Vandermonde Matrices with Nodes in the Unit Disk and the Large Sieve

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Abstract

We derive bounds on the extremal singular values and the condition number of $N \times K$, with $N \geqslant K$, Vandermonde matrices with nodes in the unit disk. The mathematical techniques we develop to prove our main results are inspired by a link—first established by Selberg [1] and later extended by Moitra [2]—between the extremal singular values of Vandermonde matrices with nodes on the unit circle and large sieve inequalities. Our main conceptual contribution lies in establishing a connection between the extremal singular values of Vandermonde matrices with nodes in the unit disk and a novel large sieve inequality involving polynomials in $z \in \mathbb{C}$ with $|z| \leqslant 1$. Compared to Bazán's upper bound on the condition number [3], which, to the best of our knowledge, constitutes the only analytical result—available in the literature—on the condition number of Vandermonde matrices with nodes in the unit disk, our bound not only takes a much simpler form, but is also sharper for certain node configurations. Moreover, the bound we obtain can be evaluated consistently in a numerically stable fashion, whereas the evaluation of Bazán's bound requires the solution of a linear system of equations which has the same condition number as the Vandermonde matrix under consideration and can therefore lead to numerical instability in practice. As a byproduct, our result—when particularized to the case of nodes on the unit circle—slightly improves upon the Selberg–Moitra bound.

Keywords: Vandermonde matrices, extremal singular values, condition number, unit disk, large sieve, Hilbert's inequality

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1. Introduction

Vandermonde matrices arise in many fields of applied mathematics and engineering such as interpolation and approximation theory [4, 5], differential equations [6], control theory [7], sampling theory [8, 9, 10, 11], subspace methods for parameter estimation [2, 12, 13, 14, 15, 16], line spectral estimation [17], and fast evaluation of linear combinations of radial basis functions using the fast Fourier transform for non-equispaced knots [18, 19].

It is well known that the condition number of real square Vandermonde matrices grows exponentially in the dimension of the matrix [20, 21]. Complex Vandermonde matrices, on the other hand, can be well-conditioned depending on the locations of the nodes in the complex plane. There exists significant literature on the condition number of Vandermonde matrices with nodes on the unit circle. Specifically, it is shown in [3, 15, 22, 23, 24, 25] that $N \times K$, with $N \geqslant K$, Vandermonde matrices with nodes $e^{2\pi i \xi_k}$, where $\xi_k \in [0, 1)$, for $k \in \{1, 2, \ldots, K\}$, are well-conditioned provided that the minimum wrap-around distance between the node frequencies ξ_k is large enough. On the other hand, the literature on $N \times K$, with $N \geqslant K$, Vandermonde matrices with nodes z_k in the unit disk, i.e., $|z_k| \leqslant 1$, for $k \in \{1, 2, \ldots, K\}$, is very scarce. In fact, the only result along these lines that we are aware of is Bazán's upper bound on the spectral condition number [3]. This bound is, however, implicit as it depends on a quantity whose computation requires the solution of the linear system of equations generated by the Vandermonde matrix under consideration. As the numerical results in Section 6 demonstrate, the evaluation of this bound can therefore be numerically unstable in practice.

Contributions. We derive a lower bound on the minimum singular value and an upper bound on the maximum singular value of $N \times K$ ($N \geqslant K$) Vandermonde matrices with general nodes $z_k = |z_k| e^{2\pi i \xi_k}$ in the unit disk, i.e., $|z_k| \leqslant 1$ and $\xi_k \in [0, 1)$, for $k \in \{1, 2, \ldots, K\}$. Based on these bounds we get an upper bound on the spectral condition number. Our bounds depend on N, the minimum wrap-around distance between the ξ_k, and the moduli $|z_k|$ of the nodes. In particular, the upper bound on the spectral condition number we report is of much simpler form than Bazán's bound, and for certain node configurations also sharper. The mathematical techniques we develop to prove our main results are inspired by a link—first established by Selberg [1] and later extended by Moitra [2]—between the extremal singular values of Vandermonde matrices with nodes on the unit circle and large sieve inequalities [26, 27, 28, 29, 30, 31]. The Selberg-Moitra approach employs Fourier-analytic techniques and the Poisson summation formula and therefore does not seem to be amenable to an extension to the case of nodes in the unit disk. Our main conceptual contribution lies in establishing a connection between the extremal singular values of Vandermonde matrices with nodes in the unit disk and a novel large sieve inequality involving polynomials in $z \in \mathbb{C}$ with $|z| \leqslant 1$. This is accomplished by first recognizing that the Selberg–Moitra connection can alternatively be established based on the Montgomery–Vaughan proof [32] of the large sieve inequality, and then extending this alternative connection from the unit circle to the unit disk. We also demonstrate how Cohen's dilatation trick, described in [33, p. 559] and originally developed for the large sieve inequality on the unit circle, can be applied to refine our bounds valid for nodes in the unit disk. As a byproduct, our result—when particularized to the unit circle—slightly improves upon the Selberg–Moitra upper bound. This improved bound also applies to the square case, $N = K$, not covered by the Selberg–Moitra result.

The numerical evaluation of Bazán's bound requires the solution of a linear system of equations which has the same condition number as the Vandermonde matrix under consideration; this can lead to numerical instability in practice. We provide numerical results demonstrating that our bound can not only be evaluated consistently in a numerically stable fashion, but is, in certain cases, also tighter than Bazán's bound.

Notation. The complex conjugate of $z \in \mathbb{C}$ is denoted by $\overline{z}$. The hyperbolic sine function is defined as $\sinh(z) := (e^z - e^{-z})/2$, for $z \in \mathbb{C}$. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the largest integer not exceeding x, $\lceil x \rceil$ stands for the smallest integer larger than x, and $[x]$ denotes the integer closest to x. Lowercase boldface letters designate (column) vectors and uppercase boldface letters denote matrices. The superscripts ^{T} and ^{H} refer to transposition and Hermitian transposition, respectively. For a vector $\boldsymbol{x} := \{x_k\}_{k=1}^K \in \mathbb{C}^K$, we write $\|\boldsymbol{x}\|_p$ for its ℓ^p-norm, $p \in [1, \infty]$, that is, $\|\boldsymbol{x}\|_p := \left(\sum_{k=1}^K |x_k|^p\right)^{1/p}$, for $p \in [1, \infty)$, and $\|\boldsymbol{x}\|_\infty := \max_{1 \leqslant k \leqslant K} |x_k|$. The Moore-Penrose pseudo-inverse of the full-rank matrix $\mathbf{A} \in \mathbb{C}^{M \times N}$ is $\mathbf{A}^\dagger = \mathbf{A}^H(\mathbf{A}\mathbf{A}^H)^{-1}$, if $M < N$, and $\mathbf{A}^\dagger = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$, if $M \geqslant N$. We denote the smallest and largest singular value of $\mathbf{A} \in \mathbb{C}^{M \times N}$ by $\sigma_{\min}(\mathbf{A})$ and $\sigma_{\max}(\mathbf{A})$, respectively, and for $p \in [1, \infty]$, we let $\|\mathbf{A}\|_p := \max\{\|\mathbf{A}\boldsymbol{x}\|_p : \boldsymbol{x} \in \mathbb{C}^N, \|\boldsymbol{x}\|_p = 1\}$. In particular, we have $\|\mathbf{A}\|_2 = \sigma_{\max}(\mathbf{A})$ and $\|\mathbf{A}\|_\infty = \max_{1 \leqslant m \leqslant M} \sum_{n=1}^N |a_{m,n}|$. For $\mathbf{A} \in \mathbb{C}^{M \times N}$ with columns $\boldsymbol{a}_n$, $n \in \{1, 2, \ldots, N\}$, we let $\operatorname{vec}(\mathbf{A}) := \left(\boldsymbol{a}_1^T \ \boldsymbol{a}_2^T \ \ldots \ \boldsymbol{a}_N^T\right)^T$, and $\overline{\mathbf{A}}$ denotes the matrix obtained by element-wise complex conjugation of $\mathbf{A}$.

2. Problem statement

We consider Vandermonde matrices of the form

$$\mathbf{V}_{N \times K} := \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ z_1 & z_2 & \ldots & z_{K-1} & z_K \\ z_1^2 & z_2^2 & \ldots & z_{K-1}^2 & z_K^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_1^{N-1} & z_2^{N-1} & \ldots & z_{K-1}^{N-1} & z_K^{N-1} \end{pmatrix} \in \mathbb{C}^{N \times K},$$

where $N \geqslant K$, and $z_1, z_2, \ldots, z_K \in \mathbb{C}$ are referred to as the "nodes" of $\mathbf{V}_{N \times K}$. Throughout the paper, we take the nodes to be non-zero and pairwise distinct, i.e., $z_{k_1} \neq z_{k_2}$, for $k_1 \neq k_2$, which ensures that the matrix $\mathbf{V}_{N \times K}$ has full rank.

