UNIFIED THEORY OF DISPLACEMENT-COVARIANT TIME-FREQUENCY ANALYSIS*

Franz Hlawatsch and Helmut Bölcskei

INTHFT, Technische Universität Wien, Gusshausstrasse 25/389, A-1040 Vienna, Austria
email address: fhlawats@email.tuwien.ac.at

Abstract—We present a theory of linear and quadratic time-frequency representations (TFRs) that are covariant to *time-frequency displacement operators*. The theory unifies important TFR classes (short-time Fourier transform, wavelet transform; Cohen's, affine, hyperbolic, and power classes), and it allows the systematic construction of new TFRs that are covariant to a given operator.

1 INTRODUCTION

Most of the known classes of linear and quadratic time-frequency representations (TFRs) [1, 2] can be defined axiomatically by *covariance properties*. In what follows, $x(t)$ is a signal, t and f denote time and frequency, respectively, and integrations are over the signals' support.

Linear TFRs. The TFR class of *short-time Fourier transforms* (STFT) [1, 2]

$$\mathrm{STFT}_x(t,f) = \int_{t'} x(t')\, h^*(t'-t)\, e^{-j2\pi f t'}\, dt', \qquad (1)$$

where $h(t)$ is a fixed function, can be shown to consist of all linear TFRs that are covariant, up to a phase factor, to time-frequency (TF) shifts:

$$\mathrm{STFT}_{\mathbf{S}_{\tau,\nu}x}(t,f) = e^{-j2\pi\tau(f-\nu)}\, \mathrm{STFT}_x(t-\tau, f-\nu) \qquad (2)$$

with $(\mathbf{S}_{\tau,\nu}x)(t) = x(t-\tau)\, e^{j2\pi\nu t}$. Similarly, the TFR class of continuous *wavelet transforms* (WT) [3, 2]

$$\mathrm{WT}_x(t,f) = \sqrt{\frac{|f|}{f_0}} \int_{t'} x(t')\, h^*\!\Big(\frac{f}{f_0}(t'-t)\Big)\, dt', \quad f \neq 0, \qquad (3)$$

where $f_0 > 0$ is a fixed reference frequency, consists of all linear TFRs covariant to time shifts and TF scalings:

$$\mathrm{WT}_{\mathbf{C}_{a,\tau}x}(t,f) = \mathrm{WT}_x\big(a(t-\tau), f/a\big) \qquad (4)$$

with $(\mathbf{C}_{a,\tau}\, x)(t) = \sqrt{|a|}\;\, x\big(a(t-\tau)\big)$, $a \neq 0$. A similar covariance is satisfied by the *hyperbolic WT* defined in [4].

Quadratic TFRs. *Cohen's class with signal-independent kernels* [5, 2, 1] (briefly called *Cohen's class* hereafter),

$$C_x(t,f) = \int_{t_1}\!\int_{t_2} x(t_1)\, x^*(t_2)\, h^*(t_1-t, t_2-t)\, e^{-j2\pi f(t_1-t_2)} dt_1 dt_2, \qquad (5)$$

where $h(t_1,t_2)$ is a fixed function, consists of all quadratic TFRs that are covariant to TF shifts,

$$C_{\mathbf{S}_{\tau,\nu}x}(t,f) = C_x(t-\tau, f-\nu)\,, \qquad (6)$$

and the *affine class* [6, 7]

$$A_x(t,f) = \frac{|f|}{f_0} \int_{t_1}\!\int_{t_2} x(t_1)\, x^*(t_2)\, h^*\!\Big(\frac{f}{f_0}(t_1-t), \frac{f}{f_0}(t_2-t)\Big) dt_1 dt_2 \qquad (7)$$

consists of all quadratic TFRs that are covariant to time shifts and TF scalings,

$$A_{\mathbf{C}_{a,\tau}x}(t,f) = A_x\big(a(t-\tau), f/a\big)\,. \qquad (8)$$

Similar covariances are satisfied by the *hyperbolic class* [4] and the *power classes* [8] of quadratic TFRs.

2 TF DISPLACEMENT OPERATORS

The TF shift operator $\mathbf{S}_{\tau,\nu}$ underlying the STFT and Cohen's class and the time shift/TF scaling operator $\mathbf{C}_{a,\tau}$ underlying the WT and the affine class are families of unitary, linear operators indexed by a 2D parameter. Both $\mathbf{S}_{\tau,\nu}$ and $\mathbf{C}_{a,\tau}$ ***displace signals in the TF plane***. We shall now establish a general framework of *TF displacement operators* (TFDOs). This will yield a unified theory of "displacement-covariant TF analysis" which includes the known classes of linear and quadratic TFRs and also provides a systematic method for constructing new displacement-covariant TFRs.

