STRUCTURED GROUP FRAMES

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ABSTRACT

We introduce and discuss the properties of a new class of finite-dimensional frames with strong symmetry properties called *geometrically uniform (GU) frames*, that are defined over a finite abelian group of unitary matrices and are generated by a single generating vector. The notion of GU frames is then extended to *compound GU (CGU)* frames which are generated by a finite abelian group of unitary matrices using *multiple* generating vectors. Finally, we discuss methods for constructing optimal GU frames.

1. INTRODUCTION AND OUTLINE

Frames are generalizations of bases which lead to redundant signal expansions [1]. A finite frame for a Hilbert space $\mathcal{H}$ is a set of vectors that are not necessarily linearly independent and span $\mathcal{H}$.

Two important classes of highly structured frames are Gabor (Weyl-Heisenberg (WH)) frames [2] and wavelet frames [3, 4]. Both classes of frames are generated by a single generating function. WH frames are obtained by translations and modulations of the generating function (referred to as the window function), and wavelet frames are obtained by shifts and dilations of the generating function (referred to as the mother wavelet). In Section 3 of this paper, we introduce a new class of finite-dimensional frames which we refer to as geometrically uniform (GU) frames, that like WH and wavelet frames are generated from a single generating vector. These frames are defined by a finite abelian group $\mathcal{Q}$ of unitary matrices, referred to as the generating group of the frame. GU frames are based on the notion of GU vector sets [5], which are known to have strong symmetry properties that may be desirable in various applications such as channel coding.

The notion of GU frames is then extended to frames that are generated by a finite abelian group $\mathcal{Q}$ of unitary matrices using *multiple* generating vectors. Such frames are not necessarily GU, but consist of subsets of GU vector sets that are each generated by $\mathcal{Q}$. We refer to this class of frames as compound GU (CGU) frames, and develop their properties in Section 5.

In Section 4, we show that the dual and canonical frame vectors associated with a GU frame are also GU, and therefore generated by a single generating vector. Furthermore, we demonstrate that the generating vector can be computed very efficiently using a Fourier transform defined over the generating group $\mathcal{Q}$ of the frame. Similarly, in Section 5 we show that the dual and canonical frame vectors associated with a CGU frame are also CGU.

An important topic in frame theory is the behavior of a frame when elements of the frame are removed. In Section 6, we show that the frame bounds of the frame resulting from removing a single vector of a GU frame are the same regardless of the particular vector removed. In this sense GU frames exhibit an interesting robustness property which is of particular importance in applications such as multiple description source coding [6].

Since GU frames have nice symmetry properties, it may be desirable to construct such a frame from a given set of frame vectors. In Section 7, we discuss methods for constructing optimal GU frames from given frame vectors.

2. FRAMES

Let $\{\phi_i, 1 \leq i \leq n\}$ denote n vectors in an m-dimensional Hilbert space $\mathcal{H}$. The vectors ϕ_i form a *frame* for $\mathcal{H}$ if there exist constants $A > 0$ and $B < \infty$ such that[1]

$$A\|x\|^2 \leq \sum_{i=1}^{n} |\langle x, \phi_i \rangle|^2 \leq B\|x\|^2, \tag{1}$$

for all $x \in \mathcal{H}$ [4]. In our development, we restrict our attention to the case where both m and n are finite. From (1) it follows that any finite set of vectors that spans $\mathcal{H}$ is a frame for $\mathcal{H}$.

The tightest possible frame bounds are $A = \min_i \lambda_i(S)$ and $B = \max_i \lambda_i(S)$, where $\{\lambda_i(S), 1 \leq i \leq m\}$ denote the eigenvalues of the frame operator $S = \sum_{i=1}^{n} \phi_i \phi_i^* = \Phi\Phi^*$. Here Φ is the matrix of columns ϕ_i, and $(\cdot)^*$ is the Hermitian transpose. If $A = B$, then the frame is a *tight frame*. If $A = B = 1$, then the frame is a *normalized tight frame*. The redundancy of the frame is defined as $r = n/m$.

