18), we obtain

$$\Delta_k = \prod_{i=1}^{k} \mathbf{y}_i^H\mathbf{a}_i = \det(\mathbf{U}_k). \tag{19}$$

Furthermore, (2), (3), and (12) imply

$$\mathbf{y}_{k'}^H\mathbf{a}_i = \sqrt{\mathbf{y}_{k'}^H\mathbf{y}_{k'}}\mathbf{q}_{k'}^H\mathbf{a}_i = [\mathbf{R}]_{k',k'}[\mathbf{R}]_{k',i}$$

which evaluates to zero for $1 \leq i < k' \leq k$ because of the upper triangularity of $\mathbf{R}$. Hence, $\mathbf{U}_k$ can be written as

$$\mathbf{U}_k = \begin{bmatrix} \mathbf{y}_1^H\mathbf{a}_1 & \mathbf{y}_1^H\mathbf{a}_2 & \cdots & \mathbf{y}_1^H\mathbf{a}_k \\ \mathbf{y}_2^H\mathbf{a}_1 & \mathbf{y}_2^H\mathbf{a}_2 & \cdots & \mathbf{y}_2^H\mathbf{a}_k \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}_k^H\mathbf{a}_1 & \mathbf{y}_k^H\mathbf{a}_2 & \cdots & \mathbf{y}_k^H\mathbf{a}_k \end{bmatrix}. \tag{20}$$

By combining (19) and (20), we obtain

$$\Delta_k = \det(\mathbf{U}_k) = \det \begin{bmatrix} \mathbf{y}_1^H\mathbf{A}_{1,k} \\ \mathbf{y}_2^H\mathbf{A}_{1,k} \\ \vdots \\ \mathbf{y}_k^H\mathbf{A}_{1,k} \end{bmatrix} = \det \begin{bmatrix} \mathbf{a}_1^H\mathbf{A}_{1,k} \\ \mathbf{a}_2^H\mathbf{A}_{1,k} \\ \vdots \\ \mathbf{a}_k^H\mathbf{A}_{1,k} \end{bmatrix} \tag{21}$$

$$= \det\big(\mathbf{A}_{1,k}^H\mathbf{A}_{1,k}\big) \tag{22}$$

where the third equality in (21) can be shown by induction as follows. We start by noting that $\mathbf{y}_1 = \mathbf{a}_1$, which implies that in the first row of $\mathbf{U}_k$, $\mathbf{y}_1$ can be replaced by $\mathbf{a}_1$. For $k' > 1$, assuming that we have already replaced $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{k'-1}$ by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{k'-1}$, respectively, we can replace $\mathbf{y}_{k'}$ by $\mathbf{a}_{k'}$ since, as a consequence of (15), the k'th row of $\mathbf{U}_k$ can be written as

$$\mathbf{y}_{k'}^H\mathbf{A}_{1,k} = \mathbf{a}_{k'}^H\mathbf{A}_{1,k} + \sum_{i=1}^{k'-1} \big(\alpha_i^{(k')}\big)^* \big(\mathbf{a}_i^H\mathbf{A}_{1,k}\big).$$

Hence, replacing $\mathbf{y}_{k'}^H \mathbf{A}_{1,k}$ by $\mathbf{a}_{k'}^H \mathbf{A}_{1,k}$ amounts to subtracting a linear combination of the first $k'-1$ rows of $\mathbf{U}_k$ from the k'th row of $\mathbf{U}_k$. This operation does not affect the value of $\det(\mathbf{U}_k)$ [13].

Similarly to what we have done for Δ_k, we will next show that $\tilde{\mathbf{q}}_k$ can be expressed in terms of $\mathbf{A}_{1,k}$ only. We start by noting that, since $\mathbf{V}_k$ is nonsingular, we can rewrite (17) as

$$\begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_k \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_k \end{bmatrix} \mathbf{V}_k^{-1}. \tag{23}$$