Consider a family of linear operators $\mathbf{D}_\theta$ defined on a linear space $\mathcal{X} \subseteq \mathcal{L}_2(\mathbb{R})$ of finite-energy signals $x(t)$, and indexed by the 2D "displacement parameter" $\theta = (\alpha,\beta) \in \mathcal{D}$ with $\mathcal{D} \subseteq \mathbb{R}^2$. We assume that there exists an operation $\circ$ such that $\mathcal{D}$ and $\circ$ form a *group* with identity element θ_0 and inverse element θ^{-1}, i.e., (i) $\theta_1 \circ \theta_2 \in \mathcal{D}$ for $\theta_1, \theta_2 \in \mathcal{D}$, (ii) $\theta_1 \circ (\theta_2 \circ \theta_3) = (\theta_1 \circ \theta_2) \circ \theta_3$, (iii) $\theta \circ \theta_0 = \theta_0 \circ \theta = \theta$, and (iv) $\theta^{-1} \circ \theta = \theta \circ \theta^{-1} = \theta_0$. It follows that $(\theta_1 \circ \theta_2)^{-1} = \theta_2^{-1} \circ \theta_1^{-1}$. We now formulate six *properties* which $\mathbf{D}_\theta$ must satisfy in order to be called a TFDO.

Property 1: For all $\theta \in \mathcal{D}$, $\mathbf{D}_\theta$ is a *unitary* operator mapping $\mathcal{X}$ onto $\mathcal{X}$, i.e.,

$$\mathbf{D}_\theta\, \mathbf{D}_\theta^* = \mathbf{D}_\theta^*\, \mathbf{D}_\theta = \mathbf{I}\,, \qquad \mathbf{D}_\theta^{-1} = \mathbf{D}_\theta^* \qquad (9)$$

where $\mathbf{D}_\theta^*$ and $\mathbf{D}_\theta^{-1}$ denote the adjoint and the inverse, respectively, of $\mathbf{D}_\theta$, and $\mathbf{I}$ is the identity operator on $\mathcal{X}$ [9]. Unitarity of $\mathbf{D}_\theta$ is a natural property since we want $\mathbf{D}_\theta$ to *displace* the signal's energy in the TF plane, but not to change the total amount of energy.

Property 2: $\mathbf{D}_\theta$ satisfies a *composition law*

$$\mathbf{D}_{\theta_2}\mathbf{D}_{\theta_1} = e^{j\psi(\theta_1,\theta_2)}\, \mathbf{D}_{\theta_1\circ\theta_2} \qquad (10)$$

where $\psi(\cdot,\cdot)$ satisfies $\psi(\theta,\theta_0) = \psi(\theta_0,\theta) = 0$ modulo 2π. Thus, a displacement by θ_1 followed by a displacement by θ_2 is equivalent, up to a phase, to a displacement by $\theta_1 \circ \theta_2$.

From the above two properties, it follows that $\mathbf{D}_{\theta_0} = \mathbf{I}$, i.e, the identity element θ_0 corresponds to *no* displacement. Furthermore,

$$\mathbf{D}_\theta^{-1} = e^{-j\psi(\theta^{-1},\theta)}\, \mathbf{D}_{\theta^{-1}}\,, \qquad (11)$$

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i.e., a TF displacement by θ can be undone, up to a phase factor, via a displacement by the inverse parameter θ^{-1}. It is also easily shown that

$$\psi(\theta^{-1},\theta)=\psi(\theta,\theta^{-1}) \text{ modulo } 2\pi\,. \qquad (12)$$

Examples. Properties 1 and 2 are satisfied by the TF shift operator $\mathbf{S}_{\tau,\nu}$ and the time shift/TF scaling operator $\mathbf{C}_{a,\tau}$. For $\mathbf{S}_{\tau,\nu}$, we have $\theta=(\tau,\nu)$, $\mathcal{D}=\mathbb{R}^2$, $(\tau_1,\nu_1)\circ(\tau_2,\nu_2)=(\tau_1+\tau_2,\,\nu_1+\nu_2)$, $\theta_0=(0,0)$, $\theta^{-1}=(-\tau,-\nu)$, and $\psi(\theta_1,\theta_2)=-2\pi\nu_1\tau_2$. For $\mathbf{C}_{a,\tau}$, we have $\theta=(a,\tau)$, $\mathcal{D}=\mathbb{R}\backslash\{0\}\times\mathbb{R}$, $(a_1,\tau_1)\circ(a_2,\tau_2)=(a_1a_2,\,\tau_1/a_2+\tau_2)$, $\theta_0=(1,0)$, $\theta^{-1}=(1/a,-a\tau)$, and $\psi(\theta_1,\theta_2)\equiv 0$. The composition law (10) is

$$\mathbf{S}_{\tau_2,\nu_2}\mathbf{S}_{\tau_1,\nu_1} = e^{-j2\pi\nu_1\tau_2}\,\mathbf{S}_{\tau_1+\tau_2,\,\nu_1+\nu_2}\,,$$
$$\mathbf{C}_{a_2,\tau_2}\mathbf{C}_{a_1,\tau_1} = \mathbf{C}_{a_1a_2,\,\tau_1/a_2+\tau_2}\,.$$