Given frame vectors ϕ_i, any $x \in \mathcal{H}$ can be expressed as $x = \sum_{i=1}^{n} a_i \phi_i$ for some coefficients a_i. If $n > m$ then the coefficients a_i are not unique. A possible choice is $a_i = \langle \bar{\phi}_i, x \rangle$ where $\bar{\phi}_i$ are the *minimal dual frame vectors* [4] and are given by $\bar{\phi}_i = S^{-1}\phi_i$. In the case of a tight frame $S = AI_m$; hence the

[1] We use the notation $\langle x, y \rangle = x^* y$ where the superscript $*$ denotes conjugate transposition.

minimal dual frame vectors are simply given by $\bar{\phi}_i = (1/A)\phi_i$. Since a tight frame expansion of a signal is very simple, it is popular in many applications. The canonical tight frame vectors associated with the vectors ϕ_i are the frame vectors $\mu_i = S^{-1/2}\phi_i$, where $S^{-1/2}$ is the positive definite square root of S. The vectors μ_i form a normalized tight frame that is closest in a least-squares sense to the vectors ϕ_i [7, 8].

In the next section, we introduce GU frames that have strong symmetry properties.

3. GEOMETRICALLY UNIFORM FRAMES

A GU frame is defined as a set of frame vectors $\mathcal{S} = \{\phi_i, 1 \leq i \leq n\}$ such that $\phi_i = U_i\phi$ where ϕ is an arbitrary *generating vector* and the matrices $\{U_i, 1 \leq i \leq n\}$ are unitary and form an abelian group $\mathcal{Q}$, which is called the *generating group* of $\mathcal{S}$ [5]. Alternatively, a vector set is a GU frame for $\mathcal{H}$ if the vectors span $\mathcal{H}$, and given any two vectors ϕ_i and ϕ_j in the set, there is an isometry that transforms ϕ_i into ϕ_j while leaving the set invariant.

Proposition 1 ([9]) *Let $\{\phi_i = U_i\phi, U_i \in \mathcal{Q}\}$ be a geometrically uniform frame with frame bounds A and B. Then $A \leq \frac{n}{m}\|\phi\|^2 \leq B$. If in addition the frame is tight, then $A = B = \frac{n}{m}\|\phi\|^2$.*

Since $U_i^* = U_i^{-1}$, we have $\langle\phi_i, \phi_j\rangle = \phi^* U_i^{-1} U_j \phi = s(U_i^{-1}U_j)$, where $s(U_i) = \phi^* U_i \phi$. For fixed i, the set $U_i^{-1}\mathcal{Q} = \{U_i^{-1}U_j, U_j \in \mathcal{Q}\}$ is a permutation of $\mathcal{Q}$ since $U_i^{-1}U_j \in \mathcal{Q}$ for all i, j [10]. Therefore, the numbers $\{s(U_i^{-1}U_j), 1 \leq j \leq n\}$ are a permutation of the numbers $\{s(U_i), 1 \leq i \leq n\}$. The same is true for fixed j. Consequently, every row and column of the Gram matrix $G = \{\langle\phi_i, \phi_j\rangle\}$ is a permutation of the numbers $\{a_i = s(U_i), 1 \leq i \leq n\}$. A matrix G whose rows (columns) are a permutation of the first row (column) will be called a *permuted matrix*. We then have the following proposition.

Proposition 2 ([9]) *The Gram matrix $G = \{\langle\phi_i, \phi_j\rangle\}$ corresponding to a geometrically uniform vector set $\mathcal{S} = \{\phi_i \in \mathcal{H}, 1 \leq i \leq n\}$ is a permuted matrix. Conversely, if the Gram matrix $G = \{\langle\phi_i, \phi_j\rangle\}$ is a permuted matrix, and $\langle\phi_i, \phi_j\rangle = \langle\phi_j, \phi_i\rangle$ for all i, j, then the vectors $\{\phi_i\}$ are geometrically uniform. If in addition the vectors $\{\phi_i\}$ span $\mathcal{H}$, then they form a geometrically uniform frame for $\mathcal{H}$.*

The Fourier transform (FT) matrix plays an important role in defining GU frames. In order to define the FT we start by replacing the multiplicative group $\mathcal{Q}$ by an additive group Q to which $\mathcal{Q}$ is isomorphic. Specifically, every finite abelian group $\mathcal{Q}$ is isomorphic to a direct product Q of a finite number of cyclic groups: $\mathcal{Q} \cong Q = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_p}$, where $\mathbb{Z}_{n_t}$ is the cyclic additive group of integers modulo n_t, and $n = \prod_t n_t$. Thus every element $U_i \in \mathcal{Q}$ can be associated with an element $q \in Q$ of the form $q = (q_1, q_2, \ldots, q_p)$, where $q_t \in \mathbb{Z}_{n_t}$; this correspondence is denoted by $U_i \leftrightarrow q$. Each vector $\phi_i = U_i\phi$ is then denoted as $\phi(q)$, where $U_i \leftrightarrow q$.