Next, from $\mathbf{V}_k = \mathbf{D}_k^{-1}\mathbf{U}_k$ we obtain that

$$\mathbf{V}_k^{-1} = \mathbf{U}_k^{-1}\mathbf{D}_k = \frac{\mathrm{adj}(\mathbf{U}_k)}{\det(\mathbf{U}_k)}\mathbf{D}_k$$

and hence, by (19), that

$$\mathbf{V}_k^{-1} = \frac{1}{\Delta_k} \underbrace{\begin{bmatrix} \Gamma_{1,1}^{(k)} & \Gamma_{2,1}^{(k)} & \cdots & \Gamma_{k,1}^{(k)} \\ 0 & \Gamma_{2,2}^{(k)} & \cdots & \Gamma_{k,2}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Gamma_{k,k}^{(k)} \end{bmatrix}}_{\mathrm{adj}(\mathbf{U}_k)} \mathbf{D}_k \tag{24}$$

where $\mathrm{adj}(\mathbf{U}_k)$ is upper triangular since $\mathbf{U}_k$ is upper triangular, and $\Gamma_{n,m}^{(k)}$ denotes the cofactor of $\mathbf{U}_k$ relative to the matrix entry $[\mathbf{U}_k]_{n,m}$ ($n = 1, 2, \ldots, k$; $m = n, n+1, \ldots, k$) [13]. Note that in order to handle the case $k = 1$ correctly, for which $\mathrm{adj}(\mathbf{U}_1) = \Gamma_{1,1}^{(1)}$, $\det(\mathbf{U}_1) = \mathbf{U}_1 = \Delta_1$, and $\mathbf{U}_1^{-1} = 1/\Delta_1$, we define $\Gamma_{1,1}^{(1)} \triangleq 1$. From (23) and (24) it follows that

$$\mathbf{y}_k = \frac{1}{\Delta_k}\mathbf{y}_k^H \mathbf{y}_k \sum_{i=1}^{k} \Gamma_{k,i}^{(k)} \mathbf{a}_i = \frac{1}{\Delta_{k-1}} \sum_{i=1}^{k} \Gamma_{k,i}^{(k)} \mathbf{a}_i$$

and therefore, by (13), we get

$$\tilde{\mathbf{q}}_k = \sum_{i=1}^{k} \Gamma_{k,i}^{(k)} \mathbf{a}_i \tag{25}$$

which evaluates to $\tilde{\mathbf{q}}_1 = \mathbf{a}_1$ for $k = 1$. Next, for $k > 1$ we denote by $\mathbf{A}_{1,k\backslash i}$ the matrix obtained by removing the ith column from $\mathbf{A}_{1,k}$, and we express $\Gamma_{k,i}^{(k)}$ as a function of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ according to

$$\Gamma_{k,i}^{(k)} = (-1)^{k+i} \det \begin{bmatrix} \mathbf{y}_1^H \mathbf{A}_{1,k\backslash i} \\ \mathbf{y}_2^H \mathbf{A}_{1,k\backslash i} \\ \vdots \\ \mathbf{y}_{k-1}^H \mathbf{A}_{1,k\backslash i} \end{bmatrix} = (-1)^{k+i} \det\left(\mathbf{A}_{1,k-1}^H \mathbf{A}_{1,k\backslash i}\right)$$

where the last equality is derived analogously to (21) and (22). Thus, (25) can be written as

$$\tilde{\mathbf{q}}_k = \begin{cases} \mathbf{a}_k, & k = 1 \\ \sum_{i=1}^{k} (-1)^{k+i} \det(\mathbf{A}_{1,k-1}^H \mathbf{A}_{1,k\setminus i}) \mathbf{a}_i, & k > 1. \end{cases} \tag{26}$$

Finally, we obtain

$$\underline{\tilde{\mathbf{r}}}_k = \tilde{\mathbf{q}}_k^H \mathbf{A} \tag{27}$$

as implied by (3), (6), and (7). The results of Step 1a are the relations (22), (26), and (27), which are valid for $(K, k) \in \mathcal{K}_1$.

Step 1b: We next show that (22), (26), and (27) hold for $(K, k) \in \mathcal{K}_2$ as well. Throughout Step 1b we assume that $(K, k) \in \mathcal{K}_2$, and, unless specified otherwise, all equations and statements involving k are valid for $k = K+1, K+2, \dots, M$. We know from Section III-A that $[\mathbf{R}]_{K+1,K+1} = 0$. According to the definition of $\mathcal{M}$, $[\mathbf{R}]_{K+1,K+1} = 0$ implies $\Delta_k = 0$, $\tilde{\mathbf{q}}_k = \mathbf{0}$, and $\underline{\tilde{\mathbf{r}}}_k = \mathbf{0}$. It is therefore to be shown that the RHS of (22) evaluates to zero, and that the RHS expressions of (26) and (27) evaluate to all-zero vectors. We start by noting that since $k > K$, $\mathbf{A}_{1,k}$ is rank-deficient. Since $\operatorname{rank}(\mathbf{A}_{1,k}^H \mathbf{A}_{1,k}) = \operatorname{rank}(\mathbf{A}_{1,k}) < k$, we obtain that $\det(\mathbf{A}_{1,k}^H \mathbf{A}_{1,k})$ on the RHS of (22) evaluates to zero. Next, for $k > \max(K, 1)$, the expression

