TIME-FREQUENCY ANALYSIS OF FRAMES*

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Abstract—The theory of *frames* is fundamental to time-frequency (TF) or time-scale signal expansions like Gabor expansions and wavelet transforms. We propose a *TF analysis of frames* via two "TF frame representations" called the *Weyl symbol and Wigner distribution of a frame*. The TF analysis shows how a frame's properties depend on the signal's TF location and on certain frame parameters.

1 INTRODUCTION

Linear time-frequency (TF) or time-scale signal expansions like the Gabor expansion or the wavelet transform [1]-[3] are often based on nonorthogonal function sets. The mathematical theory of *frames* [3]-[5] yields important insights into the properties of nonorthogonal signal expansions, as well as methods for calculating the expansion coefficients.

Review of Frame Theory. Let $\mathcal{X} \subseteq \mathcal{L}_2(\mathbb{R})$ be a Hilbert space of finite-energy signals, with dimension $D_{\mathcal{X}}$ that may be ∞. A set of functions $\mathcal{G} = \{g_k(t)\}$ with $g_k(t) \in \mathcal{X}$ is a *frame* for $\mathcal{X}$ if for every signal $x(t) \in \mathcal{X}$

$$A_{\mathcal{G}} \|x\|^2 \leq \sum_k \left|\langle x, g_k \rangle\right|^2 \leq B_{\mathcal{G}} \|x\|^2 \qquad (1)$$

with $0 < A_{\mathcal{G}} \leq B_{\mathcal{G}} < \infty$. Here, $\langle x, g_k \rangle = \int_t x(t)\, g_k^*(t)\, dt$ is the inner product[1] of $x(t)$ with $g_k(t)$, and $\|x\|^2 = \langle x, x \rangle$ is the energy of $x(t)$. The constants $A_{\mathcal{G}}$ and $B_{\mathcal{G}}$ are called *frame bounds*. Frame theory now shows [3] that any signal $x(t) \in \mathcal{X}$ can be expanded into the frame functions $g_k(t)$ as

$$x(t) = \sum_k \alpha_k\, g_k(t) \qquad \text{with } \alpha_k = \langle x, \tilde{g}_k \rangle \qquad (2)$$

where

$$\tilde{g}_k(t) = \left(\mathbf{G}^{-1} g_k\right)(t) \in \mathcal{X}\,. \qquad (3)$$

Here, the *frame operator* $\mathbf{G}$ is defined as

$$(\mathbf{G}x)(t) = \sum_k \langle x, g_k \rangle\, g_k(t) = \int_{t'} G(t,t')\, x(t')\, dt'$$

with the kernel

$$G(t,t') = \sum_k g_k(t)\, g_k^*(t')\,. \qquad (4)$$

$\mathbf{G}$ is a self-adjoint, positive semidefinite, linear operator [6] that maps $\mathcal{L}_2(\mathbb{R})$ into $\mathcal{X}$. On $\mathcal{X}$, $\mathbf{G}$ is positive definite and invertible, i.e., $\mathbf{G}$ is also an invertible mapping from $\mathcal{X}$ onto $\mathcal{X}$. We note that $(\mathbf{G}x)(t) = 0$ for $x(t) \perp \mathcal{X}$. Eq. (1) can be rewritten as $A_{\mathcal{G}}\|x\|^2 \leq \langle \mathbf{G}x, x \rangle \leq B_{\mathcal{G}}\|x\|^2$ for all $x(t) \in \mathcal{X}$. This shows that the *tightest possible* frame bounds (denoted $A_{\mathcal{G}}^T$, $B_{\mathcal{G}}^T$) are given by the infimum and supremum, respectively, of the eigenvalues of $\mathbf{G}$.

The functions $\tilde{g}_k(t)$ in (2), (3) constitute another frame $\tilde{\mathcal{G}} = \{\tilde{g}_k(t)\}$ for $\mathcal{X}$ which is called the *dual frame*. For the dual frame, the frame bounds are $A_{\tilde{\mathcal{G}}} = 1/B_{\mathcal{G}}$ and $B_{\tilde{\mathcal{G}}} = 1/A_{\mathcal{G}}$, and the frame operator is (on $\mathcal{X}$) $\tilde{\mathbf{G}} = \mathbf{G}^{-1}$.

