EQUIVALENCE OF DFT FILTER BANKS AND GABOR EXPANSIONS*

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ABSTRACT

Recently connections between the wavelet transform and filter banks have been established. We show that similar relations exist between the Gabor expansion and DFT filter banks. We introduce the "z-Zak transform" by suitably extending the discrete-time Zak transform and show its equivalence to the polyphase representation. A systematic discussion of parallels between DFT filter banks and Weyl-Heisenberg frames (Gabor expansion theory) is then given. Among other results, it is shown that tight Weyl-Heisenberg frames correspond to paraunitary DFT filter banks.

1 INTRODUCTION AND OUTLINE

The wavelet transform, filter banks, and multiresolution signal analysis have recently been unified within a single theory.[1–6] This led to new results and deeper insights in both areas. In this paper, we show that an important linear time-frequency representation known as the *Gabor expansion*[7–9] and the computationally efficient *DFT filter banks*[10–14,6] can be unified in a similar manner.[15] We show that the theory of *Weyl-Heisenberg frames* (WHFs),[16–18] which is a fundamental concept in Gabor expansion theory, allows to establish known and new results on DFT filter banks. We also extend the *Zak transform*,[19–21] a transformation particularly useful for the Gabor expansion, to the complex plane (z-plane), and we show that the resulting "z-Zak transform" is equivalent to the *polyphase representation* used in filter bank theory.[22,10,6,12,13]

This paper is organized as follows. Section 1 gives a brief review of the discrete-time Gabor expansion and DFT filter banks, along with the associated concepts of WHFs, Zak transform, and polyphase representation. In Section 2, we introduce the z-Zak transform, show its equivalence to the polyphase representation, and consider its application to the Gabor expansion. Extending previous results,[23] we show in Section 3 that the discrete-time Gabor expansion can be interpreted as a DFT filter bank with perfect reconstruction, and we present a systematic discussion of the parallels existing between WHFs (Gabor expansion theory) and DFT filter banks. Our main results can be summarized as follows:

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- In WHF theory, the Gabor analysis and synthesis windows are related in exactly the same way as are the analysis and synthesis prototype filters in DFT filter bank theory.
- The frame operator in WHF theory is intimately related to the alias component and polyphase matrices of DFT filter banks.
- The frame bounds characterizing the numerical properties of a WHF are related to the flatness of the polyphase filters and to the eigenvalues of the alias component matrix.
- Tight WHFs correspond to paraunitary DFT filter banks.

For simplicity, we consider only critically sampled DFT filter banks and Gabor expansions although most results can be extended to the case of oversampling.[24]

1.1 Gabor Expansion, Weyl-Heisenberg Frames, and Zak Transform

In this section, we review the Gabor expansion along with the associated concepts of WHFs and the Zak transform.

1.1.1 Gabor Expansion

The Gabor expansion[7-9] is the decomposition of a signal into a set of time-frequency shifted versions of a prototype signal. The *discrete-time Gabor expansion*, with critical sampling, of a signal $x[n] \in l^2(\mathbb{Z})$ is defined as*

$$x[n] = \sum_{l=-\infty}^{\infty} \sum_{m=0}^{M-1} a_{l,m}\, g_{l,m}[n] \qquad \text{with} \quad g_{l,m}[n] = g[n - lM]\, e^{j2\pi \frac{m}{M} n}, \tag{1}$$

where $a_{l,m}$ are the Gabor coefficients, $g[n]$ is a "synthesis window," and the parameter $M \in \mathbb{N}$ is the grid constant. The Gabor coefficients can be calculated as

$$a_{l,m} = \langle x, \gamma_{l,m} \rangle = \sum_{n=-\infty}^{\infty} x[n]\, \gamma_{l,m}^{*}[n] \qquad \text{with} \quad \gamma_{l,m}[n] = \gamma[n - lM]\, e^{j2\pi \frac{m}{M} n} \tag{2}$$

with an "analysis window" $\gamma[n]$. The Gabor expansion (1), (2) exists for all signals $x[n] \in l^2(\mathbb{Z})$ only if the windows $g[n]$ and $\gamma[n]$ are chosen properly (see below). Note that (1) differs from the cyclic definition previously proposed[8] in that it applies to signals of arbitrary (potentially infinite) length.