We shall be interested in the minimum and maximum singular values and the condition number of $\mathbf{V}_{N \times K}$ with respect to the general matrix norm $\|\cdot\|$ defined [34, Eq. 5.3.7] as

$$\kappa\left(\mathbf{V}_{N \times K}, \|\cdot\|\right) := \|\mathbf{V}_{N \times K}\| \|(\mathbf{V}_{N \times K})^\dagger\|.$$

We will mostly be concerned with

$$\kappa(\mathbf{V}_{N\times K}) := \kappa(\mathbf{V}_{N\times K}, \|\cdot\|_{2,2}) = \frac{\sigma_{\max}(\mathbf{V}_{N\times K})}{\sigma_{\min}(\mathbf{V}_{N\times K})}, \tag{1}$$

often referred to as "spectral condition number".

The goal of this paper is to find lower bounds on the minimum singular value and upper bounds on the maximum singular value of Vandermonde matrices $\mathbf{V}_{N\times K}$ with nodes in the unit disk, that is, $|z_k| \leqslant 1$, for $k \in \{1, 2, \ldots, K\}$. Based on these bounds, we then establish upper bounds on $\kappa(\mathbf{V}_{N\times K})$.

3. Previous work

Before stating our main results, we summarize relevant prior work.

3.1. Vandermonde matrices with real nodes

Gautschi and Inglese [20, Thms. 2.2 and 3.1] showed that the condition number $\kappa(\mathbf{V}_{K\times K}, \|\cdot\|_\infty)$, $K \geqslant 3$, is lower-bounded by $(K-1)2^K$ when $z_1, z_2, \ldots, z_K \in \mathbb{R}_+$ and by $2^{K/2}$ when $K = 2L$, $L \in \mathbb{N}$, and the nodes $z_1, z_2, \ldots, z_K \in \mathbb{R}\backslash\{0\}$ satisfy the symmetry relationship $z_{k+L} = -z_k$, for $k \in \{1, 2, \ldots, L\}$. Beckermann [21, Thm. 4.1] found that the spectral condition number of $\mathbf{V}_{K\times K}$ satisfies

$$\frac{\sqrt{2}(1+\sqrt{2})^{K-1}}{\sqrt{K+1}} \leqslant \kappa(\mathbf{V}_{K\times K}) \leqslant (K+1)\sqrt{2}(1+\sqrt{2})^{K-1},$$

for $z_1, z_2, \ldots, z_K \in \mathbb{R}\backslash\{0\}$, and

$$\frac{C_K}{2(K+1)} \leqslant \kappa(\mathbf{V}_{K\times K}) \leqslant \frac{K+1}{2} C_K,$$

for $z_1, z_2, \ldots, z_K \in \mathbb{R}_+$, where $C_K := (1+\sqrt{2})^{2K} + (1+\sqrt{2})^{-2K}$. These results show that square Vandermonde matrices $\mathbf{V}_{K\times K}$ with real nodes necessarily become ill-conditioned as the matrix dimension grows. Specifically, the condition number grows exponentially in the matrix dimension and, in particular, does so independently of the specific values of the nodes $z_1, z_2, \ldots, z_K$.

3.2. Vandermonde matrices with complex nodes

For Vandermonde matrices with complex nodes the situation is fundamentally different. Consider, e.g., the DFT matrix $\mathbf{F}_K := \{e^{2\pi i k\ell/K}\}_{0\leqslant k,\ell\leqslant K-1}$, which is a Vandermonde matrix with nodes $z_k = e^{2\pi i(k-1)/K}$, for $k \in \{1, 2, \ldots, K\}$, and, as a consequence of $\mathbf{F}_K^H \mathbf{F}_K = K\mathbf{I}_K$, has the smallest possible spectral condition number, namely, $\kappa(\mathbf{F}_K) = 1$, and this, irrespectively of the matrix dimension K.

For general nodes $z_1, z_2, \ldots, z_K \in \mathbb{C}$, Gautschi [35, Thms. 1 and 3.1] obtained the following bounds on $\|(\mathbf{V}_{K\times K})^{-1}\|_\infty$:

$$\max_{1\leqslant k\leqslant K} \prod_{\substack{\ell=1\\ \ell\neq k}}^{K} \frac{\max\{1,|z_\ell|\}}{|z_k - z_\ell|} \leqslant \left\|(\mathbf{V}_{K\times K})^{-1}\right\|_\infty \leqslant \max_{1\leqslant k\leqslant K} \prod_{\substack{\ell=1\\ \ell\neq k}}^{K} \frac{1+|z_k|}{|z_k - z_\ell|}. \tag{2}$$

This allows us to derive bounds on $\sigma_{\min}(\mathbf{V}_{K\times K})$ and $\kappa(\mathbf{V}_{K\times K}, \|\cdot\|_\infty)$ by noting that $\|(\mathbf{V}_{K\times K})^{-1}\|_2 = \sigma_{\min}(\mathbf{V}_{K\times K})$ and

$$\frac{\|(\mathbf{V}_{K\times K})^{-1}\|_\infty}{\sqrt{K}} \leqslant \left\|(\mathbf{V}_{K\times K})^{-1}\right\|_2 \leqslant \sqrt{K}\left\|(\mathbf{V}_{K\times K})^{-1}\right\|_\infty .$$

Specifically, this results in

$$\frac{1}{\sqrt{K}} \max_{1\leqslant k\leqslant K} \prod_{\substack{\ell=1\\ \ell\neq k}}^{K} \frac{\max\{1,|z_\ell|\}}{|z_k - z_\ell|} \leqslant \sigma_{\min}(\mathbf{V}_{K\times K}) \leqslant \sqrt{K} \max_{1\leqslant k\leqslant K} \prod_{\substack{\ell=1\\ \ell\neq k}}^{K} \frac{1+|z_k|}{|z_k - z_\ell|}.$$

Combining (2) with

$$\|\mathbf{V}_{K\times K}\|_\infty = \max_{0\leqslant n\leqslant K-1} \sum_{k=1}^{K} |z_k|^n = \max\left\{\sum_{k=1}^{K} |z_k|^{K-1}, K\right\},$$

we get

$$\begin{aligned}\max\left\{\sum_{k=1}^{K} |z_k|^{K-1}, K\right\}\left(\max_{1\leqslant k\leqslant K} \prod_{\substack{\ell=1\\ \ell\neq k}}^{K} \frac{\max\{1,|z_\ell|\}}{|z_k - z_\ell|}\right) &\leqslant \kappa(\mathbf{V}_{K\times K}, \|\cdot\|_\infty) \\ &\leqslant \max\left\{\sum_{k=1}^{K} |z_k|^{K-1}, K\right\}\left(\max_{1\leqslant k\leqslant K} \prod_{\substack{\ell=1\\ \ell\neq k}}^{K} \frac{1+|z_k|}{|z_k - z_\ell|}\right). \end{aligned} \tag{3}$$

It is furthermore shown in [35, Thm. 1] that the upper bound in (2), and therefore also the upper bound in (3) are met with equality if the nodes $z_1, z_2, \ldots, z_K \in \mathbb{C}$ lie on a ray emanating from the origin, that is, if there exists a $\theta \in [0, 2\pi)$ such that $z_k = |z_k|e^{i\theta}$, for $k \in \{1, 2, \ldots, K\}$. As real nodes trivially satisfy this condition, namely with $\theta = 0$, this result confirms the worst-case condition number behavior associated with real nodes.

The remaining literature on the condition number of complex Vandermonde matrices can principally be divided into the case of all nodes lying on the unit circle and the—more general—case of nodes in the unit disk.

3.2.1. Vandermonde matrices with nodes on the unit circle

The DFT matrix having spectral condition number equal to 1, irrespectively of its dimension, indicates that Vandermonde matrices with nodes that are in some sense uniformly distributed on the unit circle could be well-conditioned in general. Inspired by this intuition, Córdova et al. [22] studied the spectral condition number of $\mathbf{V}_{K\times K}$ with nodes $z_k = e^{2\pi i c_k}$, $k \in \{1, 2, \ldots, K\}$, where c_k is the Van der Corput sequence defined as $c_k = \sum_{\ell=0}^{L-1} v_\ell^{(k)} 2^{-\ell-1}$, $L = \lfloor \log_2 k \rfloor + 1$, and $(v_0^{(k)}, v_1^{(k)}, \ldots, v_{L-1}^{(k)})$ is the binary representation of k, i.e., $k = \sum_{\ell=0}^{L-1} v_\ell^{(k)} 2^\ell$. Van der Corput sequences are used, e.g., in quasi-Monte Carlo simulation algorithms [36] and are known to have excellent uniform distribution properties. It is shown in [22, Cor. 3] that the spectral condition number of Vandermonde matrices $\mathbf{V}_{K\times K}$ built from Van der Corput sequences as described above is upper-bounded by $\sqrt{2K}$.