A frame $\mathcal{G}$ is *complete* in the space $\mathcal{X}$, but the frame functions $g_k(t)$ need not be *linearly independent*. A frame with linearly independent $g_k(t)$ (called *exact* frame) satisfies the biorthogonality relations $\langle g_k, \tilde{g}_l \rangle = \delta_{kl}$. A frame is called *tight* if $A_{\mathcal{G}} = B_{\mathcal{G}}$. Here, $\mathbf{G} = A_{\mathcal{G}}\, \mathbf{P}_{\mathcal{X}}$, where $\mathbf{P}_{\mathcal{X}}$ is the orthogonal projection operator on $\mathcal{X}$, and $\tilde{g}_k(t) = g_k(t)/A_{\mathcal{G}}$ so that calculation of the dual frame is trivial. An *orthonormal basis* is a special case of a tight frame with $A_{\mathcal{G}} = B_{\mathcal{G}} = 1$. A frame with $A_{\mathcal{G}} \approx B_{\mathcal{G}}$ is called *snug*. Closer frame bounds $A_{\mathcal{G}}$ and $B_{\mathcal{G}}$ entail better numerical properties of the expansion (2) and more efficient algorithms for calculating the dual frame. Indeed, (3) can be expanded as

$$\tilde{g}_k(t) = C \sum_{n=0}^{\infty} \left((\mathbf{I} - C\,\mathbf{G})^n g_k\right)(t)\,, \quad C = \frac{2}{A_{\mathcal{G}} + B_{\mathcal{G}}}\,, \qquad (5)$$

which converges faster for closer $A_{\mathcal{G}}$, $B_{\mathcal{G}}$ [3]. For snug frames, $\tilde{g}_k(t)$ can hence be approximated by truncating the series (5). In particular, truncation after the $n = 0$ term yields $\tilde{g}_k(t) \approx C\, g_k(t)$ and, with (2),

$$x(t) \approx x^{(0)}(t) = C \sum_k \langle x, g_k \rangle\, g_k(t)\,. \qquad (6)$$

Motivation and Outline. The frame bounds $A_{\mathcal{G}}$, $B_{\mathcal{G}}$ do not show how certain parameters of a frame could be changed in order to improve the frame's numerical properties. This information can often be obtained from the *TF analysis of frames* proposed in this paper. The TF analysis also shows how a frame's properties depend on the TF location of the signal to be expanded; in particular, a frame may be "locally snug" in restricted TF regions. We propose two TF frame representations, the *Weyl symbol* and the *Wigner distribution* of a frame, both of which generalize the *Wigner distribution of a linear signal space* [7, 8] and satisfy interesting properties. Local averages of these TF representations are bounded in terms of the frame bounds. Some examples show the usefulness of the TF analysis proposed.

Trace, Inner Product, Energy. For use in subsequent sections, we define the *trace* $T_{\mathcal{G}}$ *of a frame* $\mathcal{G}$ as the trace of the frame operator $\mathbf{G}$,

$$T_{\mathcal{G}} \triangleq \operatorname{tr}\{\mathbf{G}\} = \int_t G(t,t)\, dt = \sum_k \|g_k\|^2\,.$$

We also define the *inner product of two frames* $\mathcal{G}$ *and* $\mathcal{H}$ as

$$\langle \mathcal{G}, \mathcal{H} \rangle \triangleq \operatorname{tr}\{\mathbf{GH}\} = \int_t \int_{t'} G(t,t')\, H^*(t,t')\, dt dt' = \sum_k \sum_l \left|\langle g_k, h_l \rangle\right|^2,$$

and the *energy of a frame* $\mathcal{G}$ as

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[1] Integrals go from $-\infty$ to ∞.

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$$\|\mathcal{G}\|^2 \triangleq \operatorname{tr}\{\mathbf{G}^2\} = \langle \mathcal{G},\mathcal{G}\rangle = \int_t\int_{t'} |G(t,t')|^2\, dt dt' = \sum_k\sum_l \left|\langle g_k, g_l\rangle\right|^2 .$$