1.1.2 Weyl-Heisenberg Frames

The theory of *Weyl-Heisenberg frames* (WHFs)[16-18] yields important results about the Gabor expansion. A set of functions

$$g_{l,m}[n] = g[n - lM]\, e^{j2\pi \frac{m}{M} n}, \qquad -\infty < l < \infty, \quad 0 \le m \le M-1$$

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*Here, $l^2(\mathbb{Z})$ denotes the space of square-summable discrete-time signals, i.e., $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$ for $x[n] \in l^2(\mathbb{Z})$.

is said to be a WHF (with critical sampling) for $l^2(\mathbb{Z})$ if for all $x[n] \in l^2(\mathbb{Z})$

$$A\,\|x\|^2 \;\le\; \sum_{l=-\infty}^{\infty} \sum_{m=0}^{M-1} \left|\langle x, g_{l,m}\rangle\right|^2 \;\le\; B\,\|x\|^2 \qquad \text{with} \quad 0 < A \le B < \infty\,. \tag{3}$$

The constants $A > 0$ and $B < \infty$ are called *frame bounds*. For synthesis window $g[n]$ such that $\{g_{l,m}[n]\}$ is a WHF, the Gabor expansion (1) exists for all $x[n] \in l^2(\mathbb{Z})$, and the analysis window is derived from $g[n]$ as

$$\gamma[n] = (\mathbf{S}^{-1} g)[n]\,.$$

Here, $\mathbf{S}^{-1}$ is the inverse of the *frame operator* $\mathbf{S}$ defined as

$$(\mathbf{S}x)[n] = \sum_{l=-\infty}^{\infty} \sum_{m=0}^{M-1} \langle x, g_{l,m}\rangle\, g_{l,m}[n]\,.$$

The frame operator is a linear, positive definite operator mapping $l^2(\mathbb{Z})$ onto $l^2(\mathbb{Z})$.

If $\{g_{l,m}[n]\}$ is a WHF, then $\{\gamma_{l,m}[n]\}$ is a WHF as well (the "dual" frame). The frame bounds of $\{\gamma_{l,m}[n]\}$ are $A' = 1/B$ and $B' = 1/A$ where A, B are the frame bounds of $\{g_{l,m}[n]\}$.[17,18]

The numerical properties of the Gabor expansion will be better for closer frame bounds A and B. A WHF is called *snug* if $A \approx B$, and *tight* if $A = B$. For a tight WHF, $\mathbf{S} = A\mathbf{I}$ where $\mathbf{I}$ is the identity operator on $l^2(\mathbb{Z})$, and hence there is simply $\gamma[n] = \frac{1}{A}\, g[n]$.

In the case of critical sampling considered here, a WHF $\{g_{l,m}[n]\}$ is known to be an *exact* frame, which means that the $g_{l,m}[n]$ are linearly independent. The exactness also implies that $\{g_{l,m}[n]\}$ and $\{\gamma_{l,m}[n]\}$ are *biorthogonal*,[18,7,8] i.e., $\langle g_{l',m'}, \gamma_{l,m}\rangle = \delta[l-l']\,\delta[m-m']$ or equivalently

$$\langle g, \gamma_{l,m}\rangle = \delta[l]\,\delta[m]\,, \tag{4}$$

where $\delta[\cdot]$ denotes the unit sample.

1.1.3 Zak Transform

The *discrete-time Zak transform* (ZT) is defined as[20]

$$Z_x(n,\theta) = \sum_{l=-\infty}^{\infty} x[n+lM]\, e^{-j2\pi\theta l}\,.$$

Since the ZT is quasiperiodic in n and periodic in θ,

$$Z_x(n+lM,\theta) = e^{j2\pi l\theta}\, Z_x(n,\theta)\,, \qquad Z_x(n,\theta+l) = Z_x(n,\theta)\,, \qquad l \in \mathbb{Z}\,,$$

it suffices to calculate the ZT on the "fundamental rectangle" $(n,\theta) \in [0, M-1] \times [0,1)$. The signal $x[n]$ can be recovered from its ZT as

$$x[n] = \int_0^1 Z_x(n,\theta)\, d\theta\,. \tag{5}$$

The ZT is very useful in the context of the Gabor expansion or, equivalently, WHFs:[9,17–20]

- The effect of the frame operator $\mathbf{S}$ or inverse frame operator $\mathbf{S}^{-1}$ becomes a simple multiplication or division, respectively, in the ZT domain:

$$Z_{\mathbf{S}x}(n,\theta) = M\,|Z_g(n,\theta)|^2\, Z_x(n,\theta)\,, \qquad Z_{\mathbf{S}^{-1}x}(n,\theta) = \frac{Z_x(n,\theta)}{M\,|Z_g(n,\theta)|^2}\,. \tag{6}$$

- The frame condition (3) can be reformulated, using the ZT of the Gabor synthesis window $g[n]$, as

$$A \le M\,|Z_g(n,\theta)|^2 \le B. \tag{7}$$

In particular, $\{g_{l,m}[n]\}$ is a tight WHF (i.e. $A = B$) if and only if $|Z_g(n,\theta)|$ is constant,

$$M\,|Z_g(n,\theta)|^2 = A\,.$$

- The Gabor analysis window $\gamma[n]$ can be computed via the ZT by using

$$Z_\gamma(n,\theta) = \frac{1}{M\,Z_g^*(n,\theta)} \tag{8}$$

and inverting $Z_\gamma(n,\theta)$ according to (5).