Berman and Feuer [23, Lem. 3.1] formally confirmed the intuition, expressed in [22], that nodes distributed uniformly on the unit circle lead to small condition number. Specifically, it is shown in [23, Lem. 3.1 & Thm. 3.2] that the spectral condition number of $\mathbf{V}_{K\times K}$ with $z_k = e^{-2\pi i p_k \tau / K}$, $p_k \in \{0, 1, \ldots, M-1\}$, for $k \in \{1, 2, \ldots, K\}$, $M > K$, $\tau \in \mathbb{R}$, is equal to 1 if and only if the nodes $z_1, z_2, \ldots, z_K$ are distributed uniformly on the unit circle in the following sense: There exists a $\tau \in \mathbb{R}$ such that the spectral condition number of $\mathbf{V}_{K\times K}$ is equal to 1, irrespectively of K, if and only if $\left\{\left\langle \frac{p_k}{Q} \right\rangle\right\}_{k=1}^{K}$ is a complete residue system modulo K [37, Chap. 3, §20], where $Q := \gcd(\{p_k\}_{k=1}^K)$ and $\left\langle \frac{p_k}{Q} \right\rangle$ is the remainder after division of $\frac{p_k}{Q}$ by K.

For Vandermonde matrices $\mathbf{V}_{N\times K}$, $N \geqslant K$, with nodes of the form $z_k = e^{2\pi i \xi_k}$, where $\xi_k \in [0, 1)$, for $k \in \{1, 2, \ldots, K\}$, Ferreira [24] employed Geršgorin's disc theorem [38, Thm. 6.1.1] to derive a lower bound on $\sigma_{\min}^2(\mathbf{V}_{N\times K})$ and an upper bound on $\sigma_{\max}^2(\mathbf{V}_{N\times K})$, which when combined give

$$\kappa(\mathbf{V}_{N\times K}) \leqslant \left(\frac{N + \left([\beta] + \dfrac{\beta^2}{[\beta]} - 1 \right)}{N - \left([\beta] + \dfrac{\beta^2}{[\beta]} - 1 \right)} \right)^{1/2} =: B(N, \beta), \tag{4}$$

for $N > [\beta] + \beta^2/[\beta] - 1$. Here,

$$\beta := \frac{\pi \Delta^{(w)}}{\sqrt{3} \sin(\pi \Delta^{(w)}) \delta^{(w)}} \tag{5}$$

with the minimum wrap-around distance

$$\delta^{(w)} := \min_{\substack{1 \leqslant k, \ell \leqslant K \\ k \neq \ell}} \min_{n \in \mathbb{Z}} |\xi_k - \xi_\ell + n| \tag{6}$$

and the maximum wrap-around distance

$$\Delta^{(w)} := \max_{\substack{1\leqslant k,\ell\leqslant K\\ k\neq \ell}} \min_{n\in\mathbb{Z}} |\xi_k - \xi_\ell + n|$$

between the ξ_k. Note that $\delta^{(w)} \leqslant 1/K$ as the maximum is achieved for K uniformly spaced nodes. Bazán [3]—also based on Geršgorin's disc theorem—derived a lower bound on $\sigma^2_{\min}(\mathbf{V}_{N\times K})$ and an upper bound on $\sigma^2_{\max}(\mathbf{V}_{N\times K})$ based on which one gets

$$\kappa(\mathbf{V}_{N\times K}) \leqslant \sqrt{\frac{N + (2K-2)/\sigma}{N-(2K-2)/\sigma}}, \tag{7}$$

for $N > 2(K-1)/\sigma$, where σ is the minimum (Euclidean) distance between the nodes z_k defined as

$$\sigma := \min_{\substack{1\leqslant k,\ell\leqslant K\\ k\neq \ell}} |z_k - z_\ell|. \tag{8}$$

Negreanu and Zuazua [39] and Liao and Fannjiang [15, Thm. 2] discovered discrete versions of Ingham's inequalities [40]. Besides the performance analysis of the MUSIC algorithm conducted in [15], these discrete Ingham inequalities also find application in the finite-difference discretization of homogeneous 1D wave equations [39]. In the present context, they provide a lower bound on $\sigma^2_{\min}(\mathbf{V}_{N\times K})$ and an upper bound on $\sigma^2_{\max}(\mathbf{V}_{N\times K})$, which, taken together, yield

$$\kappa(\mathbf{V}_{N\times K}) \leqslant \left(\frac{\dfrac{8\sqrt{2}\,\lceil (N-1)/2\rceil}{\pi} + \dfrac{\sqrt{2}}{2\pi\,\lceil (N-1)/2\rceil\,(\delta^{(w)})^2} + 3\sqrt{2}}{\dfrac{2(N-1)}{\pi} - \dfrac{2}{\pi(N-1)(\delta^{(w)})^2} - 4} \right)^{1/2}, \tag{9}$$

for $N \geqslant 7$ and

$$\delta^{(w)} > \frac{1}{N}\sqrt{\frac{2}{\pi}}\left(\frac{2}{\pi} - \frac{4}{N}\right)^{-1/2}.$$

Another upper bound on $\kappa(\mathbf{V}_{N\times K})$ was recently reported by Moitra in [25]. As Moitra's result is closely related to our main result, we review it in detail separately in Section 4.2.

3.2.2. Vandermonde matrices with nodes in the unit disk

For nodes z_k in the unit disk, i.e., $|z_k| \leqslant 1$, for all $k \in \{1, 2, \ldots, K\}$, Gautschi's upper bound (3) becomes

$$\kappa(\mathbf{V}_{K\times K}, \|\cdot\|_\infty) \leqslant K(2/\sigma)^{K-1}.$$

This result holds, however, for square Vandermonde matrices only. To the best of our knowledge, the only analytical result available on the condition number of rectangular (i.e., $N \geqslant K$) Vandermonde matrices with nodes in the unit disk is due to Bazán [3]. We review Bazán's result in Section 5.3 in the course of a comparison to our results.

4. Vandermonde matrices with nodes on the unit circle and the large sieve

The proof of our main result is inspired by a link—first established by Selberg [1, pp. 213–226] and later extended by Moitra [2, Thm. 2.3]—between the extremal singular values of Vandermonde matrices with nodes on the unit circle and the "large sieve" [26, 27, 28, 29, 30, 31], a family of inequalities involving polynomials in $e^{2\pi i\xi}$, $\xi \in [0, 1)$, originally developed in the field of analytic number theory [41, 42].

4.1. A brief introduction to the large sieve

We start with a brief introduction to the large sieve emphasizing the aspects relevant to the problem at hand. Specifically, we shall work with the definition of the large sieve as put forward by Davenport and Halberstam [43, Thm. 1].

Definition 1 (Large sieve inequality)**.** *Let* $\boldsymbol{y} := \{y_n\}_{n=0}^{N-1} \in \mathbb{C}^N$*. Define the trigonometric polynomial*

$$\forall \xi \in \mathbb{R}, \qquad S_{\boldsymbol{y},N}(\xi) := \sum_{n=0}^{N-1} y_n e^{-2\pi i n \xi}. \tag{10}$$

Let $\xi_1, \xi_2, \ldots, \xi_K \in [0, 1)$ *be such that the minimum wrap-around distance satisfies*

$$\delta^{(w)} := \min_{\substack{1 \leqslant k,\ell \leqslant K \\ k \neq \ell}} \min_{n \in \mathbb{Z}} |\xi_k - \xi_\ell + n| > 0.$$

A large sieve inequality is an inequality of the form

$$\sum_{k=1}^{K} |S_{\boldsymbol{y},N}(\xi_k)|^2 \leqslant \Delta(N, \delta^{(w)}) \sum_{n=0}^{N-1} |y_n|^2, \tag{11}$$

where $\Delta(N, \delta^{(w)})$ *depends on* N *and* $\delta^{(w)}$ *only.*

The large sieve inequality (11) essentially says that the energy contained in the samples $S_{\boldsymbol{y},N}(\xi_k)$, $k \in \{1, 2, \ldots, K\}$, of the trigonometric polynomial $S_{\boldsymbol{y},N}$ is bounded by the total energy of $S_{\boldsymbol{y},N}$ (given by $\sum_{n=0}^{N-1} |y_n|^2$) multiplied by a factor that depends on N and the minimum wrap-around distance between the ξ_k only.