The FT of a complex-valued function $\varphi : Q \rightarrow \mathbb{C}$ defined on $Q = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_p}$ is the complex-valued function $\hat{\varphi} : Q \rightarrow \mathbb{C}$ defined by

$$\hat{\varphi}(h) = \frac{1}{\sqrt{n}} \sum_{q \in Q} \langle h, q\rangle \varphi(q), \tag{2}$$

where $\langle h, q\rangle = \prod_{t=1}^{p} e^{-2\pi j h_t q_t / n_t}$. Here h_t and q_t are the kth components of h and q respectively, and the product $h_t q_t$ is taken as an ordinary integer modulo n_t. Correspondingly, the FT matrix over Q is defined as the $n \times n$ matrix $\mathcal{F} = \{\frac{1}{\sqrt{n}}\langle h, q\rangle, h, q \in Q\}$. The FT of a column vector $\varphi = \{\varphi(q), q \in Q\}$ is the vector $\hat{\varphi} = \{\hat{\varphi}(h), h \in Q\}$ given by $\hat{\varphi} = \mathcal{F}\varphi$. Since $\mathcal{F}$ is unitary, we obtain the inverse FT formula as $\varphi = \mathcal{F}^*\hat{\varphi}$.

In the following theorem we show that the FT matrix can be used to define GU frames.

Theorem 1 ([9]) *A set of vectors $\{\phi_i, 1 \leq i \leq n\}$ in an m-dimensional Hilbert space $\mathcal{H}$ is geometrically uniform if and only if the Gram matrix $G = \{\langle\phi_i, \phi_j\rangle\}$ is diagonalized by a Fourier transform matrix $\mathcal{F}$ over a finite product of cyclic groups Q.*

4. DUAL AND CANONICAL TIGHT FRAMES ASSOCIATED WITH GU FRAMES

For a GU vector set with generating group $\mathcal{Q}$, the frame operator $S = \Phi\Phi^*$ commutes with each of the unitary matrices $U_i \in \mathcal{Q}$ [9]. Therefore, S^{-1} and $S^{-1/2}$ also commute with U_j for all j, so that

$$\bar{\phi}_i = S^{-1}\phi_i = S^{-1}U_i\phi = U_i S^{-1}\phi = U_i\bar{\phi}, \tag{3}$$

where $\bar{\phi} = S^{-1}\phi$, which shows that the minimal dual frame vectors $\{\bar{\phi}_i = S^{-1}\phi_i\}$ are GU with generating group $\mathcal{Q}$. Similarly,

$$\mu_i = S^{-1/2}\phi_i = S^{-1/2}U_i\phi = U_i S^{-1/2}\phi = U_i\mu, \tag{4}$$

where $\mu = S^{-1/2}\phi$, which shows that the canonical tight frame vectors $\{\mu_i = S^{-1/2}\phi_i\}$ are also GU with generating group $\mathcal{Q}$.

Therefore, in order to compute the dual frame vectors or the canonical tight frame vectors all we need is to compute the generating vectors $\bar{\phi}$ and μ, respectively. The remaining frame vectors are then obtained by applying the group $\mathcal{Q}$ to the corresponding generating vectors.

When the group $\mathcal{Q}$ is abelian, the generating vectors can be computed very efficiently using the FT. Specifically we show in [9] that $\bar{\phi} = (1/\sqrt{n})\sum_{h \in \mathcal{I}}(1/\sigma(h))u(h)$, where $\{\sigma(h) = n^{1/4}\sqrt{\hat{s}(h)}, h \in Q\}$ are the singular values of Φ, $\{\hat{s}(h), h \in Q\}$ is the FT of the inner-product sequence $\{\langle\phi(0), \phi(q)\rangle, q \in Q\}$, $\mathcal{I}$ is the set of indices $h \in Q$ for which $\sigma(h) \neq 0$, $u(h) = \hat{\phi}(h)/\sigma(h)$ for $h \in \mathcal{I}$, and $\{\hat{\phi}(h), h \in Q\}$ is the FT of $\{\phi(q), q \in Q\}$. Similarly, $\mu = (1/\sqrt{n})\sum_{h \in \mathcal{I}} u(h)$. The frame bounds of the frame $\{\phi_i, 1 \leq i \leq n\}$ are given by $A = \sqrt{n}\min_{h \in \mathcal{I}} \hat{s}(h)$ and $B = \sqrt{n}\max_{h \in \mathcal{I}} \hat{s}(h)$.