An extension of the ZT will be considered in Section 2.

1.2 DFT Filter Banks and Polyphase Representation

We now review the second area considered in this paper, namely, DFT filter banks and the associated concept of polyphase representation.

1.2.1 DFT Filter Banks

We consider an M-channel maximally decimated (critically sampled) DFT filter bank[10–14,6] with perfect reconstruction and zero delay, such that $\hat{x}[n] = x[n]$ (see Fig. 1). The transfer functions (z-transforms) of the analysis and synthesis filters are frequency-translated versions of the transfer functions of two baseband "prototype" filters $H(z)$ and $F(z)$, respectively, i.e.,

$$H_m(z) = H(zW^{-m})\,, \qquad F_m(z) = F(zW^{-m})\,, \qquad 0 \le m \le M-1\,,$$

with $W = e^{-j2\pi/M}$. Equivalently, the filters' impulse responses are

$$h_m[n] = h[n]\,W^{mn}\,, \qquad f_m[n] = f[n]\,W^{mn}, \qquad 0 \le m \le M-1.$$

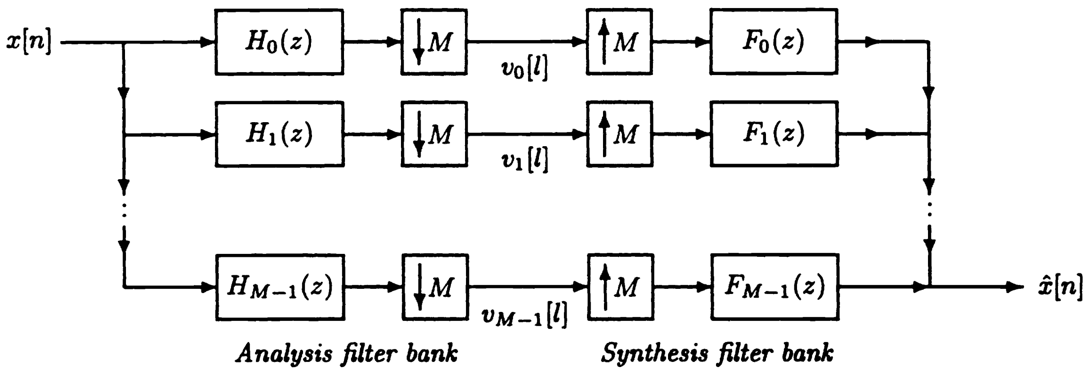

Fig. 1: M-channel maximally decimated filter bank.

The subband signals (see Fig. 1) are given by

$$v_m[l] = \sum_{n=-\infty}^{\infty} h_m[lM - n]\, x[n]\,, \qquad 0 \le m \le M-1\,, \tag{9}$$

and the signal is reconstructed from the subband signals as

$$x[n] = \hat{x}[n] = \sum_{l=-\infty}^{\infty} \sum_{m=0}^{M-1} f_m[n - lM]\, v_m[l]\,. \tag{10}$$

A necessary and sufficient condition for perfect reconstruction is that, for given analysis prototype filter $H(z)$, the synthesis prototype filter $F(z)$ satisfies the linear system of equations[6,11]

$$\sum_{l=0}^{M-1} H(zW^{m-l})\, F(zW^{-l}) = M\delta[m]\,, \qquad 0 \le m \le M-1\,. \tag{11}$$

In compact vector-matrix notation, this reads[11–14,6]

$$\boldsymbol{H}(z)\, \boldsymbol{f}(z) = \boldsymbol{c}(z)\,, \tag{12}$$

with the *alias component matrix* (AC matrix)[6]

$$\boldsymbol{H}(z) = \begin{pmatrix} H(z) & H(zW^{-1}) & \dots & H(zW^{-(M-1)}) \\ H(zW^{-(M-1)}) & H(z) & \dots & H(zW^{-(M-2)}) \\ \vdots & \vdots & \vdots & \\ H(zW^{-1}) & H(zW^{-2}) & \dots & H(z) \end{pmatrix} \tag{13}$$

and the vectors (the superscript ^{T} denotes transposition)

$$\boldsymbol{f}(z) = \Big(F(z)\;\; F(zW^{-1})\;\; \dots\;\; F(zW^{-(M-1)})\Big)^T\,, \qquad \boldsymbol{c}(z) = (M\;0\;\dots\;0)^T\,.$$

1.2.2 Polyphase Representation

The *type-1 polyphase representation*[22,10,6,12,13] of the analysis prototype filter $H(z)$ reads

$$H(z) = \sum_{n=0}^{M-1} z^{-n} E_n(z^M) \qquad \text{with} \quad E_n(z) = \sum_{l=-\infty}^{\infty} h[n + lM]\, z^{-l}\,. \tag{14}$$