Davenport and Halberstam [43, Thm. 1] established (11) with $\Delta(N, \delta^{(w)}) = 2.2 \times \max\{N, 1/\delta^{(w)}\}$, Gallagher [44] with $\Delta(N, \delta^{(w)}) = \pi N + 1/\delta^{(w)}$, Liu [45] with $\Delta(N, \delta^{(w)}) = 2 \max\{N, 1/\delta^{(w)}\}$, Bombieri and Davenport with $\Delta(N, \delta^{(w)}) =$

$(\sqrt{N}+1/\sqrt{\delta^{(w)}})^2$ in [46] and with $\Delta(N,\delta^{(w)}) = N+5/\delta^{(w)}$ in [47]. Montgomery and Vaughan [32, Thm. 1] proved (11) with $\Delta(N,\delta^{(w)}) = N+1/\delta^{(w)}$, later improved to $\Delta(N,\delta^{(w)}) = N-1+1/\delta^{(w)}$ by Cohen and independently by Selberg [33, Thm. 3]. In particular, Cohen used a "dilatation trick" to replace N in the Montgomery–Vaughan result [32, Thm. 1] by $N-1$, while Selberg's improvement [1, pp. 213–226] relies on the construction of an extremal majorant of the characteristic function χ_E of the interval $E := [0,(N-1)\delta^{(w)}]$. An extremal majorant of a function $\psi\colon \mathbb{R} \to \mathbb{R}$ is an entire function $M_\psi\colon \mathbb{C} \to \mathbb{C}$ of exponential type at most 2π [48, p. 839] which majorizes ψ along the real axis, i.e., $\psi(u) \leqslant M_\psi(u)$, for all $u \in \mathbb{R}$, and at the same time minimizes the integral $\int_{-\infty}^{\infty}(M_\psi(u) - \psi(u))\mathrm{d}u$.

4.2. Extremal singular values of Vandermonde matrices with nodes on the unit circle and the large sieve

For $N \times K$, $N \geqslant K$, Vandermonde matrices with nodes $e^{2\pi i \xi_k}$, $\xi_k \in [0,1)$, $k \in \{1,2,\ldots,K\}$, and minimum wrap-around distance $\delta^{(w)}$, Moitra [2, Thm. 2.3] showed that

$$\kappa(\mathbf{V}_{N\times K}) \leqslant \sqrt{\frac{N-1+1/\delta^{(w)}}{N-1-1/\delta^{(w)}}}, \tag{12}$$

for $N > 1+1/\delta^{(w)}$. This result is obtained from the upper bound on $\sigma^2_{\max}(\mathbf{V}_{N\times K})$ reported by Selberg in [1] and a new lower bound on $\sigma^2_{\min}(\mathbf{V}_{N\times K})$ derived by Moitra in [25].

Moitra's main insight was to recognize that replacing the extremal majorant of χ_E in Selberg's proof of the large sieve inequality by the extremal minorant of χ_E readily leads to a lower bound on $\sigma^2_{\min}(\mathbf{V}_{N\times K})$. We note that the condition $N > 1+1/\delta^{(w)}$ for (12) to hold excludes the case of square Vandermonde matrices, that is, $N = K$, because $N > 1+1/\delta^{(w)} \geqslant K+1$ as a consequence of $\delta^{(w)} \leqslant 1/K$.

We proceed to explaining in detail how (12) is obtained and to this end start by briefly reviewing Selberg's proof of the large sieve inequality. Selberg starts by considering the extremal majorant

$$\forall z \in \mathbb{C}, \qquad C_E(z) := \frac{1}{2}\big(B((N-1)\delta^{(w)} - z) + B(z)\big)$$

of the characteristic function χ_E of the interval $E = [0,(N-1)\delta^{(w)}]$, where B stands for Beurling's extremal majorant of the signum function given by [49]

$$\forall z \in \mathbb{C}, \quad B(z) := \left(\frac{\sin(\pi z)}{\pi}\right)^2 \left(\sum_{n=0}^{\infty} \frac{1}{(z-n)^2} - \sum_{n=-\infty}^{-1} \frac{1}{(z-n)^2} + \frac{2}{z}\right). \tag{13}$$

An important property of C_E is

$$\int_{-\infty}^{\infty} (C_E(u) - \chi_E(u))\,\mathrm{d}u = 1. \tag{14}$$

Letting $\mathbf{V}_{N\times K}$ be the Vandermonde matrix with nodes $z_k = e^{2\pi i \xi_k}$, $\xi_k \in [0,1)$, for $k \in \{1,2,\ldots,K\}$, Selberg first notes that

$$\sum_{k=1}^{K} |S_{\boldsymbol{y},N}(\xi_k)|^2 = \left\|(\mathbf{V}_{N\times K})^H \boldsymbol{y}\right\|_2^2, \tag{15}$$

for all $\boldsymbol{y} := \{y_n\}_{n=0}^{N-1} \in \mathbb{C}^N$. This implies that the large sieve inequality holds with every $\Delta(N,\delta^{(w)})$ satisfying

$$\Delta(N,\delta^{(w)}) \geqslant \sigma_{\max}^2((\mathbf{V}_{N\times K})^H) = \sigma_{\max}^2(\mathbf{V}_{N\times K}). \tag{16}$$

Conversely, every $\Delta(N,\delta^{(w)})$ such that (11) holds for all $\boldsymbol{y} := \{y_n\}_{n=0}^{N-1} \in \mathbb{C}^N$ must satisfy (16). Selberg goes on to derive an upper bound on $\sigma_{\max}^2(\mathbf{V}_{N\times K})$ as follows. Let $\boldsymbol{x} := \{x_k\}_{k=1}^{K}$, $\psi_{\boldsymbol{x}}(u) := \sum_{k=1}^{K} x_k e^{2\pi i \xi_k u}$, for all $u \in \mathbb{R}$, and note that

$$\begin{aligned}
\|\mathbf{V}_{N\times K}\boldsymbol{x}\|_2^2 &= \sum_{n=0}^{N-1} |\psi_{\boldsymbol{x}}(n)|^2 \\
&= \sum_{n=-\infty}^{\infty} \chi_E(\delta^{(w)} n)\, |\psi_{\boldsymbol{x}}(n)|^2 \\
&\leqslant \sum_{n=-\infty}^{\infty} C_E(\delta^{(w)} n)\, |\psi_{\boldsymbol{x}}(n)|^2 && (17)\\
&= \sum_{n=-\infty}^{\infty} C_E(\delta^{(w)} n) \sum_{k,\ell=1}^{K} x_k \overline{x_\ell} e^{2\pi i(\xi_k-\xi_\ell)n}. && (18)
\end{aligned}$$

C_E is integrable over $\mathbb{R}$, thanks to $C_E \geqslant 0$ and (14), and therefore the Fourier transform $\widehat{C}_E$ of its restriction to $\mathbb{R}$ is continuous. Moreover, as C_E is an entire function of exponential type at most 2π, $\widehat{C}_E$ is supported on $[-1,1]$. The Poisson summation formula then yields

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} C_E(\delta^{(w)} n) e^{2\pi i(\xi_k-\xi_\ell)n} &= (\delta^{(w)})^{-1} \sum_{n=-\infty}^{\infty} \widehat{C}_E((\delta^{(w)})^{-1}(n-(\xi_k-\xi_\ell))) \\
&= \begin{cases} (\delta^{(w)})^{-1}\widehat{C}_E(0), & k=\ell \\ 0, & \text{otherwise}, \end{cases} && (19)
\end{aligned}$$

where (19) follows from $|n-(\xi_k-\xi_\ell)| \geqslant \delta^{(w)}$, for all $n \in \mathbb{Z}$, and all $k,\ell \in \{1,2,\ldots,K\}$ such that $k \neq \ell$, and the fact that $\widehat{C}_E$ is a continuous function supported on $[-1,1]$ (which implies $\widehat{C}_E(-1) = \widehat{C}_E(1) = 0$). Note that the conditions for the application of the Poisson summation formula are met as C_E is integrable over $\mathbb{R}$, which, combined with the fact that C_E is an entire function of exponential type at most 2π, implies that C'_E is integrable over $\mathbb{R}$ [50, Pt. 2, Sec. 3., Prob. 7] and $C_E(u) \to 0$ as $|u| \to \infty$. From (14) we therefore get

$$\widehat{C}_E(0) = \int_{-\infty}^{\infty} C_E(u)\mathrm{d}u = 1 + \int_{-\infty}^{\infty} \chi_E(u)\mathrm{d}u = 1 + (N-1)\delta^{(w)}. \tag{20}$$

Combining (18), (19), and (20) thus yields

$$\|\mathbf{V}_{N\times K}\boldsymbol{x}\|_2^2 \leqslant \big(N - 1 + 1/\delta^{(w)}\big)\|\boldsymbol{x}\|_2^2\,. \tag{21}$$

As (21) holds for all $\boldsymbol{x} \in \mathbb{C}^K$, we can conclude that $\sigma^2_{\max}(\mathbf{V}_{N\times K}) \leqslant N-1+1/\delta^{(w)}$, which, thanks to (16), yields the large sieve inequality with $\Delta(N, \delta^{(w)}) = N - 1 + 1/\delta^{(w)}$. Bombieri and Davenport [47] showed that the large sieve inequality with $\Delta(N, \delta^{(w)}) = N - 1 + 1/\delta^{(w)}$ is tight by constructing an explicit example saturating (11) with $\Delta(N, \delta^{(w)}) = N - 1 + 1/\delta^{(w)}$.