The canonical tight frame vectors associated with a GU frame have the property that among all normalized tight frame vectors they maximize $R_{\phi\mu} = \sum_{i=1}^{n} |\langle\phi_i, \mu_i\rangle|^2$. Maximizing $R_{\phi\mu}$ may be of interest in various applications. For example, in a matched-filter detection problem considered in [11], $R_{\phi\mu}$ represents the total output signal-to-noise ratio. As another example, in a multiuser detection problem considered in [12], maximizing $R_{\phi\mu}$ has the effect of minimizing the multiple-access interference at the input to the proposed detector.

5. COMPOUND GU FRAMES

A CGU frame is defined as a set of frame vectors $\{\phi_{ik}, 1 \le i \le l, 1 \le k \le r\}$ such that $\phi_{ik} = U_i \phi_k$ for some generating vectors $\{\phi_k, 1 \le k \le r\}$, where the matrices $\{U_i, 1 \le i \le l\}$ are unitary and form an abelian group $\mathcal{Q}$. A CGU frame is in general not GU. However, for every k, the vectors $\{\phi_{ik}, 1 \le i \le l\}$ are GU with generating group $\mathcal{Q}$.

A special case of CGU frames are filter bank frames [13], in which $\mathcal{Q}$ is the group of translations by integer multiples of the subsampling factor, and the generating vectors are the filter bank synthesis filters.

Proposition 3 ([9]) *Let $\{\phi_{ik} = U_i\phi_k, 1 \le i \le l, 1 \le k \le r\}$ be a compound geometrically uniform frame with frame bounds A and B, where $\{\phi_k, 1 \le k \le r\}$ is an arbitrary set of generating vectors. Then $A \le \frac{l}{m}\sum_{k=1}^{r} \|\phi_k\|^2 \le B$. If in addition the frame is tight, then $A = \frac{l}{m}\sum_{k=1}^{r} \|\phi_k\|^2$.*

5.1. Example of a CGU Frame

An example of a CGU frame is illustrated in Fig. 1. Here $\{\phi_{ik}, 1 \le i, k \le 2\}$ where $\{\phi_{ik} = U_i\phi_k, U_i \in \mathcal{G}\}$, $\mathcal{G} = \{I_2, U\}$ and U represents a reflection about the dashed line in Fig. 1, and is given by

$$U = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}. \quad (5)$$

The generators of the CGU frame are $\phi_1 = (1/\sqrt{2})[1 \ \ 1]^*$ and $\phi_2 = (1/\sqrt{2})[1 \ \ -1]^*$. The vector ϕ_{21} is obtained by reflecting the generator ϕ_{11} about the dashed line, and similarly the vector ϕ_{22} is obtained by reflecting the generator ϕ_{12} about this line. As can be seen from the figure, the frame is not GU. In particular, there is no unitary transformation that transforms ϕ_{11} into ϕ_{12} while leaving the set invariant. However, the sets $\mathcal{S}_1 = \{\phi_{11}, \phi_{21}\}$ and $\mathcal{S}_2 = \{\phi_{12}, \phi_{22}\}$ are both GU with generating group $\mathcal{G}$; both sets are invariant under a reflection about the dashed line.

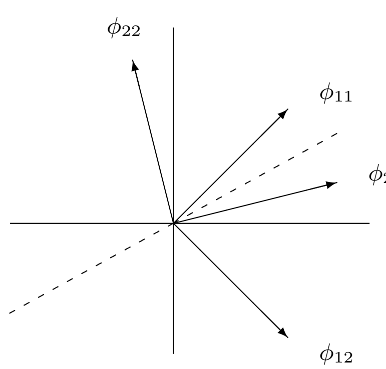

Fig. 1. A compound geometrically uniform frame.