$$\sum_{i=1}^{k} (-1)^{k+i} \det\bigl(\mathbf{A}_{1,k-1}^H \mathbf{A}_{1,k\setminus i}\bigr) \mathbf{a}_i \tag{28}$$

on the RHS of (26) is a vector whose pth component can be written, by inverse Laplace expansion [13], as

$$\sum_{i=1}^{k} (-1)^{k+i} \det\bigl(\mathbf{A}_{1,k-1}^H \mathbf{A}_{1,k\setminus i}\bigr) [\mathbf{A}]_{p,i} = \det \begin{bmatrix} \mathbf{A}_{1,k-1}^H \mathbf{a}_1 & \mathbf{A}_{1,k-1}^H \mathbf{a}_2 & \cdots & \mathbf{A}_{1,k-1}^H \mathbf{a}_k \\ [\mathbf{A}]_{p,1} & [\mathbf{A}]_{p,2} & \cdots & [\mathbf{A}]_{p,k} \end{bmatrix} \tag{29}$$

for all $p = 1, 2, \dots, P$. Now, again for $k > \max(K, 1)$, since $\mathbf{A}_{1,k}$ is rank-deficient, $\mathbf{a}_k$ can be written as a linear combination

$$\mathbf{a}_k = \sum_{k'=1}^{k-1} \beta^{(k')} \mathbf{a}_{k'}$$

(for some coefficients $\beta^{(k')}$, $k' = 1, 2, \dots, k-1$) which implies that, for all $p = 1, 2, \dots, P$, the argument of the determinant on the RHS of (29) has

$$\begin{bmatrix} \mathbf{A}_{1,k-1}^H \mathbf{a}_k \\ [\mathbf{A}]_{p,k} \end{bmatrix} = \sum_{k'=1}^{k-1} \beta^{(k')} \begin{bmatrix} \mathbf{A}_{1,k-1}^H \mathbf{a}_{k'} \\ [\mathbf{A}]_{p,k'} \end{bmatrix}$$

as its last column. Since this column is a linear combination of the first $k-1$ columns, the determinant on the RHS of (29) is equal to zero for all $p = 1, 2, \dots, P$, and hence the expression in (28) is equal to an all-zero vector for $k > \max(K, 1)$. Moreover, if $K = 0$ and $k = 1$, we have $\mathbf{a}_1 = \mathbf{0}$ on the RHS of (26). Hence, the RHS of (26) evaluates to an all-zero vector for all $(K, k) \in \mathcal{K}_2$. Thus, (26) simplifies to $\tilde{\mathbf{q}}_k = \mathbf{0}$, which in turn implies that the RHS of (27) evaluates to an all-zero vector as well. We have therefore shown that (22), (26), and (27) hold for $(K, k) \in \mathcal{K}_2$. Finally, since $\mathcal{K}_1 \cup \mathcal{K}_2 = \mathcal{K}$, the results of Steps 1a and 1b imply that (22), (26), and (27) are valid for $(K, k) \in \mathcal{K}$.

Step 2: We note that the derivations presented in Steps 1a and 1b for a given $s_0 \in \mathcal{U}$ do not depend on s_0 and can hence be carried over to all $s_0 \in \mathcal{U}$. Thus, we can rewrite (22), (26), and (27), respectively, as

$$\Delta_k(s) = \det\left(\mathbf{A}_{1,k}^H(s)\mathbf{A}_{1,k}(s)\right) \tag{30}$$

$$\tilde{\mathbf{q}}_k(s) = \begin{cases} \mathbf{a}_k(s), & k = 1 \\ \sum_{i=1}^{k} (-1)^{k+i} \det(\mathbf{A}_{1,k-1}^H(s)\mathbf{A}_{1,k\setminus i}(s))\mathbf{a}_i(s), & k > 1 \end{cases} \tag{31}$$

$$\underline{\tilde{\mathbf{r}}}_k(s) = \tilde{\mathbf{q}}_k^H(s)\mathbf{A}(s) \tag{32}$$

for $k = 1, 2, \ldots, M$ and $s \in \mathcal{U}$.