Displacement Function. The primary effect of a TFDO $\mathbf{D}_\theta$ is a *TF displacement*: if $x(t)$ is localized about a TF point $z=(t,f)$, then $(\mathbf{D}_\theta\,x)(t)$ will be localized about some other TF point $z'=(t',f')$. Here, z' depends on the original TF point z and the displacement parameter θ,

$$z'=d(z,\theta)\,,$$

which is short for $t'=d_1(t,f;\alpha,\beta)$, $f'=d_2(t,f;\alpha,\beta)$. We call $d(\cdot,\cdot)$ the *displacement function* (DF) of the TFDO $\mathbf{D}_\theta$. For example, the DF of the TF shift operator $\mathbf{S}_{\tau,\nu}$ is easily seen to be $t'=d_1(t,f;\tau,\nu)=t+\tau$, $f'=d_2(t,f;\tau,\nu)=f+\nu$. In the following, we present a systematic procedure for constructing the DF of a given TFDO $\mathbf{D}_\theta$, and we formulate some additional TFDO properties. The procedure has been introduced in [10] in a related context.

Let $\mathcal{Z}\subseteq\mathbb{R}^2$ (where $\mathbb{R}^2$ stands for the entire TF plane) denote the set of TF points $z=(t,f)$ underlying our TF analysis[1]. Suppose that $x(t)$ is localized about a TF point $z_x=(t_x,f_x)\in\mathcal{Z}$ as shown in *Fig. 1*. Let $\delta_{t_x}(t)=\delta(t-t_x)$ and $e_{f_x}(t)=e^{j2\pi f_x t}$. In the TF plane, $\delta_{t_x}(t)$ is localized along the straight line $t=t_x$, and $e_{f_x}(t)$ is localized along the straight line $f=f_x$ (see Fig. 1). The TF point $z_x=(t_x,f_x)$ is the *intersection* of these lines.

We wish to find the TF point $z'=(t',f')$ about which the displaced signal $(\mathbf{D}_\theta\,x)(t)$ is located. Consider the signals $\tilde{\delta}_{t_x,\theta}(t)=(\mathbf{D}_\theta\,\delta_{t_x})(t)$ and $\tilde{e}_{f_x,\theta}(t)=(\mathbf{D}_\theta\,e_{f_x})(t)$, and let $\tau_{t_x,\theta}(f)$ be the group delay[2] of $\tilde{\delta}_{t_x,\theta}(t)$ and $\nu_{f_x,\theta}(t)$ be the instantaneous frequency[3] of $\tilde{e}_{f_x,\theta}(t)$. The signal $\tilde{\delta}_{t_x,\theta}(t)$ is localized in the TF plane along the group delay curve $t=\tau_{t_x,\theta}(f)$, while $\tilde{e}_{f_x,\theta}(t)$ is localized along the instantaneous frequency curve $f=\nu_{f_x,\theta}(t)$. Hence, $z'=(t',f')$ will be the intersection of these curves (see Fig. 1), i.e., the solution to the system of equations $\tau_{t_x,\theta}(f')=t'$, $\nu_{f_x,\theta}(t')=f'$. This solution $z'=(t',f')$ depends on $z_x=(t_x,f_x)$ and on θ, i.e., $z'=d(z_x,\theta)$. This defines the DF $d(\cdot,\cdot)$ of $\mathbf{D}_\theta$, provided that the following property is satisfied. (Below, we write $z=(t,f)$ instead of $z_x=(t_x,f_x)$.)

Property 3: The intersection equation

$$\tau_{t,\theta}(f')=t'\,,\qquad \nu_{f,\theta}(t')=f' \qquad (13)$$

has a unique solution $z'=(t',f')\in\mathcal{Z}$ for any $z=(t,f)\in\mathcal{Z}$ and for any $\theta\in\mathcal{D}$.

[1] Note that the TF set $\mathcal{Z}$ is related to the signal space $\mathcal{X}$.

[2] The group delay of $\tilde{\delta}_{t_x,\theta}(t)$ is $\tau_{t_x,\theta}(f)=-\frac{1}{2\pi}\frac{d}{df}\Phi(f)$ where $\Phi(f)$ is the phase of the Fourier transform of $\tilde{\delta}_{t_x,\theta}(t)$.

[3] The instantaneous frequency of $\tilde{e}_{f_x,\theta}(t)$ is $\nu_{f_x,\theta}(t)=\frac{1}{2\pi}\frac{d}{dt}\phi(t)$ where $\phi(t)$ is the phase of $\tilde{e}_{f_x,\theta}(t)$.

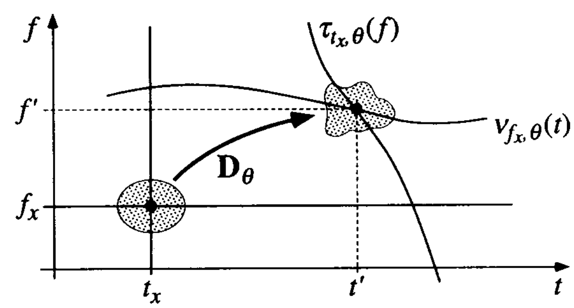

Fig. 1: Construction of the displacement function.