$E_n(z)$ is the transfer function of the nth *polyphase filter*. The eigenvalues of the AC matrix $\boldsymbol{H}(z)$ can be expressed in terms of the polyphase filters as[11]

$$\Lambda_n(z) = M\, z^n\, E_{-n}(z^M) = M\, z^{-(M-n)}\, E_{M-n}(z^M)\,, \qquad 0 \le n \le M-1\,. \tag{15}$$

If $E_n(z_0) = 0$ for some n, then $\Lambda_{M-n}(z_0^{1/M}) = 0$, causing $\boldsymbol{H}(z)$ to become singular at $z = z_0^{1/M}$. The type-1 polyphase representation of the synthesis prototype filter is

$$F(z) = \sum_{n=0}^{M-1} z^{-n} R_n(z^M) \qquad \text{with} \quad R_n(z) = \sum_{l=-\infty}^{\infty} f[n + lM]\, z^{-l}\,. \tag{16}$$

The perfect reconstruction property implies that the polyphase filters are inverse filters,[10–14]

$$E_n(z) = z \frac{1}{M\, R_{M-n}(z)} . \tag{17}$$

Besides the AC matrix $\boldsymbol{H}(z)$ in (12), (13), another matrix important in filter bank theory is the *polyphase matrix* $\boldsymbol{E}(z)$ defined as[6]

$$\boldsymbol{E}(z) = \mathbf{W}\ \mathrm{diag}\{E_n(z)\}_{n=0}^{M-1} ,$$

where $\mathbf{W}$ is the $M \times M$ DFT matrix with elements $(\mathbf{W})_{kl} = W^{kl} = e^{-j2\pi kl/M}$, and $\mathrm{diag}\{E_n(z)\}_{n=0}^{M-1}$ is the diagonal $M \times M$ matrix whose diagonal elements are the analysis filter polyphase components $E_0(z), E_1(z), ..., E_{M-1}(z)$. The polyphase matrix is essentially equivalent to the AC matrix since the two matrices are related as[6]

$$\boldsymbol{H}(z) = \mathbf{W}^H \boldsymbol{D}(z)\, \boldsymbol{E}^T(z^M) , \tag{18}$$

where $\boldsymbol{D}(z) = \mathrm{diag}\{z^{-n}\}_{n=0}^{M-1}$, and ^{T} and ^{H} denote transposition and conjugate transposition, respectively.

2 THE z-ZAK TRANSFORM: EXTENDING THE ZAK TRANSFORM TO THE z-PLANE

The ZT is a powerful mathematical tool for WHFs and the Gabor expansion. In order to achieve a closer relation to the theory of DFT filter banks, it is advantageous to extend the ZT to the complex plane (z-plane).

2.1 The z-Zak Transform

We define the *z-Zak transform* (ZZT) of a discrete-time signal $x[n] \in l^2(\mathbb{Z})$ as

$$\mathcal{Z}_x(n, z) \stackrel{\mathrm{def}}{=} \sum_{l=-\infty}^{\infty} x[n + lM]\, z^{-l}. \tag{19}$$

The ZZT is a discrete-time version of the so-called *modified z-transform.*[25–27] The ZT reviewed in Section 1.1.3 is the ZZT evaluated on the unit circle,

$$Z_x(n, \theta) = \mathcal{Z}_x(n, e^{j2\pi\theta}) .$$

The ZZT can be expressed in terms of the z-transform $X(z)$ of $x[n]$ as

$$\mathcal{Z}_x(n, z) = \frac{1}{M}\, z^{n/M} \sum_{k=0}^{M-1} X\left(z^{1/M}\, W^{-k}\right) W^{-kn} . \tag{20}$$

If $r_1 < |z| < r_2$ with $0 \le r_1 < r_2 < \infty$ is the region of convergence of the z-transform $X(z)$, then it follows from (20) that the ZZT's region of convergence is $r_1^M < |z| < r_2^M$. If $X(z)$ converges at ∞, the same need not be true for $\mathcal{Z}_x(n, z)$ due to the factor $z^{n/M}$ in (20). Some basic properties of the ZZT are listed in the following (cf. Section 1.1.3). The ZZT is quasiperiodic in n,

$$\mathcal{Z}_x(n + lM, z) = z^l\, \mathcal{Z}_x(n, z) ,$$

so that it suffices to calculate the ZZT in the fundamental time interval $0 \le n \le M - 1$. Time reversal plus conjugation of the signal changes the ZZT as follows:

$$\tilde{x}[n] = x^*[-n] \qquad \Longrightarrow \qquad \mathcal{Z}_{\tilde{x}}(n, z) = \mathcal{Z}_x^*\left(-n, \frac{1}{z^*}\right) = z\, \mathcal{Z}_x^*\left(M - n, \frac{1}{z^*}\right) . \tag{21}$$