Step 3: For $k = 1, 2, \ldots, M$, we note that $\mathbf{A}(s) \sim (V_1, V_2)$, along with $V = V_1 + V_2$, implies $\mathbf{A}_{1,k}^H(s)\mathbf{A}_{1,k}(s) \sim (V, V)$. Now, the determinant on the RHS of (30) can be expressed through Laplace expansion as a sum of products of k entries of $\mathbf{A}_{1,k}^H(s)\mathbf{A}_{1,k}(s) \sim (V, V)$. Therefore, we get $\Delta_k(s) \sim (kV, kV)$ for $k = 1, 2, \ldots, M$. By analogous arguments, for $k = 2, 3, \ldots, M$ we obtain $\det(\mathbf{A}_{1,k-1}^H(s)\mathbf{A}_{1,k\setminus i}(s)) \sim ((k-1)V, (k-1)V)$. The latter result, combined with $\mathbf{a}_i(s) \sim (V_1, V_2)$ in (31) yields $\tilde{\mathbf{q}}_k(s) \sim ((k-1)V + V_1, (k-1)V + V_2)$, which holds for $k = 1$ as well as a trivial consequence of (31) and $\mathbf{a}_1(s) \sim (V_1, V_2)$. Finally, from (32) and $\tilde{\mathbf{q}}_k(s) \sim ((k-1)V + V_1, (k-1)V + V_2)$, using $\mathbf{A}(s) \sim (V_1, V_2)$ and $V = V_1 + V_2$, we obtain $\underline{\tilde{\mathbf{r}}}_k(s) \sim (kV, kV)$ for $k = 1, 2, \ldots, M$. ∎

D. Extension to Regularized QR Decomposition

In the following, we briefly show how the results of Section III-B can be extended to the regularized QR decomposition of $\mathbf{A}(s) \sim (V_1, V_2)$. To this end, we consider the augmented matrix $\bar{\mathbf{A}}(s) = [\mathbf{A}^T(s) \quad \alpha\mathbf{I}_M]^T$ with the QR factors $\bar{\mathbf{Q}}(s)$ and $\mathbf{R}(s)$. From $\alpha\mathbf{I}_M \sim (0, 0)$ and $\mathbf{A}(s) \sim (V_1, V_2)$ we get $\bar{\mathbf{A}}(s) \sim (V_1, V_2)$. Therefore, the results in Section III-B (including, in particular, Theorem 1 and the definition of $\mathcal{M}$ and $\mathcal{M}^{-1}$) can be applied to the QR decomposition $\bar{\mathbf{A}}(s) = \bar{\mathbf{Q}}(s)\mathbf{R}(s)$.

The following discussion pertains to a single sample $\mathbf{A}(s_0)$, $s_0 \in \mathcal{U}$. To keep the notation compact, as done in Section III-B, we omit the dependence of all involved quantities on s_0. Since $\mathbf{Q} = \bar{\mathbf{Q}}^{1,P}$, by Definition 2, applying the mapping $\mathcal{M}$ to the regularized QR factors $\mathbf{Q}$ and $\mathbf{R}$ of $\mathbf{A}$ according to (5)–(7) yields the same results as the application of $\mathcal{M}$ to the QR factors $\bar{\mathbf{Q}}$ and $\mathbf{R}$ of the augmented matrix $\bar{\mathbf{A}} = [\,\mathbf{A}^T \quad \alpha\mathbf{I}_M\,]^T$ to obtain $\tilde{\bar{\mathbf{Q}}}$ and $\tilde{\mathbf{R}}$ followed by extracting $\tilde{\mathbf{Q}} = \tilde{\bar{\mathbf{Q}}}^{1,P}$. With this insight, it is straightforward to restate Theorem 1 for the case of regularized QR decomposition.