The following bounds and relations can be shown:

$$A_\mathcal{G} D_\mathcal{X} \le T_\mathcal{G} \le B_\mathcal{G} D_\mathcal{X} ,$$
$$\max\{A_\mathcal{H} T_\mathcal{G},\, A_\mathcal{G} T_\mathcal{H}\} \le \langle \mathcal{G},\mathcal{H}\rangle \le \min\{B_\mathcal{H} T_\mathcal{G},\, B_\mathcal{G} T_\mathcal{H}\},$$
$$A_\mathcal{G}^2 D_\mathcal{X} \le A_\mathcal{G} T_\mathcal{G} \le \|\mathcal{G}\|^2 \le B_\mathcal{G} T_\mathcal{G} \le B_\mathcal{G}^2 D_\mathcal{X} ,$$
$$\|\mathcal{G}\|^2 \le T_\mathcal{G}^2, \qquad \langle \mathcal{G},\tilde{\mathcal{G}}\rangle = D_\mathcal{X} .$$

For a *tight* frame, we have $T_\mathcal{G} = A_\mathcal{G} D_\mathcal{X}$ and $\|\mathcal{G}\|^2 = A_\mathcal{G}^2 D_\mathcal{X}$.

2 WEYL SYMBOL OF A FRAME

The *Weyl symbol* (WS) is an important TF representation of linear operators [9, 10]. We define the *Weyl symbol* $L_\mathcal{G}(t,f)$ *of a frame* $\mathcal{G}$ as the WS of the frame operator $\mathbf{G}$,

$$L_\mathcal{G}(t,f) \triangleq \int_\tau G\Big(t+\frac{\tau}{2}, t-\frac{\tau}{2}\Big)\, e^{-j2\pi f\tau}\, d\tau .$$

This is a realvalued but (in general) not everywhere nonnegative function of time t and frequency f. With (4),

$$L_\mathcal{G}(t,f) = \sum_k W_{g_k}(t,f)$$

where $W_{g_k}(t,f) = \int_\tau g_k(t+\tau/2)\, g_k^*(t-\tau/2)\, e^{-j2\pi f\tau}\, d\tau$ is the Wigner distribution (WD) of $g_k(t)$ [11]. Hence, the WS of $\mathcal{G}$ is simply the sum of the WDs of all frame functions $g_k(t)$, and thus indicates the frame's TF location.

Tight Frames. For a *tight* frame, we have

$$L_\mathcal{G}(t,f) = A_\mathcal{G}\, W_\mathcal{X}(t,f), \qquad L_{\tilde{\mathcal{G}}}(t,f) = W_\mathcal{X}(t,f)/A_\mathcal{G} ,$$

where $W_\mathcal{X}(t,f)$ is the *WD of the space* $\mathcal{X}$ [7, 8]. For an orthonormal basis (or, more generally, any tight frame with $A_\mathcal{G}=1$), there is $L_\mathcal{G}(t,f) = L_{\tilde{\mathcal{G}}}(t,f) = W_\mathcal{X}(t,f)$. If $\mathcal{G}$ is a tight frame for $\mathcal{X} = \mathcal{L}_2(\mathbb{R})$, then $W_\mathcal{X}(t,f) \equiv 1$ [7] and the WS is constant over the entire TF plane,

$$L_\mathcal{G}(t,f) \equiv A_\mathcal{G}, \qquad L_{\tilde{\mathcal{G}}}(t,f) \equiv 1/A_\mathcal{G} .$$

Integral Relations and Bounds. The WS can be considered as a *TF distribution of the frame's trace* since

$$\int_t\int_f L_\mathcal{G}(t,f)\, dt df = T_\mathcal{G} .$$

The inner product of the WSs of two frames equals the inner product of the frames,

$$\langle L_\mathcal{G}, L_\mathcal{H}\rangle = \int_t\int_f L_\mathcal{G}(t,f)\, L_\mathcal{H}(t,f)\, dt df = \langle \mathcal{G},\mathcal{H}\rangle ,$$

which will be zero for frames $\mathcal{G}$ and $\mathcal{H}$ whose underlying spaces are orthogonal; the WSs are here orthogonal as well. The squared norm of the WS equals the frame's energy,

$$\|L_\mathcal{G}\|^2 = \int_t\int_f L_\mathcal{G}^2(t,f)\, dt df = \|\mathcal{G}\|^2 .$$