The signal $x[n]$ can be recovered from its ZZT according to

$$x[n] = \frac{1}{2\pi j} \oint \mathcal{Z}_x(n,z)\, z^{-1}\, dz\,,$$

where the contour of integration is counterclockwise, encloses the origin, and lies in the region of convergence of $\mathcal{Z}_x(n,z)$. The z-transform of $x[n]$ can be recovered from the ZZT as

$$X(z) = \sum_{n=0}^{M-1} \mathcal{Z}_x(n, z^M)\, z^{-n}.$$

2.2 ZZT and Polyphase Representation

A comparison of (19) with (14) and (16) shows that the transfer function of the nth polyphase filter equals the ZZT, at time n, of the corresponding filter impulse response:

$$E_n(z) = \mathcal{Z}_h(n,z)\,, \qquad R_n(z) = \mathcal{Z}_f(n,z)\,. \tag{22}$$

The polyphase representation is thus equivalent to the ZZT, and the polyphase representation evaluated on the unit circle is equivalent to the ZT.[15] It is interesting that the equivalent concepts of polyphase representation and ZT have been developed independently in filter bank theory and Gabor expansion theory, respectively. While only the case of critical sampling will be considered in this paper, we note that the polyphase representation plays an important role in integer oversampled DFT filter banks as well,[28,29] and that the ZT has also been applied to integer oversampled Gabor expansions.[9]

2.3 ZZT and Gabor Expansion

The application of the ZT to the Gabor expansion has been summarized in Section 1.1.3. The relevant equations will now be reformulated using the ZZT.

- In the ZZT domain, the effect of the frame operator $\mathbf{S}$ or inverse frame operator $\mathbf{S}^{-1}$ becomes a simple multiplication or division, respectively (cf. (6)),

$$\mathcal{Z}_{\mathbf{S}x}(n,z) = M\,[\mathcal{Z}_g(n,z)\,\tilde{\mathcal{Z}}_g(n,z)]\,\mathcal{Z}_x(n,z)\,, \qquad \mathcal{Z}_{\mathbf{S}^{-1}x}(n,z) = \frac{\mathcal{Z}_x(n,z)}{M\,\mathcal{Z}_g(n,z)\,\tilde{\mathcal{Z}}_g(n,z)}\,, \tag{23}$$

  where

$$\tilde{\mathcal{Z}}_g(n,z) \stackrel{\text{def}}{=} \mathcal{Z}_g^*\left(n, \frac{1}{z^*}\right) \tag{24}$$

  is the *paraconjugate*[6] of $\mathcal{Z}_g(n,z)$. On the unit circle, paraconjugation simply means conjugation, i.e., $\tilde{\mathcal{Z}}_g(n, e^{j2\pi\theta}) = \mathcal{Z}_g^*(n, e^{j2\pi\theta})$.

- The frame condition (7) becomes

$$A \;\le\; M\mathcal{Z}_g(n,z)\,\tilde{\mathcal{Z}}_g(n,z) \;\le\; B\,.$$

  We conclude that the set $\{g_{l,m}[n]\}$ is a *tight* WHF if and only if

$$M\mathcal{Z}_g(n,z)\,\tilde{\mathcal{Z}}_g(n,z) = A\,.$$

- The ZZT of the analysis window $\gamma[n]$ is given by (cf. (8))

$$\mathcal{Z}_\gamma(n,z) = \frac{1}{M\tilde{\mathcal{Z}}_g(n,z)}\,. \tag{25}$$

3 GABOR EXPANSION AS A DFT FILTER BANK

In Section 1, we reviewed two distinct areas—the Gabor expansion and DFT filter banks—that have been developed independently of each other. We shall now show that these areas are essentially equivalent, and that important results derived independently in each of the two areas are equivalent as well. Furthermore, we shall also see that in each area additional insights can be gained from the respective other area.

3.1 Fundamental Equivalences

Comparing the Gabor analysis and synthesis formulas (2) and (1) with the filter bank analysis and synthesis formulas (9) and (10), respectively, we see that these formulas are fully equivalent if only the Gabor analysis and synthesis windows are related to the analysis and synthesis prototype filters, respectively, according to

$$\gamma[n] = h^*[-n]\,, \qquad g[n] = f[n]\,. \tag{26}$$

In this case, the Gabor coefficients equal the filter bank subband signals,

$$a_{l,m} = v_{M-m}[l]\,, \qquad 0 \le m \le M-1\,.$$

Hence, ***the Gabor expansion can be interpreted as a perfect-reconstruction DFT filter bank.***[15,23]

With (22), (26), (21), and (24), it furthermore follows that the transfer functions of the analysis and synthesis polyphase filters are essentially equal to the ZZTs of the Gabor analysis and synthesis windows, respectively:

$$E_n(z) = \mathcal{Z}_h(n,z) = z\tilde{\mathcal{Z}}_\gamma(M-n,z)\,, \qquad R_n(z) = \mathcal{Z}_f(n,z) = \mathcal{Z}_g(n,z)\,. \tag{27}$$