Examples. Property 3 is satisfied for the TF shift operator $\mathbf{S}_{\tau,\nu}$ and the time shift/TF scaling operator $\mathbf{C}_{a,\tau}$. For $\mathbf{S}_{\tau,\nu}$, $\mathcal{Z}$ is $\mathbb{R}^2$ (the entire TF plane) and the DF is obtained from (13) as $t'=d_1(t,f;\tau,\nu)=t+\tau$, $f'=d_2(t,f;\tau,\nu)=f+\nu$. For $\mathbf{C}_{a,\tau}$, $\mathcal{Z}$ is $\mathbb{R}\times\mathbb{R}\backslash\{0\}$ (the entire TF plane minus the line $f=0$) and the DF is obtained from (13) as $t'=d_1(t,f;a,\tau)=t/a+\tau$, $f'=d_2(t,f;a,\tau)=af$.

Induced TFDO. The DF expresses a *TF coordinate transform*. Let $T(z)=T(t,f)\in\mathcal{L}_2(\mathcal{Z})$ be a square-integrable TF function defined for $z\in\mathcal{Z}$, and consider the coordinate transform operator $\tilde{\mathbf{D}}_\theta$ defined on $\mathcal{L}_2(\mathcal{Z})$ as

$$\big(\tilde{\mathbf{D}}_\theta\,T\big)(z)=T\big(d(z,\theta^{-1})\big)\,.$$

The operator family $\tilde{\mathbf{D}}_\theta$ will be called the *induced TFDO* (ITFDO) associated to $\mathbf{D}_\theta$. While the TFDO acts on a signal, the ITFDO acts on a TF function (which may be the TFR of a signal). The ITFDO is a linear operator even though the TF coordinate transform $z'=d(z,\theta)$ may be nonlinear. The ITFDOs associated to $\mathbf{S}_{\tau,\nu}$ and $\mathbf{C}_{a,\tau}$ are $\big(\tilde{\mathbf{S}}_{\tau,\nu}\,T\big)(t,f)=T(t-\tau,f-\nu)$ and $\big(\tilde{\mathbf{C}}_{a,\tau}\,T\big)(t,f)=T\big(a(t-\tau),f/a\big)$. We now formulate three further properties which concern the DF or, equivalently, the ITFDO.

Property 4: For any $\theta\in\mathcal{D}$, the TF coordinate transform $z'=d(z,\theta)$ is an *invertible, area-preserving mapping of* $\mathcal{Z}$ *onto* $\mathcal{Z}$. This implies that the Jacobian of the vector function $z\rightarrow z'=d(z,\theta)$ is 1 for any $\theta\in\mathcal{D}$. Equivalently, the ITFDO $\tilde{\mathbf{D}}_\theta$ is *unitary* on $\mathcal{L}_2(\mathcal{Z})$, i.e.,

$$\tilde{\mathbf{D}}_\theta\,\tilde{\mathbf{D}}_\theta^*=\tilde{\mathbf{D}}_\theta^*\,\tilde{\mathbf{D}}_\theta=\tilde{\mathbf{I}}\,,\qquad \tilde{\mathbf{D}}_\theta^{-1}=\tilde{\mathbf{D}}_\theta^*$$

where $\tilde{\mathbf{I}}$ is the identity operator on $\mathcal{L}_2(\mathcal{Z})$.

Property 5: The DF and the ITFDO satisfy the (equivalent) *composition laws*

$$d\big(d(z,\theta_1),\theta_2\big)=d(z,\,\theta_1\circ\theta_2)\,,\qquad \tilde{\mathbf{D}}_{\theta_2}\,\tilde{\mathbf{D}}_{\theta_1}=\tilde{\mathbf{D}}_{\theta_1\circ\theta_2}\,.$$

From properties 4 and 5, it follows that the TF coordinate transform corresponding to the identity element θ_0 is the identity transform, i.e., $d(z,\theta_0)=z$ or equivalently $\tilde{\mathbf{D}}_{\theta_0}=\tilde{\mathbf{I}}$. Furthermore, a coordinate transform by θ can be undone by a coordinate transform by θ^{-1}: if $z'=d(z,\theta)$, then $z=d(z',\theta^{-1})$. Equivalently,

$$d\big(d(z,\theta),\theta^{-1}\big)=z\,,\qquad \tilde{\mathbf{D}}_\theta^{-1}=\tilde{\mathbf{D}}_{\theta^{-1}}\,.$$

Parameter Function. We finally postulate that, from any given TF point z, we can reach any other TF point z' via a suitable TF displacement:

Property 6: The equation $d(z,\theta)=z'$ has a solution $\theta\in\mathcal{D}$ for any $z,z'\in\mathcal{Z}$.