The inner product of the WS of a frame $\mathcal{G}$ with the WD of a signal $x(t) \in \mathcal{L}_2(\mathbb{R})$ is

$$\langle L_\mathcal{G}, W_x\rangle = \langle \mathbf{G}x, x\rangle = \sum_k \left|\langle x, g_k\rangle\right|^2 ,$$

which will be zero for $x(t) \perp \mathcal{X}$. The WS satisfies the following bounds and relations:

$$A_\mathcal{G} D_\mathcal{X} \le \int_t\int_f L_\mathcal{G}(t,f)\, dt df \le B_\mathcal{G} D_\mathcal{X} ,$$
$$\max\{A_\mathcal{H} T_\mathcal{G},\, A_\mathcal{G} T_\mathcal{H}\} \le \langle L_\mathcal{G}, L_\mathcal{H}\rangle \le \min\{B_\mathcal{H} T_\mathcal{G},\, B_\mathcal{G} T_\mathcal{H}\},$$
$$A_\mathcal{G}^2 D_\mathcal{X} \le A_\mathcal{G} T_\mathcal{G} \le \|L_\mathcal{G}\|^2 \le B_\mathcal{G} T_\mathcal{G} \le B_\mathcal{G}^2 D_\mathcal{X} ,$$
$$\|L_\mathcal{G}\|^2 \le \left(\int_t\int_f L_\mathcal{G}(t,f)\, dt df\right)^2, \qquad \langle L_\mathcal{G}, L_{\tilde{\mathcal{G}}}\rangle = D_\mathcal{X} ,$$
$$A_\mathcal{G}\, \|x\|^2 \le \langle L_\mathcal{G}, W_x\rangle \le B_\mathcal{G}\, \|x\|^2 \quad \text{for } x(t) \in \mathcal{X} . \qquad (7)$$

For a *tight* frame, we have $\int_t\int_f L_\mathcal{G}(t,f)\, dt df = A_\mathcal{G} D_\mathcal{X}$, $\|L_\mathcal{G}\|^2 = A_\mathcal{G}^2 D_\mathcal{X}$, and $\langle L_\mathcal{G}, W_x\rangle = A_\mathcal{G}\, \|x\|^2$ for $x(t) \in \mathcal{X}$.

Local Averages and Frame Bounds. The inequality (7) relates the WS with the frame bounds. Let $h(t) \in \mathcal{X}$ be a normalized "test signal" which is well localized about a given TF point (t_0, f_0). The WD of $h(t)$ is then normalized as $\int_t\int_f W_h(t,f)\, dt df = 1$, well localized about (t_0, f_0), and predominantly nonnegative. Thus, the inner product $\langle L_\mathcal{G}, W_h\rangle = \int_t\int_f L_\mathcal{G}(t,f)\, W_h(t,f)\, dt df$ can be interpreted as a *local average* of the WS $L_\mathcal{G}(t,f)$ over a TF region of area ≈ 1, centered about (t_0, f_0). Due to (7), we have

$$A_\mathcal{G} \le \langle L_\mathcal{G}, W_h\rangle \le B_\mathcal{G} . \qquad (8)$$

While this bound does not say anything about the *pointwise* behavior of the WS, it shows that the WS may not be consistently $< A_\mathcal{G}$ or $> B_\mathcal{G}$ in any TF region with area ≈ 1. In this sense, the WS indicates the "snugness" and numerical properties of a frame. In particular, if the WS consistently assumes low values in a TF region of area ≥ 1 and high values in another TF region of area ≥ 1, then we know that the frame bounds must be widely different and the frame is not snug. Conversely, if the WS is approximately constant over the entire TF region corresponding to the underlying space $\mathcal{X}$, then the frame is guaranteed to be snug. This interpretation will be refined in Section 4.

If λ and $u(t)$ denote the eigenvalues and normalized eigenfunctions, respectively, of the frame operator $\mathbf{G}$, then

$$\langle L_\mathcal{G}, W_u\rangle = \langle \mathbf{G}u, u\rangle = \lambda ,$$

and it follows that the tightest possible frame bounds can be obtained from the frame's WS according to

$$A_\mathcal{G}^T = \inf \lambda = \inf_u \langle L_\mathcal{G}, W_u\rangle, \quad B_\mathcal{G}^T = \sup \lambda = \sup_u \langle L_\mathcal{G}, W_u\rangle .$$