This shows another important equivalence. WHF theory requires the Gabor analysis and synthesis windows to be related by (25), while the perfect reconstruction property requires the analysis and synthesis prototype filters to be related by (17). Due to (27), these relations are equivalent. Hence, ***WHF theory and DFT filter bank theory lead to the same relation between the analysis and synthesis windows and the analysis and synthesis prototype filters, respectively.***

This can also be interpreted from a different viewpoint. We have mentioned in Section 1.1.2 that the Gabor analysis and synthesis windows satisfy the biorthogonality relation (4), $\langle g, \gamma_{l,m}\rangle = \delta[l]\,\delta[m]$. We shall now show that this biorthogonality relation is equivalent to the central analysis-synthesis filter bank relation (11). Indeed, applying the inverse z-transform to (11) yields

$$\sum_{n=-\infty}^{\infty} h[lM-n]\, f[n]\, W^{mn} = \delta[l]\,\delta[m]\,.$$

With (26), this becomes (4). Conversely, (11) is obtained by applying the z-transform to (4).

3.2 Weyl-Heisenberg Frames and DFT Filter Banks

With these basic equivalences established, some important results from WHF theory can now be applied to DFT filter banks. This will permit us to re-obtain some known facts about DFT filter banks, and to establish new insights as well.

Frame Operator, Polyphase Matrix, and AC Matrix. The polyphase matrix $\boldsymbol{E}(z)$ and the AC matrix $\boldsymbol{H}(z)$ are intimately connected with the frame operator S. With $E_n(z) = z\tilde{\mathcal{Z}}_\gamma(M-n,z) = z\frac{1}{M\tilde{\mathcal{Z}}_g(M-n,z)}$ (see

(25)), the first equation in (23) can be reformulated in vector-matrix notation as

$$\begin{pmatrix} \mathcal{Z}_{\mathbf{S}x}(M,z) \\ \mathcal{Z}_{\mathbf{S}x}(M-1,z) \\ \vdots \\ \mathcal{Z}_{\mathbf{S}x}(1,z) \end{pmatrix} = \left(\tilde{\boldsymbol{E}}(z)\boldsymbol{E}(z)\right)^{-1} \begin{pmatrix} \mathcal{Z}_{x}(M,z) \\ \mathcal{Z}_{x}(M-1,z) \\ \vdots \\ \mathcal{Z}_{x}(1,z) \end{pmatrix}, \tag{28}$$

where $\tilde{\boldsymbol{E}}(z) = \boldsymbol{E}^H\left(\frac{1}{z^*}\right)$. It is easily shown that $\tilde{\boldsymbol{E}}(z)\boldsymbol{E}(z)$ is a diagonal matrix given by

$$\tilde{\boldsymbol{E}}(z)\boldsymbol{E}(z) = M\,\mathrm{diag}\{\tilde{E}_n(z)\,E_n(z)\}_{n=0}^{M-1} = M\,\mathrm{diag}\{\mathcal{Z}_\gamma(M-n,z)\tilde{\mathcal{Z}}_\gamma(M-n,z)\}_{n=0}^{M-1}. \tag{29}$$

Eq. (28) shows that the diagonal matrix $\left(\tilde{\boldsymbol{E}}(z)\boldsymbol{E}(z)\right)^{-1}$ can be interpreted as a *matrix representation* of the frame operator S; the underlying transform mapping the frame operator S onto its matrix representation $\left(\tilde{\boldsymbol{E}}(z)\boldsymbol{E}(z)\right)^{-1}$ is the ZZT or, equivalently, the polyphase transform. The second equation in (23) can be reformulated in a similar manner, showing that the diagonal matrix $\tilde{\boldsymbol{E}}(z)\boldsymbol{E}(z)$ is a matrix representation of the inverse frame operator $\mathbf{S}^{-1}$. All this can equivalently be formulated in terms of the AC matrix $\tilde{\boldsymbol{H}}(z)$ since it follows with (18) that

$$\tilde{\boldsymbol{E}}(z)\boldsymbol{E}(z) = \frac{1}{M^2}\mathbf{W}^H[\tilde{\boldsymbol{H}}(z^{1/M})\boldsymbol{H}(z^{1/M})]^T\,\mathbf{W}. \tag{30}$$