5.2. Dual and canonical tight frames associated with CGU frames

With Φ denoting the matrix of columns ϕ_{ik}, it can be shown that $S = \Phi\Phi^*$ commutes with each of the matrices $U_i \in \mathcal{Q}$ [9], so that the dual frame vectors are given by

$$\bar{\phi}_{ik} = S^{-1}\phi_{ik} = S^{-1}U_i\phi_k = U_i S^{-1}\phi_k = U_i\bar{\phi}_k, \quad (6)$$

where $\bar{\phi}_k = S^{-1}\phi_k$. Therefore, the dual frame vectors are CGU with generating group $\mathcal{Q}$. Similarly,

$$\mu_{ik} = S^{-1/2}\phi_{ik} = S^{-1/2}U_i\phi_k = U_i S^{-1/2}\phi_k = U_i\mu_k, \quad (7)$$

where $\mu_k = S^{-1/2}\phi_k$, so that the canonical tight frame vectors are CGU with generating group $\mathcal{Q}$.

Therefore, in order to compute the dual frame vectors or the canonical tight frame vectors all we need is to compute the generating vectors $\{\bar{\phi}_k, 1 \le k \le r\}$ and $\{\mu_k, 1 \le k \le r\}$, respectively. The remaining frame vectors are then obtained by applying the group $\mathcal{Q}$ to the corresponding set of generating vectors.

5.3. CGU frames with GU generators

A special class of CGU frames is *CGU frames with GU generators* in which the generating vectors $\{\phi_k, 1 \le k \le r\}$ are themselves GU. Specifically, $\{\phi_k = V_k\phi\}$ for some generator ϕ, where the matrices $\{V_k, 1 \le k \le r\}$ are unitary, and form an abelian group $\mathcal{G}$.

Suppose that $U_pV_t = V_tU_pe^{j\theta(p,t)}$ for all t and p and arbitrary $\theta(p,t)$. Then V_t commutes with $S = \Phi\Phi^*$ for each t, and the dual frame vectors are given by

$$\bar{\phi}_{ik} = S^{-1}\phi_{ik} = S^{-1}U_iV_k\phi = U_iV_kS^{-1}\phi = U_iV_k\bar{\phi}, \quad (8)$$

where $\bar{\phi} = S^{-1}\phi$. Similarly,

$$\mu_{ik} = S^{-1/2}\phi_{ik} = S^{-1/2}U_iV_k\phi = U_iV_kS^{-1/2}\phi = U_iV_k\mu \quad (9)$$

where $\mu = S^{-1/2}\phi$. Thus even though the frame is not in general GU, the dual and canonical tight frame vectors can still be computed using a single generating vector. Alternatively, $\bar{\phi}_{ik} = U_i\bar{\phi}_k$ where $\bar{\phi}_k = V_k\bar{\phi}$, and $\mu_{ik} = U_i\mu_k$ where $\mu_k = V_k\mu$. Thus $\{\bar{\phi}_k, 1 \le k \le r\}$ and $\{\mu_k, 1 \le k \le r\}$ are both GU with generating group $\mathcal{G}$.

A special case of CGU frames with GU generators for which $\mathcal{Q}$ and $\mathcal{G}$ commute up to a phase factor are Weyl-Heisenberg (WH) frames [4, 3, 2]. If the WH frame is critically sampled, then $\theta(p,t) = 0$ and the WH frame reduces to a GU frame. In the more general oversampled case, $\theta(p,t) \ne 0$.

6. PRUNING GU FRAMES

In applications, such as multiple description source coding [6], it is often desirable to know or to be able to estimate the frame bounds when one or more of the frame elements are removed. The following theorem reveals an interesting symmetry property of GU frames in this context.

Theorem 2 ([9]) *Let $\mathcal{S} = \{\phi_i = U_i\phi, U_i \in \mathcal{Q}\}$ be a geometrically uniform frame generated by a finite abelian group $\mathcal{Q}$ of unitary matrices, where ϕ is an arbitrary generating vector. Let Φ be the matrix of columns ϕ_i, and let $S = \Phi\Phi^*$ be the corresponding frame operator. Let $\mathcal{S}(j) = \{\phi_i = U_i\phi, U_i \in \mathcal{Q}, i \neq j\}$ be the pruned set obtained by removing the element ϕ_j. Then the eigenvalues of the frame operator corresponding to the pruned set do not depend on the particular element ϕ_j removed. If in addition the frame is tight, then the eigenvalues of the frame operator corresponding to the pruned set are given by $\lambda_1 = \frac{n}{m} - 1$ and $\lambda_i = \frac{n}{m}$, $2 \leq i \leq n$, independent of ϕ_j.*

An immediate consequence is that the frame bound ratio of a pruned tight frame is given by $B/A = 1/(1 - m/n)$, which is close to 1 for large redundancy $r = n/m$.