Covariance Properties. The WS of a frame is "covariant" to certain unitary transformations of a frame. Let us transform a frame $\mathcal{G} = \{g_k(t)\}$ into a new frame $\mathcal{H} = \{h_k(t)\}$ (for a transformed signal space) by TF-shifting all frame signals by time τ and frequency ν, i.e. $h_k(t) = g_k(t-\tau)\, e^{j2\pi\nu t}$. The WS of the "TF-shifted frame" is then

$$L_\mathcal{H}(t,f) = L_\mathcal{G}(t-\tau, f-\nu) .$$

For a TF-scaling $h_k(t) = \sqrt{|a|}\, g_k(at)$, we obtain

$$L_\mathcal{H}(t,f) = L_\mathcal{G}(at, f/a) .$$

Similar covariance properties exist for certain other unitary transformations, such as the multiplication or convolution by a chirp signal, the Fourier transform, etc. These frame transformations correspond to area-preserving, affine TF coordinate transforms in the WS.

Sum Property. Let $\mathcal{G} = \{g_k(t)\}$ and $\mathcal{H} = \{h_l(t)\}$ be two frames for the same signal space $\mathcal{X}$, and define the *sum* of the frames $\mathcal{G}$ and $\mathcal{H}$ as $\mathcal{G}+\mathcal{H} = \{g_k(t)\} \cup \{h_l(t)\}$. $\mathcal{G}+\mathcal{H}$ is again a frame for $\mathcal{X}$, with frame operator $\mathbf{G}+\mathbf{H}$ and WS

$$L_{\mathcal{G}+\mathcal{H}}(t,f) = L_\mathcal{G}(t,f) + L_\mathcal{H}(t,f) .$$

3 WIGNER DISTRIBUTION OF A FRAME

Besides the WS, another important TF representation of a (normal) linear operator is the operator's *Wigner distribution* (WD) [12]. We define the *Wigner distribution* $W_{\mathcal{G}}(t,f)$ *of a frame* $\mathcal{G}$ as the WD of the frame operator $\mathbf{G}$, which equals the WS of the squared frame operator $\mathbf{G}^2$,

$$W_{\mathcal{G}}(t,f) \triangleq \int_{\tau} G^{(2)}\left(t+\frac{\tau}{2}, t-\frac{\tau}{2}\right) e^{-j2\pi f\tau}\, d\tau$$

with $G^{(2)}(t,t') = \int_s G(t,s)\, G(s,t')\, ds$. $W_{\mathcal{G}}(t,f)$ is realvalued but not necessarily nonnegative. With (4), we obtain

$$W_{\mathcal{G}}(t,f) = \sum_k \sum_l \langle g_k, g_l\rangle^* \, W_{g_k,g_l}(t,f)$$

with $W_{g_k,g_l}(t,f) = \int_{\tau} g_k(t+\tau/2)\, g_l^*(t-\tau/2)\, e^{-j2\pi f\tau}\, d\tau$ [11].

Tight Frames. For a *tight* frame, we have

$$W_{\mathcal{G}}(t,f) = A_{\mathcal{G}}^2\, W_{\mathcal{X}}(t,f)\,, \qquad W_{\tilde{\mathcal{G}}}(t,f) = W_{\mathcal{X}}(t,f)/A_{\mathcal{G}}^2\,.$$

In the case of an orthonormal basis (or, more generally, any tight frame with $A_{\mathcal{G}} = 1$), the WD equals the WS and also the WD of the space $\mathcal{X}$, $W_{\mathcal{G}}(t,f) = W_{\tilde{\mathcal{G}}}(t,f) = L_{\mathcal{G}}(t,f) = L_{\tilde{\mathcal{G}}}(t,f) = W_{\mathcal{X}}(t,f)$. If $\mathcal{G}$ is a tight frame for $\mathcal{X} = \mathcal{L}_2(\mathbb{R})$, then

$$W_{\mathcal{G}}(t,f) \equiv A_{\mathcal{G}}^2\,, \qquad W_{\tilde{\mathcal{G}}}(t,f) \equiv 1/A_{\mathcal{G}}^2\,.$$

Note that here $W_{\mathcal{G}}(t,f) = [L_{\mathcal{G}}(t,f)]^2$.