Frame Bounds, Polyphase Filters, and AC Matrix. If $\{g_{l,m}[n]\}$ is a WHF with frame bounds A and B, then $\{\gamma_{l,m}[n]\}$ is a WHF with frame bounds $A' = 1/B$ and $B' = 1/A$.[17] Hence $\frac{1}{B} \le M\,\mathcal{Z}_\gamma(n,z)\tilde{\mathcal{Z}}_\gamma(n,z) \le \frac{1}{A}$ and, using (27),

$$\frac{1}{B} \;\le\; M\tilde{E}_{M-n}(z)E_{M-n}(z) \;\le\; \frac{1}{A}, \tag{31}$$

where $\tilde{E}_n(z) = E_n^*\left(\frac{1}{z^*}\right)$. On the unit circle, this becomes

$$\frac{1}{B} \;\le\; M\,|E_{M-n}(e^{j2\pi\theta})|^2 \;\le\; \frac{1}{A}. \tag{32}$$

Thus, *the numerical properties of the filter bank (as expressed by the frame bounds A, B) are determined by the magnitude responses $|E_n(e^{j2\pi\theta})|$ of the analysis polyphase filters.* According to WHF theory, good numerical properties (snug WHFs) require $A \approx B$ or equivalently $\frac{1}{B} \approx \frac{1}{A}$, which requires that the magnitude responses $|E_n(e^{j2\pi\theta})|$ of the analysis polyphase filters be *flat* (as a function of θ) and approximately equal for all n. In the case of perfect flatness (i.e. $|E_n(e^{j2\pi\theta})|$ is constant with respect to n and θ), we have $A = B$ and $\{g_{l,m}[n]\}$ is a *tight* WHF. On the other hand, if $E_n(e^{j2\pi\theta}) = 0$ for some n and θ, then $B = \infty$ and $\{g_{l,m}[n]\}$ is no longer a frame, which results in poor numerical properties. Conversely, if $\{g_{l,m}[n]\}$ is a frame, then (32) shows that $E_n(e^{j2\pi\theta})$ has no zeros.

The frame bounds A, B characterizing the numerical properties of the WHF or, equivalently, of the DFT filterbank can also be related to the eigenvalues $\Lambda_n(z)$ of the AC matrix $\boldsymbol{H}(z)$ on the unit circle. With (15), (32) can be rewritten as

$$\frac{M}{B} \;\le\; |\Lambda_n(e^{j2\pi\theta})|^2 \;\le\; \frac{M}{A},$$

which shows that the magnitudes of the eigenvalues $\Lambda_n(e^{j2\pi\theta})$ of $\boldsymbol{H}(e^{j2\pi\theta})$ are bounded in terms of the frame bounds. In particular, if $\{g_{l,m}[n]\}$ is a WHF, then $\Lambda_n(e^{j2\pi\theta}) \ne 0$. Hence, *if $\{g_{l,m}[n]\}$ is a WHF, then the AC matrix $\boldsymbol{H}(e^{j2\pi\theta})$ is nonsingular for all θ.* This has an interesting consequence. Let us, for the moment, consider a DFT filterbank that does not necessarily satisfy the perfect-reconstruction property. Usually, the first step in the design of the filter bank is to make sure that aliasing distortion is eliminated[6]; the filter bank can then be viewed as an LTI system whose transfer function will be denoted as $T(e^{j2\pi\theta})$. Then, if the AC matrix $\boldsymbol{H}(e^{j2\pi\theta})$ is nonsingular for all θ, it can be shown[6] that $T(e^{j2\pi\theta}) \ne 0$, provided that the synthesis prototype filter $F(e^{j2\pi\theta})$ is chosen such that not all of the synthesis filter transfer functions $F_m(e^{j2\pi\theta})$ have a zero at the same frequency. (Of course, if $F(e^{j2\pi\theta})$ is chosen such that the perfect-reconstruction property is satisfied, then $T(e^{j2\pi\theta}) \equiv 1$.)

Simple Approximation. With (26), frame theory suggests the following approximation[17] of the synthesis prototype filter:

$$f[n] \approx f_0[n] \stackrel{\text{def}}{=} \frac{A+B}{2} h^*[-n].$$

The resulting reconstruction error, $\varepsilon[n] = \hat{x}[n] - x[n]$, is bounded as[17]

$$\|\varepsilon\| \le \frac{B-A}{B+A} \|x\|.$$

It will be small for polyphase filters with flat magnitude responses (where $A \approx B$ as discussed above). For perfect flatness, $A = B$ and $f[n] = f_0[n] = A\,h^*[-n]$; the approximation is here exact and perfect reconstruction is obtained. In the general case, higher-order approximations with smaller reconstruction errors are also available.[17]

Tight WHFs and Paraunitary DFT Filter Banks. For many reasons, *paraunitary* filter banks are of particular importance.[6] The AC matrix of a paraunitary filter bank satisfies

$$\tilde{\boldsymbol{H}}(z)\,\boldsymbol{H}(z) = K\,\mathbf{I}_M$$

for all z, where $\mathbf{I}_M$ is the $M \times M$ identity matrix and $K > 0$. We now show[15] that *a DFT filter bank is paraunitary if and only if the corresponding Gabor synthesis set* $\{g_{l,m}[n]\}$ *is a tight WHF* *.