We are now ready to review Moitra's lower bound on $\sigma^2_{\min}(\mathbf{V}_{N\times K})$. Specifically, Moitra recognized that Selberg's idea for upper-bounding $\sigma^2_{\max}(\mathbf{V}_{N\times K})$ can also be applied to lower-bound $\sigma^2_{\min}(\mathbf{V}_{N\times K})$, simply by working with the extremal minorant of χ_E, constructed by Selberg in [1], instead of the extremal majorant. An extremal minorant of a function $\psi\colon \mathbb{R} \to \mathbb{R}$ is an entire function $m_\psi\colon \mathbb{C} \to \mathbb{C}$ of exponential type at most 2π which minorizes ψ along the real axis, i.e., $m_\psi(u) \leqslant \psi(u)$, for all $u \in \mathbb{R}$, and at the same time minimizes the integral $\int_{-\infty}^{\infty}(\psi(u) - m_\psi(u))\mathrm{d}u$. The extremal minorant of χ_E constructed by Selberg is

$$\forall z \in \mathbb{C}, \qquad c_E(z) := -\frac{1}{2}\big(B(z - (N-1)\delta^{(w)}) + B(-z)\big),$$

where B was defined in (13). By construction, c_E satisfies

$$\int_{-\infty}^{\infty}(\chi_E(u) - c_E(u))\mathrm{d}u = 1. \tag{22}$$

Moitra showed that $\sigma^2_{\min}(\mathbf{V}_{N\times K}) \geqslant N - 1 - 1/\delta^{(w)}$ by replacing $\leqslant$ in (17) and C_E in (17)-(18) by $\geqslant$ and c_E, respectively, and employing arguments similar to those in (19) and (20) with c_E in place of C_E. The final result (12) then follows by using this lower bound in conjunction with the Selberg upper bound on $\sigma^2_{\max}(\mathbf{V}_{N\times K})$ in (1).

4.3. Relation to other bounds in the literature

We now put the Selberg–Moitra bound into perspective with respect to other bounds (for nodes on the unit circle) available in the literature. Both the Selberg–Moitra bound (12) as well as the Liao-Fannjiang bound (9) depend neither on the maximum wrap-around distance $\Delta^{(w)}$, as Ferreira's bound (4) does, nor do they exhibit a dependence on K as is the case for Bazán's bound (7). While (4), (9), and (12) depend on the minimum wrap-around distance $\delta^{(w)}$, Bazán's bound

(7) is in terms of the minimum distance σ between the nodes z_k. However, as $\sigma = 2\sin(\pi\delta^{(w)})$ (which follows from a simple geometric argument) and $2x/\pi \leqslant \sin(x) \leqslant x$, for $x \in [0, \pi/2)$, we get $4\delta^{(w)} \leqslant \sigma \leqslant 2\pi\delta^{(w)}$, so that the bounds (4), (9), and (12) can readily be expressed in terms of σ.

We next analyze Ferreira's bound (4). As $\delta^{(w)} \leqslant 1/2$ and $2x/\pi \leqslant \sin(x) \leqslant x$, for $x \in [0, \pi/2)$, it follows that β in (5) satisfies

$$1 \leqslant \frac{1}{\sqrt{3}\delta^{(w)}} \leqslant \beta \leqslant \frac{\pi}{2\sqrt{3}\delta^{(w)}} \leqslant \frac{1}{\delta^{(w)}}.$$

Further, we have

$$\frac{17}{18\delta^{(w)}} - \frac{1}{2} \leqslant [\beta] + \frac{\beta^2}{[\beta]} \leqslant \frac{7}{3\delta^{(w)}} + \frac{1}{2}, \tag{23}$$

where we used $x - 1/2 \leqslant [x] \leqslant x + 1/2$, the fact that the functions $x \mapsto x^2/(x - 1/2) + x + 1/2$ and $x \mapsto x - 1/2 + x^2/(x + 1/2)$ are non-decreasing on $[1, \infty)$, and $\delta^{(w)} \leqslant 1/2$. Employing (23) in Ferreira's bound (4), we get

$$\sqrt{\frac{N - 3/2 + 17/(18\delta^{(w)})}{N + 3/2 - 17/(18\delta^{(w)})}} \leqslant B(N, \beta) \leqslant \sqrt{\frac{N - 1/2 + 7/(3\delta^{(w)})}{N + 1/2 - 7/(3\delta^{(w)})}},$$

which shows that Ferreira's bound (4) exhibits the same structure as the Selberg–Moitra bound (12).

5. Main result

The main conceptual contribution of the present paper is an extension of the connection between the extremal singular values of Vandermonde matrices and the large sieve principle from the unit circle to the unit disk. As a byproduct, we find a new large sieve-type inequality involving polynomials in $z \in \mathbb{C}$ with $|z| \leqslant 1$ instead of trigonometric polynomials (i.e., polynomials in $e^{2\pi i\xi}$). This generalization can not be deduced from the Selberg–Moitra result whose proof relies on Fourier-analytic techniques and the Poisson summation formula and is hence restricted to nodes on the unit circle and to the classical large sieve inequality involving polynomials in variables that take value on the unit circle. It turns out, however, that an alternative connection between the extremal singular values of Vandermonde matrices with nodes on the unit circle and the large sieve can be obtained based on the Montgomery–Vaughan proof [32] of the large sieve inequality. The key insight now is that this alternative connection—thanks to being built on generalizations of Hilbert's inequality—can be extended from the unit circle to the unit disk. As a byproduct, the corresponding result—when particularized to the unit circle—slightly improves upon the Selberg–Moitra upper bound.

For pedagogical reasons, we start by explaining our approach for the special case of nodes on the unit circle. The general case of nodes in the unit disk is presented in Section 5.2.

5.1. An alternative connection for nodes on the unit circle

The Montgomery–Vaughan proof of the large sieve inequality with $\Delta(N, \delta^{(w)}) = N + 1/\delta^{(w)}$ is based on a generalization of Hilbert's inequality [51], which in its original form states that[1]

$$\left| \sum_{\substack{k,\ell=1 \\ k\neq\ell}}^{K} \frac{x_k \overline{x_\ell}}{k-\ell} \right| \leqslant \pi \sum_{k=1}^{K} |x_k|^2 , \tag{24}$$

for arbitrary $\boldsymbol{x} := \{x_k\}_{k=1}^K \in \mathbb{C}^K$. Specifically, Montgomery and Vaughan generalize (24) as follows.

Theorem 1 (Generalization of Hilbert's inequality, [51, Thms. 1 & 2])**.** *Let* $K \in \mathbb{N}\backslash\{0\}$.

a) For all $\boldsymbol{u} := \{u_k\}_{k=1}^K \in \mathbb{R}^K$ *such that*

$$\delta := \min_{\substack{1\leqslant k,\ell\leqslant K \\ k\neq\ell}} |u_k - u_\ell| > 0,$$

we have

$$\left| \sum_{\substack{k,\ell=1 \\ k\neq\ell}}^{K} \frac{\alpha_k \overline{\alpha_\ell}}{2\pi(u_k - u_\ell)} \right| \leqslant \frac{1}{2\delta} \sum_{k=1}^{K} |\alpha_k|^2 , \tag{25}$$

for all $\boldsymbol{\alpha} := \{\alpha_k\}_{k=1}^K \in \mathbb{C}^K$.

b) For all $\boldsymbol{\xi} := \{\xi_k\} \in \mathbb{R}^K$ *such that*

$$\delta^{(w)} := \min_{\substack{1\leqslant k,\ell\leqslant K \\ k\neq\ell}} \min_{n\in\mathbb{Z}} |\xi_k - \xi_\ell + n| > 0,$$

we have

$$\left| \sum_{\substack{k,\ell=1 \\ k\neq\ell}}^{K} \frac{a_k \overline{a_\ell}}{\sin(\pi(\xi_k - \xi_\ell))} \right| \leqslant \frac{1}{\delta^{(w)}} \sum_{k=1}^{K} |a_k|^2 , \tag{26}$$

[1] Hilbert actually proved (24) with a factor of 2π instead of π. Later Schur [52] replaced the factor 2π by the best possible constant π, but the inequality (24) has come to be referred to as "Hilbert's inequality".

for all $\boldsymbol{a} := \{a_k\}_{k=1}^K \in \mathbb{C}^K$.