Integral Relations and Bounds. The WD of a frame is a *TF distribution of the frame's energy* since

$$\int_t \int_f W_{\mathcal{G}}(t,f)\, dt df = \|\mathcal{G}\|^2\,.$$

The inner product of the WDs of two frames is

$$\langle W_{\mathcal{G}}, W_{\mathcal{H}}\rangle = \mathrm{tr}\{\mathbf{G}^2\mathbf{H}^2\} = \sum_k \sum_l \sum_m \sum_n \langle g_k, g_l\rangle^* \langle h_m, h_n\rangle \langle g_k, h_m\rangle \langle g_l, h_n\rangle^*$$

which will be zero for frames $\mathcal{G}$ and $\mathcal{H}$ whose underlying spaces are orthogonal; the WDs are here orthogonal as well. The squared norm of the WD is

$$\|W_{\mathcal{G}}\|^2 = \mathrm{tr}\{\mathbf{G}^4\}\,.$$

The inner product of the WD of a frame $\mathcal{G}$ with the WD of a signal $x(t) \in \mathcal{L}_2(\mathbb{R})$ is the energy of the signal $(\mathbf{G}x)(t)$,

$$\langle W_{\mathcal{G}}, W_x\rangle = \|\mathbf{G}x\|^2\,,$$

which will be zero for $x(t) \perp \mathcal{X}$. The WD satisfies the following bounds and relations:

$$A_{\mathcal{G}}^2 D_{\mathcal{X}} \le A_{\mathcal{G}} T_{\mathcal{G}} \le \int_t\int_f W_{\mathcal{G}}(t,f)\, dt df \le B_{\mathcal{G}} T_{\mathcal{G}} \le B_{\mathcal{G}}^2 D_{\mathcal{X}}\,,$$

$$\langle W_{\mathcal{G}}, W_{\mathcal{H}}\rangle \le T_{\mathcal{G}}^2 T_{\mathcal{H}}^2\,, \quad \|W_{\mathcal{G}}\|^2 \le T_{\mathcal{G}}^4\,, \quad \langle W_{\mathcal{G}}, W_{\tilde{\mathcal{G}}}\rangle = D_{\mathcal{X}}\,,$$

$$\int_t\int_f W_{\mathcal{G}}(t,f)\, dt df \le \left(\int_t\int_f L_{\mathcal{G}}(t,f)\, dt df\right)^2,$$

$$A_{\mathcal{G}}^2\, \|x\|^2 \le \langle W_{\mathcal{G}}, W_x\rangle \le B_{\mathcal{G}}^2\, \|x\|^2 \quad \text{for } x(t) \in \mathcal{X}\,. \qquad (9)$$

For a *tight* frame, we have $\int_t \int_f W_{\mathcal{G}}(t,f)\, dt df = A_{\mathcal{G}}^2 D_{\mathcal{X}}$, $\|W_{\mathcal{G}}\|^2 = A_{\mathcal{G}}^4 D_{\mathcal{X}}$, and $\langle W_{\mathcal{G}}, W_x\rangle = A_{\mathcal{G}}^2\, \|x\|^2$ for $x(t) \in \mathcal{X}$.

Local Averages and Frame Bounds. For a normalized "test signal" $h(t) \in \mathcal{X}$ localized about a TF point (t_0, f_0), the inner product $\langle W_{\mathcal{G}}, W_h\rangle = \int_t \int_f W_{\mathcal{G}}(t,f)\, W_h(t,f)\, dt df$ is a local average of the WD $W_{\mathcal{G}}(t,f)$ about the TF point (t_0, f_0). With (9), this local average is bounded as

$$A_{\mathcal{G}}^2 \le \langle W_{\mathcal{G}}, W_h\rangle \le B_{\mathcal{G}}^2\,.$$

The discussion of this result is completely analogous to that of the WS result (8). We have furthermore

$$\langle W_{\mathcal{G}}, W_u\rangle = \langle \mathbf{G}^2 u, u\rangle = \lambda^2$$

where λ and $u(t)$ are the eigenvalues and normalized eigenfunctions, respectively, of $\mathbf{G}$. Thus, the tightest possible frame bounds are obtained from the WD as

$$A_{\mathcal{G}}^T = \inf_u \sqrt{\langle W_{\mathcal{G}}, W_u\rangle}\,, \qquad B_{\mathcal{G}}^T = \sup_u \sqrt{\langle W_{\mathcal{G}}, W_u\rangle}\,.$$

Covariance Properties. The WD of a frame satisfies the same covariance properties as the WS, i.e. frame transformations by TF shifts or scalings, multiplication or convolution by chirp signals, Fourier transform etc. correspond to area-preserving affine TF coordinate transforms in the WD.