This solution can be written as

$$\theta = p(z', z),$$

which is short for $\alpha = p_1(t', f'; t, f)$, $\beta = p_2(t', f'; t, f)$. We call $p(\cdot, \cdot)$ the *parameter function* (PF) of the TFDO $\mathbf{D}_\theta$. Note that $p(z, z) = \theta_0$ and $p(z', z) = \theta \Rightarrow p(z, z') = \theta^{-1}$. Furthermore, it can be shown that

$$p\big(d(z', \theta), z\big) = p(z', z) \circ \theta. \tag{14}$$

Examples. The properties 4-6 are satisfied in the case of $\mathbf{S}_{\tau,\nu}$ and $\mathbf{C}_{a,\tau}$. The PF of $\mathbf{S}_{\tau,\nu}$ is $\tau = p_1(t', f'; t, f) = t' - t$, $\nu = p_2(t', f'; t, f) = f' - f$, and the PF of $\mathbf{C}_{a,\tau}$ is $a = p_1(t', f'; t, f) = f'/f$, $\tau = p_2(t', f'; t, f) = t' - (f/f')t$.

3 DISPLACEMENT-COVARIANT TFRs

In the previous section, we formulated six properties which define a TFDO. We now consider linear and quadratic TFRs which are covariant to a TFDO.

Linear TFRs. A linear TFR (LTFR) $T_x(t, f) = T_x(z)$ will be called *covariant to a TFDO* $\mathbf{D}_\theta$ if

$$T_{\mathbf{D}_\theta x}(z) = e^{j\epsilon(z,\theta)} \big(\tilde{\mathbf{D}}_\theta T_x\big)(z) \tag{15}$$

with

$$\epsilon(z, \theta) = \psi(\theta^{-1}, \theta) - \psi\big(p(z, z_0), \theta^{-1}\big), \tag{16}$$

where $z_0 \in \mathcal{Z}$ is an arbitrary fixed reference TF point. (We use this particular phase function $\epsilon(z, \theta)$ since other definitions would lead to an additional phase factor in (17).) The next theorem characterizes all covariant LTFRs.

Theorem 1. All LTFRs covariant to a TFDO $\mathbf{D}_\theta$ can be written as the inner product

$$T_x(z) = \big\langle x, \mathbf{D}_{p(z,z_0)} h \big\rangle = \int_{t'} x(t') \big(\mathbf{D}_{p(z,z_0)} h\big)^*(t')\, dt', \tag{17}$$

where $h(t)$ is an arbitrary function (independent of $x(t)$) and z_0 is the reference TF point used in (16). Conversely, all LTFRs of the form (17) are covariant to $\mathbf{D}_\theta$.

Examples. For $\mathbf{D}_\theta = \mathbf{S}_{\tau,\nu}$ and $z_0 = (0, 0)$, (15) becomes the TF shift covariance property (2), and (17) becomes the STFT defined in (1). For $\mathbf{D}_\theta = \mathbf{C}_{a,\tau}$ and $z_0 = (0, f_0)$, (15) becomes the time shift/TF scaling covariance property (4) and (17) becomes the WT in (3).

Proof of Theorem 1. Any LTFR can be written as

$$T_x(z) = \big\langle x, k_z \big\rangle = \int_{t'} x(t')\, k_z^*(t')\, dt', \tag{18}$$

where the function $k_z(t)$ depends on z but not on $x(t)$. With (18), the LHS of (15) is $T_{\mathbf{D}_\theta x}(z) = \big\langle \mathbf{D}_\theta x, k_z \big\rangle = \big\langle x, \mathbf{D}_\theta^* k_z \big\rangle = \big\langle x, \mathbf{D}_\theta^{-1} k_z \big\rangle$, where (9) has been used, and the RHS is $e^{j\epsilon(z,\theta)} \big(\tilde{\mathbf{D}}_\theta T_x\big)(z) = e^{j\epsilon(z,\theta)}\, T\big(d(z, \theta^{-1})\big) = e^{j\epsilon(z,\theta)} \big\langle x, k_{d(z,\theta^{-1})} \big\rangle$. Hence, (15) is satisfied if and only if $k_z(t)$ satisfies $\big(\mathbf{D}_\theta^{-1} k_z\big)(t') = e^{j\epsilon(z,\theta)}\, k_{d(z,\theta^{-1})}(t')$ or

$$k_z(t') = e^{j\epsilon(z,\theta)} \big(\mathbf{D}_\theta\, k_{d(z,\theta^{-1})}\big)(t') \qquad \forall\, z, \theta, t'. \tag{19}$$

Consider now a fixed reference TF point z_0. Due to Property 6, there exists a θ for any z such that $d(z, \theta^{-1}) = z_0$; this θ is given by $\theta^{-1} = p(z_0, z)$ or $\theta = p(z, z_0)$. For this specific θ, (19) becomes

$$k_z(t') = e^{j\epsilon(z, p(z,z_0))} \big(\mathbf{D}_{p(z,z_0)}\, k_{z_0}\big)(t'). \tag{20}$$

Note that, for fixed z_0, this is now only a *necessary* condition since we picked a specific θ whereas (19) must be satisfied for all θ. With (16), we have $\epsilon(z, p(z, z_0)) = \psi(\theta^{-1}, \theta) - \psi(\theta, \theta^{-1}) = 0$ modulo 2π, where $\theta = p(z, z_0)$ and (12) have been used. Hence, (20) simplifies to

$$k_z(t') = \big(\mathbf{D}_{p(z,z_0)}\, k_{z_0}\big)(t') = \big(\mathbf{D}_{p(z,z_0)}\, h\big)(t'), \tag{21}$$

with $h(t) \triangleq k_{z_0}(t)$. Inserting (21) in (18) gives (17).