Corollary 1: Given $\mathbf{A} : \mathcal{U} \to \mathbb{C}^{P\times M}$ with $P \geq M$, such that $\mathbf{A}(s) \sim (V_1, V_2)$ with maximum degree $V = V_1 + V_2$, and the regularization parameter $\alpha > 0$. The functions $\Delta_k(s)$, $\tilde{\mathbf{q}}_k(s)$, and $\underline{\tilde{\mathbf{r}}}_k(s)$, obtained by applying the mapping $\mathcal{M}$ as in (5)–(7) to the regularized QR factors $\mathbf{Q}(s)$ and $\mathbf{R}(s)$ of $\mathbf{A}(s)$ for all $s \in \mathcal{U}$, satisfy the following properties:

1) $\Delta_k(s) \sim (kV, kV)$
2) $\tilde{\mathbf{q}}_k(s) \sim ((k-1)V + V_1, (k-1)V + V_2)$
3) $\underline{\tilde{\mathbf{r}}}_k(s) \sim (kV, kV)$.

Finally, we mention that, since the augmented matrix $\bar{\mathbf{A}}$ has full rank, the derivation in Section III-B implies that the mapping $\mathcal{M}$ is invertible and that the regularized QR factors $\mathbf{Q} = \bar{\mathbf{Q}}^{1,P}$ and $\mathbf{R}$ of the matrix $\mathbf{A}$ can be obtained from $\tilde{\mathbf{Q}} = \tilde{\bar{\mathbf{Q}}}^{1,P}$ and $\tilde{\mathbf{R}}$ using (8)–(10).

IV. Discussion

In this section, we briefly discuss the application of our main result to interpolation-based QR decomposition of LP matrices. For simplicity, we restrict ourselves to QR decomposition, noting that the extension to regularized QR decomposition is straightforward.

Theorem 1 implies that the LP matrices $\tilde{\mathbf{Q}}(s)$ and $\tilde{\mathbf{R}}(s)$ have maximum degree $(2M-1)V$ and $2MV$, respectively. Knowledge of $2MV + 1$ samples of the LP matrices $\tilde{\mathbf{Q}}(s)$ and $\tilde{\mathbf{R}}(s)$ is therefore sufficient for computing the samples $\tilde{\mathbf{Q}}(s_0)$ and $\tilde{\mathbf{R}}(s_0)$ for any $s_0 \in \mathcal{U}$ by interpolation. Motivated by this insight, we can formulate an interpolation-based algorithm that computes QR factors of $N > 2MV + 1$ samples of $\mathbf{A}(s)$ on $\mathcal{U}$. The algorithm starts by performing QR decomposition on $2MV + 1$ samples of $\mathbf{A}(s)$, and then applies the mapping $\mathcal{M}$ to the resulting QR factors to obtain $2MV + 1$ corresponding samples of $\tilde{\mathbf{Q}}(s)$ and $\tilde{\mathbf{R}}(s)$. QR factors $\mathbf{Q}(s_0)$ and $\mathbf{R}(s_0)$ of every additional sample $\mathbf{A}(s_0)$, $s_0 \in \mathcal{U}$, can be obtained by first computing $\tilde{\mathbf{Q}}(s_0)$ and $\tilde{\mathbf{R}}(s_0)$ through interpolation from the $2MV + 1$ known samples of $\tilde{\mathbf{Q}}(s)$ and $\tilde{\mathbf{R}}(s)$, and then applying the inverse mapping $\mathcal{M}^{-1}$ to $\tilde{\mathbf{Q}}(s_0)$ and $\tilde{\mathbf{R}}(s_0)$. This algorithm finds applications in the context of transmit precoding schemes and detection algorithms for MIMO-OFDM systems [1], which involve computing QR factors of a sequence of matrices obtained by oversampling a polynomial matrix on the unit circle. Interpolation-based QR decomposition was shown in [8], using a complexity metric relevant for very large scale integration (VLSI) implementations, to yield (often significant) computational complexity savings over per-sample QR decomposition, for a wide range of MIMO-OFDM system parameters.

The fact that the maximum degrees of the LP matrices $\tilde{\mathbf{Q}}(s)$ and $\tilde{\mathbf{R}}(s)$ produced by the mapping $\mathcal{M}$ are $(2M-1)V$ and $2MV$, respectively, although the matrix $\mathbf{A}(s)$ has maximum degree V, naturally raises the question of whether the mapping $\mathcal{M}$ is optimal in the sense of producing LP matrices with the lowest possible maximum degrees. We note that an alternative mapping can be formulated on the basis of the scaled and decoupled QR decomposition proposed in [14]. Specifically, by extending the results in [14], formulated for a constant matrix $\mathbf{A}$, to an LP matrix $\mathbf{A}(s)$, we obtain a mapping from $\mathbf{Q}(s)$ and $\mathbf{R}(s)$ to LP matrices. These LP matrices have significantly higher maximum degrees than the LP matrices $\tilde{\mathbf{Q}}(s)$ and $\tilde{\mathbf{R}}(s)$.