Sum Property and Cross-WD. The WD of the sum of two frames $\mathcal{G}$ and $\mathcal{H}$ for the same signal space is

$$W_{\mathcal{G}+\mathcal{H}}(t,f) = W_{\mathcal{G}}(t,f) + W_{\mathcal{H}}(t,f) + 2\,\mathrm{Re}\{W_{\mathcal{G},\mathcal{H}}(t,f)\}$$

where the *cross-WD* $W_{\mathcal{G},\mathcal{H}}(t,f)$ is defined as the WS of the composite operator $\mathbf{GH}$. It follows that

$$W_{\mathcal{G},\mathcal{H}}(t,f) = \sum_k \sum_l \langle g_k, h_l\rangle^* \, W_{g_k,h_l}(t,f)\,,$$

$$W_{\mathcal{H},\mathcal{G}}(t,f) = W_{\mathcal{G},\mathcal{H}}^*(t,f)\,, \qquad W_{\mathcal{G},\mathcal{G}}(t,f) = W_{\mathcal{G}}(t,f)\,,$$

$$\int_t \int_f W_{\mathcal{G},\mathcal{H}}(t,f)\, dt df = \langle \mathcal{G}, \mathcal{H}\rangle\,, \qquad W_{\mathcal{G},\tilde{\mathcal{G}}}(t,f) = W_{\mathcal{X}}(t,f)\,.$$

4 EXAMPLES

We shall illustrate the TF analysis of frames by some examples. Due to space restrictions, only the WS will be considered; however, a similar discussion applies to the WD.

Improving Frame Snugness. The TF analysis can yield valuable information on how to change the parameters of a frame in order to improve the frame's snugness. *Fig. 1(a)* shows a segment of the WS of a Weyl-Heisenberg (WH) frame [3] for $\mathcal{X} = \mathcal{L}_2(\mathbb{R})$, i.e., $\mathcal{G} = \{g_{kl}(t)\}$ with the "Gabor logons" $g_{kl}(t) = g(t-kT)\, e^{j2\pi lFt}$ $(-\infty < k,l < \infty)$, where $g(t)$ is a suitable function and $TF \le 1$ [3]. The WS of $\mathcal{G}$ is

$$L_{\mathcal{G}}(t,f) = \sum_k \sum_l W_{g_{kl}}(t,f) = \sum_k \sum_l W_g(t-kT, f-lF)\,,$$

which is T-periodic in t and F-periodic in f. The WH frame in Fig. 1(a) uses a Gaussian $g(t)$ and $TF = 1/2$. The large dynamic range of the WS ($\max L_{\mathcal{G}}(t,f)/\min L_{\mathcal{G}}(t,f) = 2.4323$) indicates that the frame is not snug. Indeed, the ratio of the tightest possible frame bounds (calculated via the Zak transform [3, 13]) is $B_{\mathcal{G}}^T/A_{\mathcal{G}}^T = 2.4323$, which equals[2] $\max L_{\mathcal{G}}(t,f)/\min L_{\mathcal{G}}(t,f)$. The variations of the WS in the t direction indicate that the logons' time spacing T is too large, causing "energy gaps" between logons adjacent with respect to t. *Fig. 1(b)* depicts the WS of a WH frame with the same $g(t)$ but T reduced by one half (i.e. $TF = 1/4$). The WS is practically constant, indicating that this frame

[2]For $TF = 1/(2n)$ with $n \in \mathbb{N}$, one can show [14] a relation of the WS with the Zak transform, from which it follows that $\min L_{\mathcal{G}}(t,f) = A_{\mathcal{G}}^T$ and $\max L_{\mathcal{G}}(t,f) = B_{\mathcal{G}}^T$.