We have finally to show that the form (21) or, equivalently, (17) is also *sufficient* for the covariance (15). Using (17), (15) is proved as follows:

$$\begin{aligned}
T_{\mathbf{D}_\theta x}(z) &= \big\langle \mathbf{D}_\theta\, x, \mathbf{D}_{p(z,z_0)} h \big\rangle = \big\langle x, \mathbf{D}_\theta^*\, \mathbf{D}_{p(z,z_0)} h \big\rangle \\
&= \big\langle x, \mathbf{D}_\theta^{-1}\, \mathbf{D}_{p(z,z_0)} h \big\rangle = \big\langle x, e^{-j\psi(\theta^{-1},\theta)}\, \mathbf{D}_{\theta^{-1}}\, \mathbf{D}_{p(z,z_0)} h \big\rangle \\
&= e^{j\psi(\theta^{-1},\theta)} \big\langle x, e^{j\psi(p(z,z_0),\theta^{-1})}\, \mathbf{D}_{p(z,z_0)\circ\theta^{-1}} h \big\rangle \\
&= e^{j[\psi(\theta^{-1},\theta) - \psi(p(z,z_0),\theta^{-1})]} \big\langle x, \mathbf{D}_{p(d(z,\theta^{-1}),z_0)} h \big\rangle \\
&= e^{j\epsilon(z,\theta)}\, T_x\big(d(z, \theta^{-1})\big) = e^{j\epsilon(z,\theta)} \big(\tilde{\mathbf{D}}_\theta T_x\big)(z),
\end{aligned}$$

where (9), (11), (10), (14), and (16) have been used. ∎

TF Localization. The form (17), besides being necessary and sufficient for the covariance property (15), also guarantees correct TF localization of the LTFR $T_x(z)$ if only $h(t')$ is TF-localized about z_0. In this case, $\big(\mathbf{D}_{p(z,z_0)} h\big)(t')$ is TF-localized about z. Thus, at a given analysis TF point z, $T_x(z)$ is formed by correlating $x(t')$ with a "test signal" $\big(\mathbf{D}_{p(z,z_0)} h\big)(t')$ correctly localized about z.

Quadratic TFRs. A quadratic TFR (QTFR) $T_x(t, f) = T_x(z)$ will be called *covariant to a TFDO* $\mathbf{D}_\theta$ if

$$T_{\mathbf{D}_\theta x}(z) = \big(\tilde{\mathbf{D}}_\theta T_x\big)(z). \tag{22}$$

This differs from the covariance (15) by the absence of a phase factor. The next theorem characterizes all covariant QTFRs. In what follows, $x^{\otimes}(t_1, t_2) = x(t_1)\, x^*(t_2)$ denotes the outer product of the signal $x(t)$ by itself, and $\mathbf{D}_\theta^{\otimes}$ denotes the outer product of the operator $\mathbf{D}_\theta$ by itself[4].

Theorem 2. All QTFRs covariant to a TFDO $\mathbf{D}_\theta$ can be written as the 2D inner product

$$\begin{aligned}
T_x(z) &= \big\langle\!\big\langle x^{\otimes}, \mathbf{D}^{\otimes}_{p(z,z_0)} h \big\rangle\!\big\rangle \\
&= \int_{t_1}\!\int_{t_2} x(t_1)\, x^*(t_2) \big(\mathbf{D}^{\otimes}_{p(z,z_0)} h\big)^*(t_1, t_2)\, dt_1 dt_2
\end{aligned} \tag{23}$$

where $h(t_1, t_2)$ is an arbitrary 2D function (independent of $x(t)$) and $z_0 \in \mathcal{Z}$ is an arbitrary reference TF point. Conversely, all QTFRs (23) are covariant to $\mathbf{D}_\theta$.

Examples. For $\mathbf{D}_\theta = \mathbf{S}_{\tau,\nu}$ and $z_0 = (0, 0)$, (22) becomes the TF shift covariance property (6) and (23) becomes Cohen's class defined in (5). For $\mathbf{D}_\theta = \mathbf{C}_{a,\tau}$ and $z_0 = (0, f_0)$, (22) becomes the time shift/TF scaling covariance property (8) and (23) becomes the affine class in (7).