Setting $u_k = k$, for $k \in \{1, 2, \dots, K\}$ in (25), we recover (24) since

$$\delta = \min_{\substack{1\leqslant k,\ell\leqslant K\\ k\neq \ell}} |k-\ell| = 1.$$

Based on Theorem 1, Montgomery and Vaughan [32, Thm. 1] proved the large sieve inequality with $\Delta(N, \delta^{(w)}) = N + 1/\delta^{(w)}$, which, thanks to (16), implies $\sigma_{\max}^2(\mathbf{V}_{N\times K}) \leqslant \Delta(N, \delta^{(w)})$. We next adapt the methodology of the proof of [32, Thm. 1] to derive a lower bound on $\sigma_{\min}^2(\mathbf{V}_{N\times K})$, and, en route, present the proof of $\sigma_{\max}^2(\mathbf{V}_{N\times K}) \leqslant N + 1/\delta^{(w)}$ provided in [32]. Improving the Montgomery–Vaughan result, by way of Cohen's dilatation trick, to $\sigma_{\max}^2(\mathbf{V}_{N\times K}) \leqslant N - 1 + 1/\delta^{(w)}$ and combining the result thereof with our new lower bound yields an improvement of the Selberg–Moitra result.

Let $\boldsymbol{x} := (x_1\ x_2\ \dots\ x_K)^T \in \mathbb{C}^K$. For $z_k = e^{2\pi i \xi_k}$, $k \in \{1, 2, \dots, K\}$, we have

$$\begin{aligned}
\|\mathbf{V}_{N\times K}\boldsymbol{x}\|_2^2 &= \sum_{n=0}^{N-1}\left|\sum_{k=1}^{K} x_k z_k^n\right|^2 = \sum_{n=0}^{N-1}\sum_{k,\ell=1}^{K} x_k\overline{x_\ell}e^{2\pi i(\xi_k-\xi_\ell)n}\\
&= \sum_{n=0}^{N-1}\left(\sum_{k=1}^{K}|x_k|^2 + \sum_{\substack{k,\ell=1\\k\neq\ell}}^{K} x_k\overline{x_\ell}e^{2\pi i(\xi_k-\xi_\ell)n}\right)\\
&= N\sum_{k=1}^{K}|x_k|^2 + \sum_{\substack{k,\ell=1\\k\neq\ell}}^{K} x_k\overline{x_\ell}\left(\sum_{n=0}^{N-1} e^{2\pi i(\xi_k-\xi_\ell)n}\right)\\
&= N\|\boldsymbol{x}\|_2^2 + \sum_{\substack{k,\ell=1\\k\neq\ell}}^{K} x_k\overline{x_\ell}\,\frac{1-e^{2\pi i(\xi_k-\xi_\ell)N}}{1-e^{2\pi i(\xi_k-\xi_\ell)}}\\
&= N\|\boldsymbol{x}\|_2^2 - \sum_{\substack{k,\ell=1\\k\neq\ell}}^{K} x_k\overline{x_\ell}\,\frac{1-e^{2\pi i(\xi_k-\xi_\ell)N}}{2ie^{\pi i(\xi_k-\xi_\ell)}\sin(\pi(\xi_k-\xi_\ell))}\\
&= N\|\boldsymbol{x}\|_2^2 - \underbrace{\sum_{\substack{k,\ell=1\\k\neq\ell}}^{K} \frac{x_k\overline{x_\ell}e^{-\pi i(\xi_k-\xi_\ell)}}{2i\sin(\pi(\xi_k-\xi_\ell))}}_{=:\,X_1} + \underbrace{\sum_{\substack{k,\ell=1\\k\neq\ell}}^{K} \frac{x_k\overline{x_\ell}e^{\pi i(\xi_k-\xi_\ell)(2N-1)}}{2i\sin(\pi(\xi_k-\xi_\ell))}}_{=:\,X_2}. \qquad (27)
\end{aligned}$$

As the nodes $z_1, z_2, \dots, z_K$ are, by assumption, pairwise distinct, we have

$$\delta^{(w)} = \min_{\substack{1\leqslant k,\ell\leqslant K\\k\neq\ell}} \min_{n\in\mathbb{Z}} |\xi_k - \xi_\ell + n| > 0, \qquad (28)$$

so that application of Theorem 1b) first with $a_k := x_k e^{-\pi i \xi_k}$, $k \in \{1, 2, \ldots, K\}$, yields

$$|X_1| \leqslant \frac{1}{2\delta^{(w)}} \sum_{k=1}^{K} \left| x_k e^{\pi i \xi_k} \right|^2 = \frac{\|\boldsymbol{x}\|_2^2}{2\delta^{(w)}} \tag{29}$$

and then with $a_k := x_k e^{\pi i \xi_k (2N-1)}$, $k \in \{1, 2, \ldots, K\}$, results in

$$|X_2| \leqslant \frac{1}{2\delta^{(w)}} \sum_{k=1}^{K} \left| x_k e^{\pi i \xi_k (2N-1)} \right|^2 = \frac{\|\boldsymbol{x}\|_2^2}{2\delta^{(w)}}. \tag{30}$$

Combining (27), (29), and (30), and using the forward and the reverse triangle inequality, we obtain

$$\left(N - 1/\delta^{(w)}\right) \|\boldsymbol{x}\|_2^2 \leqslant \|\mathbf{V}_{N\times K}\boldsymbol{x}\|_2^2 \leqslant \left(N + 1/\delta^{(w)}\right) \|\boldsymbol{x}\|_2^2, \tag{31}$$

for all $\boldsymbol{x} \in \mathbb{C}^K$. The lower and upper bounds in (31) therefore yield

$$\sigma_{\min}^2(\mathbf{V}_{N\times K}) \geqslant N - 1/\delta^{(w)} \tag{32}$$

$$\sigma_{\max}^2(\mathbf{V}_{N\times K}) \leqslant N + 1/\delta^{(w)}. \tag{33}$$

The upper bound $N + 1/\delta^{(w)}$ in (33) can be refined to $N - 1 + 1/\delta^{(w)}$ through Cohen's dilatation trick, explained for the general case of nodes in the unit disk in Section 5.2 (proof of Theorem 5). In summary, we get

$$\kappa(\mathbf{V}_{N\times K}) = \frac{\sigma_{\max}(\mathbf{V}_{N\times K})}{\sigma_{\min}(\mathbf{V}_{N\times K})} \leqslant \sqrt{\frac{N - 1 + 1/\delta^{(w)}}{N - 1/\delta^{(w)}}}, \tag{34}$$

for $N > 1/\delta^{(w)}$, which constitutes a slight improvement over the Selberg–Moitra bound (12).

5.2. Extremal singular values of Vandermonde matrices with nodes in the unit disk

We are now ready to proceed to our main result, namely a lower bound on $\sigma_{\min}^2(\mathbf{V}_{N\times K})$ and an upper bound on $\sigma_{\max}^2(\mathbf{V}_{N\times K})$ for nodes in the unit disk. Extending the connection between the extremal singular values of Vandermonde matrices and the Montgomery–Vaughan proof of the large sieve inequality from the unit circle to the unit disk requires a further generalization of Hilbert's inequality as follows.