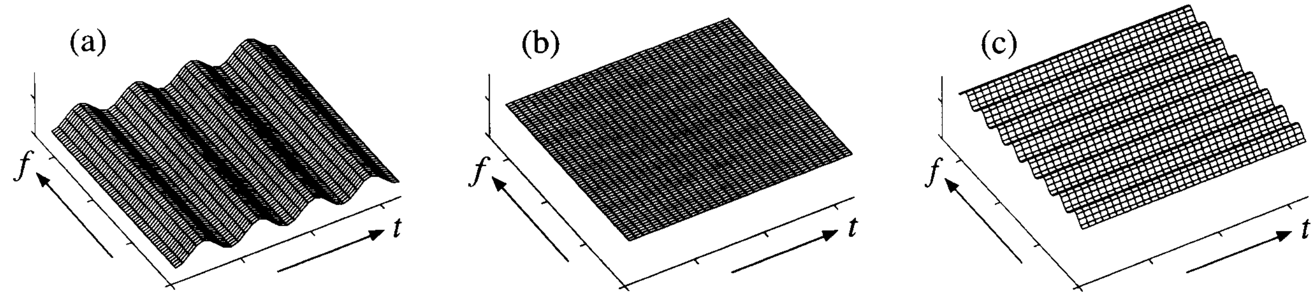

Fig. 1: WS (segment) of a WH frame with (a) $TF = 1/2$, (b) $TF = 1/4$ and "correct" logon spread, and (c) $TF = 1/4$ and "incorrect" logon spread.

is snug; indeed, $\max L_G(t,f)/\min L_G(t,f) = B_G^T/A_G^T = 1.0151 \approx 1$. Finally, *Fig. 1(c)* shows the WS of a WH frame with T, F as in Fig. 1(b); however, the Gaussian $g(t)$ now has a larger time spread so that its effective bandwidth is too small as compared to the logons' frequency spacing F. This is correctly indicated by the WS variations in the f direction. Indeed, $\max L_G(t,f)/\min L_G(t,f) = B_G^T/A_G^T = 1.1892$, which means poorer snugness even though the oversampling factor $1/(TF) = 4$ is the same as before.

Local Snugness. The TF analysis of frames also leads to the new concept of "local snugness." *Fig. 2(a)* shows the WS of a frame for a finite-dimensional signal space. This frame is not snug globally but *locally snug* in the sense that the frame's WS is nearly constant in a specific TF region $\mathcal{R}$. We hypothesize that, for a signal $x(t)$ concentrated in $\mathcal{R}$, the numerical properties of the frame expansion are as if the frame were snug in the entire TF plane. This hypothesis was verified by calculating the "zero-order expansions" $x_i^{(0)}(t) = C_i \sum_k \langle x_i, g_k \rangle g_k(t)$ (cf. (6)) of two Gaussian signals $x_1(t)$ and $x_2(t)$ located inside and outside $\mathcal{R}$, respectively (see *Fig. 2(b)*). The factors C_i were chosen as $C_i = \arg\min_C \|x_i^{(0)} - x_i\|$ in order to optimally approximate the true signals $x_i(t)$. The normalized approximation errors $\epsilon_i = \|x_i^{(0)} - x_i\|/\|x_i\|$ were obtained as $\epsilon_1 = 0.015$ and $\epsilon_2 = 0.804$. As expected, the error is very small for $x_1(t)$ (localized in $\mathcal{R}$) but large for $x_2(t)$ (localized outside $\mathcal{R}$).

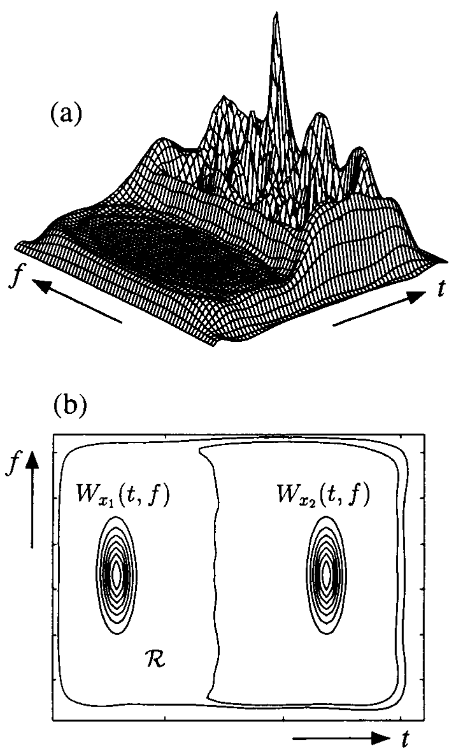

Fig. 2: (a) WS (slightly smoothed) of a "locally snug" frame, (b) WDs of Gaussian signals used for verifying the concept of local snugness.

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