The proof of Theorem 2 is structurally analogous to that of

[4] If $\mathbf{D}_\theta$ acts on a 1D function $x(t)$ as $\big(\mathbf{D}_\theta\, x\big)(t) = \int_{t'} D_\theta(t, t')\, x(t')\, dt'$ (where $D_\theta(t, t')$ is the kernel of $\mathbf{D}_\theta$), then $\mathbf{D}_\theta^{\otimes}$ acts on a 2D function $y(t_1, t_2)$ as $\big(\mathbf{D}_\theta^{\otimes}\, y\big)(t_1, t_2) = \int_{t_1'}\int_{t_2'} D_\theta(t_1, t_1')\, D_\theta^*(t_2, t_2')\, y(t_1', t_2')\, dt_1' dt_2'$. For example, $(\mathbf{S}^{\otimes}_{\tau,\nu}\, y)(t_1, t_2) = y(t_1 - \tau, t_2 - \tau)\, e^{j2\pi\nu(t_1 - t_2)}$ and $(\mathbf{C}^{\otimes}_{a,\tau}\, y)(t_1, t_2) = |a|\, y\big(a(t_1 - \tau), a(t_2 - \tau)\big)$.

Theorem 1 and will not be included. Correct TF localization of the QTFR (23) is guaranteed if a (suitably defined) TF representation of the kernel $h(t_1, t_2)$ is localized about the reference TF point z_0 used in (23). Generalized *marginal properties* are considered in [11].

4 EXAMPLES

We now apply our theory to three TFDOs which are less trivial than the TFDOs $\mathbf{S}_{\tau,\nu}$ and $\mathbf{C}_{a,\tau}$ considered so far.

Example 1. The TFDO $\mathbf{H}_{a,c}$ is defined on the space $\mathcal{H}$ of analytic signals as

$$(\mathbf{H}_{a,c}\, x)(t) = \mathcal{F}^{-1}\Big\{ \frac{1}{\sqrt{a}}\, X\Big(\frac{f}{a}\Big)\, e^{-j2\pi c \ln(f/f_0)} \Big\}, \quad a > 0\,,$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform operator and $X(f)$ is the Fourier transform of $x(t)$. $\mathbf{H}_{a,c}$ consists of a TF scaling and a "hyperbolic time shift" [4]. We have $\theta = (a, c)$, $\mathcal{D} = \mathbb{R}_+ \times \mathbb{R}$, $(a_1, c_1) \circ (a_2, c_2) = (a_1 a_2,\, c_1 + c_2)$, $\theta_0 = (1, 0)$, $\theta^{-1} = (1/a, -c)$, and $\psi(\theta_1, \theta_2) = 2\pi c_1 \ln a_2$. The DF, defined on $\mathcal{Z} = \mathbb{R} \times \mathbb{R}_+$, is obtained as $t' = d_1(t, f; a, c) = (t + c/f)/a$, $f' = d_2(t, f; a, c) = af$, and the PF is $a = p_1(t', f'; t, f) = f'/f$, $c = p_2(t', f'; t, f) = t'f' - tf$. Setting $z_0 = (0, f_0)$, the LTFR covariance property (15) becomes

$$T_{\mathbf{H}_{a,c}\, x}(t,f) = e^{j2\pi(tf - c)\ln a}\, T_x\big(a(t - c/f), f/a\big)\,. \qquad (24)$$

Applying Theorem 1, it follows that all LTFRs satisfying this covariance are given by

$$T_x(t,f) = \sqrt{\frac{f_0}{f}} \int_{f'} X(f')\, H^*\Big(\frac{f_0}{f}\, f'\Big)\, e^{j2\pi t f \ln(f'/f_0)}\, df'\,,$$

which is the *hyperbolic WT* introduced in [4]. The QTFR covariance property is (24) without the phase factor. Due to Theorem 2, all covariant QTFRs are given by

$$T_x(t,f) = \frac{f_0}{f} \int_{f_1}\!\int_{f_2} X(f_1)\, X^*(f_2)\, H^*\Big(\frac{f_0}{f} f_1, \frac{f_0}{f} f_2\Big)\, e^{j2\pi t f \ln(f_1/f_2)}\, df_1 df_2\,,$$

which is the *hyperbolic class* introduced in [4].

Example 2. The TFDO $\mathbf{P}_{a,c}$ defined on $\mathcal{X} = \mathcal{L}_2(\mathbb{R})$ as

$$(\mathbf{P}_{a,c}\, x)(t) = \mathcal{F}^{-1}\Big\{ \frac{1}{\sqrt{|a|}}\, X\Big(\frac{f}{a}\Big)\, e^{-j2\pi c\, \xi_\kappa(f/f_0)} \Big\}$$

