# A Geometrical Investigation of the Rank-1 Ricean MIMO Channel at High SNR

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## I. Introduction

We investigate the high-SNR mutual information (MI) of the Ricean MIMO channel for i.i.d. gaussian code books, no channel state information (CSI) at the transmitter, perfect CSI at the receiver, and the case where the channel's deterministic component has rank 1.

**Results.** Our main results can be summarized as follows: 1) Using an illustrative geometrical technique, we decompose the MI into the sum of the MIs of one single-input single-output (SISO) Ricean channel and several SISO Rayleigh fading channels with different diversity orders (cf. Theorem 1). 2) We derive an analytical approximation for the probability density function (pdf) of the MI that reveals the gaussian nature of MI both in the Rayleigh and the rank-1 Ricean cases (cf. Theorem 2). 3) We provide accurate approximations for the mean and the variance of the MI (cf. Theorem 2) which allow to analytically quantify the impact of the K-factor on capacity.

**Notation.** $\|\mathbf{a}\|$ denotes the $l^2$-norm of the vector $\mathbf{a}$, $|.|$ stands for the absolute value of a complex number, $\sim$ denotes equivalence in distribution, $n_T$ and $n_R$ stand for the number of transmit and receive antennas, respectively, $L^- = \min(n_T, n_R)$ and $L^+ = \max(n_T, n_R)$, and the superscript $^*$ denotes conjugate transpose. A circularly symmetric complex gaussian random variable is a random variable $Z = X + jY \sim \mathcal{CN}(0, \sigma^2)$, where $X$ and $Y$ are i.i.d. $\mathcal{N}(0, \sigma^2/2)$. Rank$(\mathbf{A})$, $\det(\mathbf{A})$ and $\|\mathbf{A}\|_F^2$ denote the rank, determinant, and squared Frobenius norm of the matrix $\mathbf{A}$, respectively.

## II. Results

We decompose the $n_R \times n_T$ channel matrix $\mathbf{H} = [\mathbf{h}_1\ \mathbf{h}_2\ \ldots\ \mathbf{h}_{n_T}]$ according to

$$\mathbf{H} = \sqrt{\frac{K}{K+1}}\bar{\mathbf{H}} + \sqrt{\frac{1}{K+1}}\tilde{\mathbf{H}} \qquad (1)$$

where $\tilde{\mathbf{H}}$ consists of i.i.d. $\mathcal{CN}(0,1)$ elements, rank$(\bar{\mathbf{H}}) = 1$ with $\|\bar{\mathbf{H}}\|_F^2 = n_T n_R$, and $K$ denotes the Ricean K-factor. For $n_R \geq n_T$ and high SNR, the MI of $\mathbf{H}$, under the assumptions stated in Sec. I, satisfies[2]

$$I \approx \log_2 \det\left(\frac{\rho}{n_T}\mathbf{H}^*\mathbf{H}\right) \qquad (2)$$

where $\rho$ denotes the SNR per receive antenna.

It is known in geometry [1] that $\det(\mathbf{H}^*\mathbf{H})$ equals the square of the volume of the parallelotope spanned by the vectors $\mathbf{h}_i$. Computing the high-SNR MI in (2) therefore amounts to determining the volume of the parallelotope spanned by the matrix $\gamma\left(\sqrt{K}\,\bar{\mathbf{H}} + \tilde{\mathbf{H}}\right)$ with $\gamma = \sqrt{\frac{\rho}{n_T(K+1)}}$. Since rank$(\bar{\mathbf{H}}) = 1$, we can equivalently consider the volume of $\gamma\left(\sqrt{K}\mathbf{\Phi}\bar{\mathbf{H}}\mathbf{\Psi}^* + \mathbf{\Phi}\tilde{\mathbf{H}}\mathbf{\Psi}^*\right)$ where the unitary matrices $\mathbf{\Phi}$ and $\mathbf{\Psi}$ have been chosen such that $\mathbf{\Phi}\bar{\mathbf{H}}\mathbf{\Psi}^*$ has only one non-zero element equal to $\sqrt{n_T n_R}$. We have therefore transformed the problem into computing the volume of a random parallelotope with i.i.d. complex gaussian vertices one of which is non-central and the others are central. Using results from [2] we arrive at

**Theorem 1.** *The high-SNR MI of the rank-1 Ricean MIMO channel in (1) satisfies*

$$I \sim \sum_{i=0}^{L^- -1} \log_2\left(\frac{\rho}{2n_T(K+1)}g_i\right) \qquad (3)$$

*where $g_0$ is non-central chi-square distributed with non-centrality parameter $\Delta = K n_T n_R$ and $2L^+$ degrees of freedom (d.o.f.) and the $g_i$ $(i = 1, 2, \ldots, L^- - 1)$ are central chi-square with $2(L^+ - i)$ d.o.f.*

The gaussian behavior of MI is revealed through an expansion of the characteristic function of $I$ in (3) in terms of Bernoulli polynomials and truncation after the second term for the pdf and after the first term for its mean and variance:

**Theorem 2.** *The pdf of MI of the rank-1 Ricean MIMO channel can be approximated as*

$$p_I(x) = \phi\left(\frac{x - \tilde{\mu}_I}{\tilde{\sigma}_I}\right) + \frac{1}{2}\sum_{i=1}^{L^- -1}\left(\frac{1}{L^+ - L^- + i}\right)^2 \times\left(\frac{1}{6}\phi'\left(\frac{x-\tilde{\mu}_I}{\tilde{\sigma}_I}\right) + \frac{1}{2}\phi''\left(\frac{x-\tilde{\mu}_I}{\tilde{\sigma}_I}\right) + \frac{1}{3}\phi'''\left(\frac{x-\tilde{\mu}_I}{\tilde{\sigma}_I}\right)\right) \qquad (4)$$

*where $\phi(x)$ denotes the pdf of the real-valued gaussian distribution with mean 0 and unit variance and $\phi'(x)$, $\phi''(x)$, and $\phi'''(x)$ stands for the first, second, and third derivative with respect to $x$, respectively. The mean $\tilde{\mu}_I$ and the variance $\tilde{\sigma}_I^2$ are*

$$\tilde{\mu}_I = L^- \log_2 \rho - L^- \log_2\left(n_T(K+1)\right) + \log_2\left(1 + L^- K\right) + \frac{\theta}{2L^+} + \sum_{i=1}^{L^-}\left(\log_2(L^+ - L^- + i) - \frac{\theta}{2(L^+ - L^- + i)}\right),$$

$$\tilde{\sigma}_I^2 = \theta^2\left(\sum_{i=1}^{L^- -1}\frac{1}{L^+ - L^- + i} + \frac{1}{L^+}\frac{1 + 2L^- K}{(1 + L^- K)^2}\right)$$

where $\theta = \frac{1}{\log_e 2}$. Eq. (4) shows that a gaussian approximation of the pdf of $I$ in (3) becomes more precise for large $(L^+ - L^-)$.

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[2]For $n_R < n_T$, MI is given by (2) with $\mathbf{H}^*\mathbf{H}$ replaced by $\mathbf{H}\mathbf{H}^*$.

## References

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[2] H. Ruben, "The volume of an isotropic random parallelotope," *Advances in Applied Probability*, vol. 16, pp. 84–94, 1979.

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