$\boldsymbol{H}(z)$ is a right-circulant matrix and therefore can be written as

$$\boldsymbol{H}(z) = \frac{1}{M}\mathbf{W}\boldsymbol{\Lambda}(z)\mathbf{W}^H,$$

where $\boldsymbol{\Lambda}(z) = \mathrm{diag}\{\Lambda_n(z)\}_{n=0}^{M-1}$ and $\mathbf{W}$ is the M-point DFT matrix.[6] It follows further that

$$\tilde{\boldsymbol{H}}(z)\,\boldsymbol{H}(z) = \frac{1}{M}\mathbf{W}\tilde{\boldsymbol{\Lambda}}(z)\,\boldsymbol{\Lambda}(z)\,\mathbf{W}^H = \frac{1}{M}\mathbf{W}\;\mathrm{diag}\{\tilde{\Lambda}_n(z)\Lambda_n(z)\}_{n=0}^{M-1}\;\mathbf{W}^H. \tag{33}$$

Now if $\{g_{l,m}[n]\}$ is a tight WHF, then $\tilde{\Lambda}_n(z)\Lambda_n(z) = M^2\,\tilde{\mathcal{Z}}_\gamma(n,z)\mathcal{Z}_\gamma(n,z) \equiv \frac{M}{A}$. Inserting in (33) yields the paraunitarity condition,

$$\tilde{\boldsymbol{H}}(z)\,\boldsymbol{H}(z) = \frac{1}{A}\,\mathbf{W}\,\mathbf{W}^H = K\,\mathbf{I}_M$$

with $K = \frac{M}{A}$. This proof can be reversed so that, conversely, the paraunitarity condition $\tilde{\boldsymbol{H}}(z)\,\boldsymbol{H}(z) = K\,\mathbf{I}_M$ implies a tight WHF with $A = \frac{M}{K}$.

For a tight WHF, $\gamma[n] = \frac{1}{A}g[n]$ or equivalently $f[n] = Ah^*[-n]$, which means that the synthesis prototype filter follows trivially from the analysis prototype filter. Furthermore, if $h[n]$ is FIR, then $f[n]$ is FIR with the same filter length.[6] Frame theory yields time-domain and frequency-domain conditions for $g[n]$ to induce a tight WHF or equivalently a paraunitary filter bank.[16] Also, based on the ZT a systematic procedure for the construction of tight WHFs and hence paraunitary filter banks can be derived.[17] Conversely, design techniques for paraunitary filter banks[6] can be applied to the construction of tight WHFs.

For a tight WHF, the Gabor frame operator is the identity operator up to a constant factor, $\mathbf{S} = A\mathbf{I}$, and the inverse frame operator is $\mathbf{S}^{-1} = \frac{1}{A}\mathbf{I}$. We shall now show that in the paraunitary case the filterbank counterpart of the inverse frame operator, namely the diagonal matrix $\tilde{\boldsymbol{E}}(z)\boldsymbol{E}(z)$, is the identity matrix up to a constant factor. According to (29) we have $\tilde{\boldsymbol{E}}(z)\boldsymbol{E}(z) = M\,\mathrm{diag}\{\tilde{E}_n(z)E_n(z)\}_{n=0}^{M-1}$. Using (31) with $A = B$ immediately yields

$$\tilde{\boldsymbol{E}}(z)\boldsymbol{E}(z) = \frac{1}{A}\mathbf{I}_M.$$

Note that this result could alternatively have been derived from $\tilde{\boldsymbol{H}}(z)\,\boldsymbol{H}(z) = \frac{M}{A}\,\mathbf{I}_M$ using (30).

*A similar result has been independently reported in Ref. [30].

In the case of critical sampling considered here, tight WHFs can be shown to be orthogonal function sets. The relation between paraunitary filter banks and orthogonal wavelet bases is well known.[5,2] We have shown above that an analogous relation exists for the Gabor expansion and DFT filter banks. The equivalence of tight WHFs and paraunitary DFT filter banks can be extended to the oversampled case.[24]

4 CONCLUSION

We have shown the equivalence of the Gabor expansion and Weyl-Heisenberg frame (WHF) theory on the one hand and DFT filter banks on the other—two important fields that have been treated independently so far. Based on the equivalence of the z-Zak transform and the polyphase representation, we have derived a number of further equivalences and correspondences. Among other results, it has been shown that the transfer functions of the polyphase filters determine the frame bounds and hence the numerical properties of the filter bank, and that tight WHFs correspond to paraunitary filter banks. With these basic equivalences established, it will now be possible to reformulate other results of WHF theory in the filter bank setting, and vice versa. While only the case of critical sampling has been considered for simplicity, many of the results presented carry over to the case of oversampling.

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