with $\xi_\kappa(b) = \operatorname{sign}(b)\, |b|^\kappa$, $\kappa \in \mathbb{R}\backslash\{0\}$, consists of a TF scaling and a "power-law time shift" [8]. We have $\theta = (a, c)$, $\mathcal{D} = \mathbb{R}\backslash\{0\} \times \mathbb{R}$, $(a_1, c_1) \circ (a_2, c_2) = (a_1 a_2,\, c_1/\xi_\kappa(a_2) + c_2)$, $\theta_0 = (1, 0)$, $\theta^{-1} = (1/a, -\xi_\kappa(a)\, c)$, and $\psi(\theta_1, \theta_2) \equiv 0$. The DF, defined on $\mathcal{Z} = \mathbb{R} \times \mathbb{R}\backslash\{0\}$, is $t' = d_1(t, f; a, c) = t/a + c\, \tau_\kappa(af)$, $f' = d_2(t, f; a, c) = af$ where $\tau_\kappa(f) = (1/f_0)\, \xi_\kappa'(f/f_0) = (\kappa/f_0)\, |f/f_0|^{\kappa - 1}$. The PF is $a = p_1(t', f'; t, f) = f'/f$, $c = p_2(t', f'; t, f) = (t'f' - tf)/(f' \tau_\kappa(f'))$. Setting $z_0 = (0, f_0)$, the covariance property for LTFRs and QTFRs reads

$$T_{\mathbf{P}_{a,c}\, x}(t,f) = T_x\big(a(t - c\, \tau_\kappa(f)), f/a\big)\,.$$

By application of Theorem 1, all LTFRs satisfying this covariance are obtained as

$$T_x(t,f) = \sqrt{\frac{f_0}{|f|}} \int_{f'} X(f')\, H^*\Big(\frac{f_0}{f} f'\Big) \exp\Big\{ j2\pi \frac{t}{\tau_\kappa(f)}\, \xi_\kappa\Big(\frac{f'}{f_0}\Big) \Big\}\, df'.$$

Similarly, it follows from Theorem 2 that all QTFRs satisfying the covariance are given by

$$T_x(t,f) = \frac{f_0}{|f|} \int_{f_1}\!\int_{f_2} X(f_1)\, X^*(f_2)\, H^*\Big(\frac{f_0}{f} f_1, \frac{f_0}{f} f_2\Big) \cdot \exp\Big\{ j2\pi \frac{t}{\tau_\kappa(f)} \Big[\xi_\kappa\Big(\frac{f_1}{f_0}\Big) - \xi_\kappa\Big(\frac{f_2}{f_0}\Big) \Big] \Big\}\, df_1 df_2\,,$$

which is the *power class* with power parameter κ [8].

Example 3. We finally define the TFDO $\mathbf{W}_{\kappa,a}$ on the space $\mathcal{X} = \mathcal{L}_2(\mathbb{R}_+)$ as

$$(\mathbf{W}_{\kappa,a}\, x)(t) = \sqrt{a\, |\kappa| \Big(\frac{at}{t_0}\Big)^{\kappa - 1}}\; x\Big(t_0 \Big(\frac{at}{t_0}\Big)^{\kappa}\Big)\,, \quad a > 0\,.$$

This TFDO is a "power-law warping" (essentially $t \to t^\kappa$) [8, 10] followed by a TF scaling. We have $\theta = (\kappa, a)$, $\mathcal{D} = \mathbb{R}\backslash\{0\} \times \mathbb{R}_+$, $(\kappa_1, a_1) \circ (\kappa_2, a_2) = (\kappa_1 \kappa_2,\, a_1^{1/\kappa_2} a_2)$, $\theta_0 = (1, 1)$, $\theta^{-1} = (1/\kappa, 1/a^\kappa)$, and $\psi(\theta_1, \theta_2) \equiv 0$. The DF, defined on $\mathcal{Z} = \mathbb{R}_+ \times \mathbb{R}\backslash\{0\}$, is $t' = d_1(t, f; \kappa, a) = (t_0/a)\, (t/t_0)^{1/\kappa}$, $f' = d_2(t, f; \kappa, a) = a\kappa f\, (t/t_0)^{1 - 1/\kappa}$. The PF is $\kappa = p_1(t', f'; t, f) = t'f'/(tf)$, $a = p_2(t', f'; t, f) = (t_0/t')\, (t/t_0)^{tf/(t'f')}$. Setting $z_0 = (t_0, 1/t_0)$, the covariance property for LTFRs and QTFRs reads

$$T_{\mathbf{W}_{\kappa,a}\, x}(t,f) = T_x\Big(t_0 \Big(\frac{at}{t_0}\Big)^{\kappa}, \frac{1}{\kappa\, a^\kappa} \Big(\frac{t}{t_0}\Big)^{1-\kappa} f\Big)\,.$$

From Theorem 1, all covariant LTFRs are obtained as

$$T_x(t,f) = \sqrt{t_0 |f|} \int_{t'} x(t') \sqrt{\Big(\frac{t'}{t}\Big)^{tf - 1}}\; h^*\Big(t_0 \Big(\frac{t'}{t}\Big)^{tf}\Big)\, dt'\,,$$

and from Theorem 2, all covariant QTFRs are obtained as

$$T_x(t,f) = t_0 |f| \int_{t_1}\!\int_{t_2} x(t_1)\, x^*(t_2) \sqrt{\Big(\frac{t_1 t_2}{t^2}\Big)^{tf - 1}} \cdot h^*\Big(t_0 \Big(\frac{t_1}{t}\Big)^{tf}, t_0 \Big(\frac{t_2}{t}\Big)^{tf}\Big)\, dt_1 dt_2\,.$$

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