We next consider the case where multiple frame elements are removed.

Corollary 1 ([9]) *Let $\mathcal{S} = \{\phi_i = U_i\phi, U_i \in \mathcal{Q}\}$ be a GU frame generated by a finite abelian group $\mathcal{Q}$ of unitary matrices, let Φ be the matrix of columns ϕ_i, and let $S = \Phi\Phi^*$ be the corresponding frame operator. Let $\mathcal{J}$ be a set of indices, and let $\mathcal{J}(k)$ denote the set of indices i such that $U_i = U_kU_j$ for fixed k and $j \in \mathcal{J}$. Let $\mathcal{S}(k) = \{\phi_i = U_i\phi, U_i \in \mathcal{Q}, i \neq \mathcal{J}(k)\}$ be a pruned set obtained by removing the elements ϕ_i with $i \in \mathcal{J}(k)$. Then the eigenvalues of the frame operator corresponding to the pruned set are independent of k.*

Corollary 1 implies that the frame bounds of the frame resulting from removing the elements ϕ_i with $i \in \mathcal{J}(k)$ are independent of k so that in the context of multiple description source coding, the quality of the reconstruction from the reduced set is independent of the elements removed.

7. CONSTRUCTING GU FRAMES

Since GU frames have nice symmetry properties, it is desirable to construct a GU frame from a given set of frame vectors. We shall next present a method for constructing a GU frame $\{\phi_i, 1 \leq i \leq n\}$ with Gram matrix $\beta^2 R$ for some $\beta > 0$, from given frame vectors $\{\varphi_i, 1 \leq i \leq n\}$, where the GU vectors ϕ_i are chosen to minimize the least-squares error $E = \sum_{i=1}^n \langle \varphi_i - \phi_i, \varphi_i - \phi_i \rangle$.

Theorem 3 ([14, 9]) *Let $\{\varphi_i\}$ be a set of n vectors in an m-dimensional Hilbert space $\mathcal{H}$ that span $\mathcal{H}$, and let F be the matrix of columns φ_i. Let $\{\hat{\phi}_i\}$ denote the n GU frame vectors that minimize the least-squares error $E = \sum_{i=1}^n \langle \varphi_i - \phi_i, \varphi_i - \phi_i \rangle$ subject to the constraint $\Phi^*\Phi = \beta^2 R$, and let $\widehat{\Phi}$ be the matrix of columns $\hat{\phi}_i$. Let $\mathcal{F}$ be the Fourier transform matrix that diagonalizes R, let A be the diagonal matrix with diagonal elements $\{\alpha_j = n^{1/2}\hat{a}_j, 1 \leq j \leq n\}$ where $\{\hat{a}_j, 1 \leq j \leq n\}$ is the FT of the first row $\{a_j, 1 \leq j \leq n\}$ of R, let Σ be an $m \times n$ diagonal matrix with m diagonal elements $\sqrt{\alpha_i}$ for values of i for which $\alpha_i \neq 0$, and let U and V be the right-hand unitary matrix and left-hand unitary matrix respectively in the singular value decomposition of $F\mathcal{F}\Sigma^*$. Then,*

1. *if $\beta = \beta_0$ is given, then we have $\widehat{\Phi} = \beta_0 UV^*\Sigma\mathcal{F}^*$. If in addition FRF^* is invertible, then $\widehat{\Phi} = \beta_0(FRF^*)^{-1/2}FR$.*
2. *if $\beta > 0$ is chosen to minimize E, then $\widehat{\Phi} = \hat{\beta}UV^*\Sigma\mathcal{F}^*$, where $\hat{\beta} = Tr(UV^*\Sigma\mathcal{F}^*)/Tr(R)$. If in addition FRF^* is invertible, then $\widehat{\Phi} = \hat{\beta}(FRF^*)^{-1/2}FR$, where $\hat{\beta} = Tr((FRF^*)^{1/2})/Tr(R)$.*

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