Theorem 2 (Further generalization of Hilbert's inequality, [53, Eq. 5.11], [54])**.** *Let* $K \in \mathbb{N} \setminus \{0\}$. *For all* $\boldsymbol{\rho} := \{\rho_k\}_{k=1}^K \in \mathbb{C}^K$ *with* $\rho_k := \lambda_k + 2\pi i u_k$, *where* $\lambda_k > 0$

and $u_k \in \mathbb{R}$ *is such that*

$$\delta_k := \min_{\substack{1\leqslant \ell\leqslant K\\ \ell\neq k}} |u_k - u_\ell| > 0,$$

for all $k \in \{1, 2, \ldots, K\}$*, we have*[2]

$$\left|\sum_{\substack{k,\ell=1\\ k\neq \ell}}^{K} \frac{\alpha_k\overline{\alpha_\ell}}{\rho_k + \overline{\rho_\ell}}\right| \leqslant \frac{42}{\pi}\sum_{k=1}^{K}\frac{|\alpha_k|^2}{\delta_k}, \tag{35}$$

for all $\boldsymbol{\alpha} := \{\alpha_k\}_{k=1}^K \in \mathbb{C}^K$*. Moreover, in the case* $\lambda_1 = \lambda_2 = \ldots = \lambda_K = \lambda > 0$*,* (35) *can be refined to*

$$\frac{1}{\delta(e^{2\lambda/\delta}-1)}\sum_{k=1}^{K}|\alpha_k|^2 \leqslant \sum_{k,\ell=1}^{K}\frac{\alpha_k\overline{\alpha_\ell}}{\rho_k+\overline{\rho_\ell}} \leqslant \frac{e^{2\lambda/\delta}}{\delta(e^{2\lambda/\delta}-1)}\sum_{k=1}^{K}|\alpha_k|^2, \tag{36}$$

for all $\boldsymbol{\alpha} := \{\alpha_k\}_{k=1}^K \in \mathbb{C}^K$*, where* $\delta := \min\limits_{1\leqslant k\leqslant K}\delta_k$*.*

As pointed out in [54] the inequalities in (36) are best possible while (35) is not. This will be seen to have important ramifications for the range of validity of our main bounds in Theorem 5 and Corollary 6. We furthermore observe that (25) can be recovered from (36) by subtracting $\sum_{k=1}^{K}|\alpha_k|^2/(2\lambda)$ in (36) and letting $\lambda \to 0$. Indeed, the lower bound on $\sum\limits_{\substack{k,\ell=1\\ k\neq\ell}}^{K}\dfrac{\alpha_k\overline{\alpha_\ell}}{2\pi i(u_k-u_\ell)}$ resulting from (25) can be obtained from (36) by noting that

$$\lim_{\lambda\to 0}\left(\frac{1}{\delta(e^{2\lambda/\delta}-1)}-\frac{1}{2\lambda}\right) = \lim_{\lambda\to 0}\frac{1}{\delta}\left(\frac{1}{e^{2\lambda/\delta}-1}-\frac{1}{2\lambda/\delta}\right) = -\frac{1}{2\delta} \tag{37}$$

and the upper bound is a consequence of

$$\lim_{\lambda\to 0}\left(\frac{e^{2\lambda/\delta}}{\delta(e^{2\lambda/\delta}-1)}-\frac{1}{2\lambda}\right) = \lim_{\lambda\to 0}\left(\frac{1}{\delta}+\frac{1}{\delta(e^{2\lambda/\delta}-1)}-\frac{1}{2\lambda}\right) = \frac{1}{2\delta}, \tag{38}$$

[2]Note that the center term in (36) is real-valued as

$$\overline{\sum_{k,\ell=1}^{M}\frac{\alpha_k\overline{\alpha_\ell}}{\rho_k+\overline{\rho_\ell}}} = \sum_{k,\ell=1}^{M}\frac{\overline{\alpha_k}\alpha_\ell}{\overline{\rho_k}+\rho_\ell} = \sum_{k,\ell=1}^{M}\frac{\alpha_k\overline{\alpha_\ell}}{\rho_k+\overline{\rho_\ell}}.$$

where (37) follows from l'Hôpital's rule applied twice.

Theorem 2 generalizes Theorem 1a) from $i\mathbb{R}$ to the complex plane, i.e., the $2\pi i u_k \in i\mathbb{R}$ in (25) are replaced by the $\rho_k = \lambda_k + 2\pi i u_k \in \mathbb{C}$ in (35). We will also need a corresponding generalization of Theorem 1b). This generalization is formalized in Theorem 3 and builds on the following result.

Proposition 1. *Let* $A\colon (0,\infty)^3 \to (0,\infty)$ *and* $B\colon (0,\infty)^3 \to (0,\infty)$ *be functions satisfying*

$$\varepsilon A(\varepsilon x, \varepsilon y, \varepsilon z) = A(x,y,z)$$
$$\varepsilon B(\varepsilon x, \varepsilon y, \varepsilon z) = B(x,y,z),$$

for all $x > 0$, $y > 0$, $z > 0$, *and* $\varepsilon > 0$. *The following statements are equivalent:*

i) For all $M \in \mathbb{N}\backslash\{0\}$ *and* $\boldsymbol{\rho} := \{\rho_k\}_{k=1}^M \in \mathbb{C}^M$ *with* $\rho_k := \lambda_k + 2\pi i u_k$, *where* $\lambda_k > 0$ *and* $u_k \in \mathbb{R}$ *is such that*

$$\delta_k := \min_{\substack{1\leqslant \ell \leqslant M \\ \ell \neq k}} |u_k - u_\ell| > 0,$$

for all $k \in \{1,2,\dots,M\}$, *we have*

$$\sum_{k=1}^M A(\lambda_k, \delta_k, \delta)\, |\alpha_k|^2 \leqslant \sum_{k,\ell=1}^M \frac{\alpha_k \overline{\alpha_\ell}}{\rho_k + \overline{\rho_\ell}} \leqslant \sum_{k=1}^M B(\lambda_k, \delta_k, \delta)\, |\alpha_k|^2, \tag{39}$$

for all $\boldsymbol{\alpha} := \{\alpha_k\}_{k=1}^M \in \mathbb{C}^M$, *where* $\delta := \min_{1\leqslant k\leqslant M} \delta_k$.

ii) For all $M \in \mathbb{N}\backslash\{0\}$ *and* $\boldsymbol{r} := \{r_k\}_{k=1}^M \in \mathbb{C}^M$ *with* $r_k := d_k + 2\pi i \xi_k$, *where* $d_k > 0$ *and* $\xi_k \in \mathbb{R}$ *is such that*

$$\delta_k^{(w)} := \min_{\substack{1\leqslant \ell \leqslant M \\ \ell \neq k}} \min_{n\in\mathbb{Z}} |\xi_k - \xi_\ell + n| > 0,$$

for all $k \in \{1,2,\dots,M\}$, *we have*

$$\sum_{k=1}^M 2A(d_k, \delta_k^{(w)}, \delta^{(w)})\, |a_k|^2 \leqslant \sum_{k,\ell=1}^M \frac{a_k \overline{a_\ell}}{\sinh((r_k + \overline{r_\ell})/2)} \leqslant \sum_{k=1}^M 2B(d_k, \delta_k^{(w)}, \delta^{(w)})\, |a_k|^2, \tag{40}$$

for all $\boldsymbol{a} := \{a_k\}_{k=1}^M \in \mathbb{C}^M$, *where* $\delta^{(w)} := \min_{1\leqslant k\leqslant M} \delta_k^{(w)}$.

Proof. See Appendix A. □

Theorem 2 provides a generalization of Theorem 1a). Combining Proposition 1 with Theorem 2, we get the following generalization of Theorem 1b).

Theorem 3. *Let $K \in \mathbb{N}\backslash\{0\}$. For all $\boldsymbol{r} := \{r_k\}_{k=1}^K$ with $r_k := d_k + 2\pi i \xi_k$, where $d_k > 0$ and $\xi_k \in \mathbb{R}$ is such that*

$$\delta_k^{(w)} := \min_{\substack{1\leqslant \ell \leqslant K \\ \ell \neq k}} \min_{n\in\mathbb{Z}} |\xi_k - \xi_\ell + n| > 0,$$

for all $k \in \{1, 2, \ldots, K\}$, we have

$$\sum_{k=1}^K \left(\frac{1}{d_k} - \frac{84}{\pi\delta_k^{(w)}}\right)|a_k|^2 \leqslant \sum_{k,\ell=1}^K \frac{a_k\overline{a_\ell}}{\sinh((r_k+\overline{r_\ell})/2)} \leqslant \sum_{k=1}^K \left(\frac{1}{d_k} + \frac{84}{\pi\delta_k^{(w)}}\right)|a_k|^2, \quad (41)$$

for all $\boldsymbol{a} = \{a_k\}_{k=1}^K \in \mathbb{C}^K$. Moreover, in the case $d_1 = d_2 = \ldots = d_K = d > 0$, (41) can be refined to

$$\frac{2}{\delta^{(w)}(e^{2d/\delta^{(w)}}-1)} \sum_{k=1}^K |a_k|^2 \leqslant \sum_{k,\ell=1}^K \frac{a_k\overline{a_\ell}}{\sinh((r_k+\overline{r_\ell})/2)} \leqslant \frac{2e^{2d/\delta^{(w)}}}{\delta^{(w)}(e^{2d/\delta^{(w)}}-1)} \sum_{k=1}^K |a_k|^2, \quad (42)$$

for all $\boldsymbol{a} = \{a_k\}_{k=1}^K \in \mathbb{C}^K$, where $\delta := \min_{1\leqslant \ell\leqslant K} \delta_k$.