We note that since $\tilde{\mathbf{q}}_k(s)$ and $\underline{\tilde{\mathbf{r}}}_k(s)$ have maximum degree $2(k-1)V$ and $2kV$, respectively, the number of samples of $\tilde{\mathbf{q}}_k(s)$ and $\underline{\tilde{\mathbf{r}}}_k(s)$ needed for interpolation can be reduced, compared to the approach described above, from $2MV+1$ to $2kV+1$. An interpolation-based QR decomposition algorithm making use of this fact is presented in [8].

We conclude by mentioning that the key ideas reported in this paper were shown in [15] to carry over to LU decomposition of LP matrices.

Acknowledgments

The authors would like to thank Jan Hansen for his contributions in early stages of this work and Patrick Kuppinger for many helpful discussions.

References

[1] A. J. Paulraj, R. U. Nabar, and D. A. Gore, *Introduction to Space-Time Wireless Communications*. Cambridge, U.K.: Cambridge Univ. Press, 2003.

[2] M. Sandell, "Design and analysis of estimators for multicarrier modulation and ultrasonic imaging," Ph.D. dissertation, Luleå University of Technology, Luleå, Sweden, 1996.

[3] P. Wolniansky, G. Foschini, G. Golden, and R. Valenzuela, "V-BLAST: An architecture for realizing very high data rates over the rich-scattering wireless channel," in *Proc. URSI Symp. Signals, Syst., Electron. (ISSSE)*, Pisa, Italy, Oct. 1998, pp. 295–300.

[4] B. Hassibi, "An efficient square-root algorithm for BLAST," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process. (ICASSP)*, vol. 2, Istanbul, Turkey, June 2000, pp. 737–740.

[5] U. Fincke and M. Pohst, "Improved methods for calculating vectors of short length in a lattice, including a complexity analysis," *Math. Comp.*, vol. 44, no. 170, pp. 463–471, Apr. 1985.

[6] E. Viterbo and E. Biglieri, "A universal decoding algorithm for lattice codes," in *Proc. GRETSI Symp. Signal and Image Process.*, Juan-les-Pins, France, Sep. 1993, pp. 611–614.

[7] C. Windpassinger, R. F. H. Fischer, T. Vencel, and J. B. Huber, "Precoding in multi-antenna and multi-user communication," *IEEE Trans. Wireless Commun.*, vol. 3, no. 4, pp. 1305–1316, July 2004.

[8] D. Cescato and H. Bölcskei, "Algorithms for interpolation-based QR decomposition in MIMO-OFDM systems," *IEEE Trans. Signal Process.*, 2009, submitted.

[9] Q. T. Nam, Y. Otha, and T. Matsumoto, "Inversion of rational matrices by using FFT algorithm," *Trans. IECE*, vol. J61-A, no. 9, pp. 732–733, 1978, in Japanese.

[10] A. Schuster and P. Hippe, "Inversion of polynomial matrices by interpolation," *IEEE Trans. Autom. Control*, vol. 37, no. 3, pp. 363–365, Mar. 1992.

[11] M. Borgmann and H. Bölcskei, "Interpolation-based efficient matrix inversion for MIMO-OFDM receivers," in *Proc. Asilomar Conf. Signals, Syst., Comput.*, Pacific Grove, CA, Nov. 2004, pp. 1941–1947.

[12] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 3rd ed. Baltimore, MD: Johns Hopkins Univ. Press, 1996.

[13] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1985.

[14] L. M. Davis, "Scaled and decoupled Cholesky and QR decompositions with application to spherical MIMO detection," in *Proc. IEEE Wireless Commun. Netw. Conf. (WCNC)*, New Orleans, LA, Mar. 2003, pp. 326–331.

[15] D. Cescato, *Interpolation-Based Matrix Arithmetics for MIMO-OFDM Systems*, 2010, Ph.D. thesis, ETH Zurich.