Proof. See Appendix B. □

We note that (26) can be recovered from (42) by subtracting the diagonal terms $\sum_{k=1}^K |a_k|^2/\sinh(d)$ in (42), letting $d \to 0$, and noting that $\sinh(i\pi\xi) = i\sin(\pi\xi)$, for $\xi \in \mathbb{R}$. Indeed, we have

$$\lim_{d\to 0}\left(\frac{2}{\delta^{(w)}(e^{2d/\delta^{(w)}}-1)} - \frac{1}{\sinh(d)}\right)$$
$$= \lim_{d\to 0} 2\left[\frac{1}{\delta^{(w)}(e^{2d/\delta^{(w)}}-1)} - \frac{1}{2d} + \frac{1}{2d}\left(1 - \frac{d}{\sinh(d)}\right)\right] = -\frac{1}{\delta^{(w)}}$$

and

$$\lim_{d\to 0}\left(\frac{2e^{2d/\delta^{(w)}}}{\delta^{(w)}(e^{2d/\delta^{(w)}}-1)} - \frac{1}{\sinh(d)}\right)$$
$$= \lim_{\lambda\to 0} 2\left[\frac{e^{2d/\delta^{(w)}}}{\delta^{(w)}(e^{2d/\delta^{(w)}}-1)} - \frac{1}{2d} + \frac{1}{2d}\left(1 - \frac{d}{\sinh(d)}\right)\right] = \frac{1}{\delta^{(w)}},$$

where we used (37), (38), and $\lim_{z\to 0}\sinh(z)/z = 1$.

We next show that the constant in the lower bound in (42) can be improved through a slight modification of a result by Graham and Vaaler [53]. This improvement is relevant as it leads to improved bounds on $\sigma^2_{\min}(\mathbf{V}_{N\times K})$ and $\kappa(\mathbf{V}_{N\times K})$ and to a more general condition for these bounds to be valid.

Corollary 4. *Let $K \in \mathbb{N}\backslash\{0\}$. For all $d > 0$ and $\boldsymbol{\xi} := \{\xi_k\}_{k=1}^K \in \mathbb{R}^K$ such that*

$$\delta^{(w)} := \min_{\substack{1 \leqslant k,\ell \leqslant K \\ k \neq \ell}} \min_{n \in \mathbb{Z}} |\xi_k - \xi_\ell + n| > 0,$$

for all $k \in \{1, 2, \ldots, K\}$, we have

$$\frac{2e^d}{\delta^{(w)}(e^{2d/\delta^{(w)}} - 1)} \sum_{k=1}^K |a_k|^2 \leqslant \sum_{k,\ell=1}^K \frac{a_k \overline{a_\ell}}{\sinh(d + \pi i(\xi_k - \xi_\ell))} \leqslant \frac{2e^{2d/\delta^{(w)}}}{\delta^{(w)}(e^{2d/\delta^{(w)}} - 1)} \sum_{k=1}^K |a_k|^2, \tag{43}$$

for all $\boldsymbol{a} = \{a_k\}_{k=1}^K \in \mathbb{C}^K$.

Proof. Based on an extremal minorant and an extremal majorant of the function

$$\forall t \in \mathbb{R}, \qquad f(t) = \begin{cases} e^{-2\delta t}, & t \geqslant 0 \\ 0, & t < 0, \end{cases}$$

Graham and Vaaler [53] showed[3] that for $d > 0$, and $\boldsymbol{\xi} := \{\xi_k\}_{k=1}^K \in \mathbb{R}^K$ such that

$$\delta^{(w)} := \min_{\substack{1 \leqslant k,\ell \leqslant K \\ k \neq \ell}} \min_{n \in \mathbb{Z}} |\xi_k - \xi_\ell + n| > 0,$$

we have

$$\frac{2e^d}{\delta^{(w)}(e^{2d/\delta^{(w)}} - 1)} \sum_{k=1}^K |a_k|^2 \leqslant \sum_{k,\ell=1}^K \frac{a_k \overline{a_\ell}}{\sinh(d + \pi i(\xi_\ell - \xi_k))} \leqslant \frac{2e^d e^{2d/\delta^{(w)}}}{\delta^{(w)}(e^{2d/\delta^{(w)}} - 1)} \sum_{k=1}^K |a_k|^2, \tag{44}$$

for all $\boldsymbol{a} = \{a_k\}_{k=1}^K \in \mathbb{C}^K$. Since $d > 0$, the lower and upper bounds in (44) are larger than those in (42). Combining the improved lower bound in (44) with the upper bound in (42) yields the desired result. □

We are now ready to establish our new bounds on $\sigma^2_{\min}(\mathbf{V}_{N \times K})$ and $\sigma^2_{\max}(\mathbf{V}_{N \times K})$ for nodes $z_1, z_2, \ldots, z_K$ in the unit disk.

Theorem 5 (Lower bound on $\sigma^2_{\min}(\mathbf{V}_{N \times K})$ and upper bound on $\sigma^2_{\max}(\mathbf{V}_{N \times K})$ for nodes in the unit disk)**.** *Let $\boldsymbol{z} := \{z_k\}_{k=1}^K \in \mathbb{C}^K$ with $z_k := |z_k| e^{2\pi i \xi_k}$ be such that*

[3] The inequalities provided in Graham and Vaaler [53] are actually given by

$$\frac{e^d}{\delta^{(w)}(e^{2d/\delta^{(w)}} - 1)} \sum_{k=1}^K |a_k|^2 \leqslant \sum_{k,\ell=1}^K \frac{a_k \overline{a_\ell}}{\sinh(d + \pi i(\xi_\ell - \xi_k))} \leqslant \frac{e^d e^{2d/\delta^{(w)}}}{\delta^{(w)}(e^{2d/\delta^{(w)}} - 1)} \sum_{k=1}^K |a_k|^2.$$

We believe, however, that there is a mathematical typo in [53] and that a factor of 2 is missing in the lower and the upper bounds.

$0 < |z_k| \leqslant 1$, $\xi_k \in [0,1)$, *and*

$$\delta_k^{(w)} := \min_{\substack{1\leqslant \ell \leqslant K\\ \ell \neq k}} \min_{n\in\mathbb{Z}} |\xi_k - \xi_\ell + n| > 0,$$

for all $k \in \{1,2,\dots,K\}$. *The extremal singular values of the Vandermonde matrix* $\mathbf{V}_{N\times K}$ *with nodes* $z_1, z_2, \dots, z_K$ *satisfy*

$$\sigma_{\min}^2(\mathbf{V}_{N\times K}) \geqslant \mathscr{L}(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)}) \tag{45}$$

$$\sigma_{\max}^2(\mathbf{V}_{N\times K}) \leqslant \min\Big\{\mathscr{U}(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)}), \mathscr{U}(N-1, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)})\Big\}, \tag{46}$$

where $|\boldsymbol{z}| := \{|z_k|\}_{k=1}^K$, $\boldsymbol{\delta}^{(w)} := \{\delta_k^{(w)}\}_{k=1}^K$, *and*

$$\mathscr{L}(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)}) := \min_{1\leqslant k\leqslant K} \left\{ \frac{1}{|z_k|}\left[\varphi_N(|z_k|) - \frac{42}{\pi\delta_k^{(w)}}\Big(1 + |z_k|^{2N}\Big)\right]\right\} \tag{47}$$

$$\mathscr{U}(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)}) := \max_{1\leqslant k\leqslant K} \left\{ \frac{1}{|z_k|}\left[\varphi_N(|z_k|) + \frac{42}{\pi\delta_k^{(w)}}\Big(1 + |z_k|^{2N}\Big)\right]\right\}, \tag{48}$$

with

$$\forall k \in \{1,2,\dots,K\}, \quad \varphi_N(|z_k|) := \begin{cases} \dfrac{|z_k|^{2N} - 1}{2\ln|z_k|}, & |z_k| < 1 \\ N, & |z_k| = 1. \end{cases} \tag{49}$$

Moreover, if $|z_1| = |z_2| = \ldots = |z_K| = A < 1$, (45) *and* (46) *can be refined to*

$$\sigma_{\min}^2(\mathbf{V}_{N\times K}) \geqslant \frac{1 - A^{2(N+1/2-1/\delta^{(w)})}}{\delta^{(w)}(A^{-2/\delta^{(w)}} - 1)A^2} \tag{50}$$

$$\sigma_{\max}^2(\mathbf{V}_{N\times K}) \leqslant \frac{A^{-2/\delta^{(w)}}\Big(1 - A^{2(N-1+1/\delta^{(w)})}\Big)}{\delta^{(w)}(A^{-2/\delta^{(w)}} - 1)}, \tag{51}$$

where $\delta^{(w)} := \min_{1\leqslant k\leqslant K} \delta_k^{(w)}$.

Proof. See Appendix C. □

The following upper bound on the condition number $\kappa(\mathbf{V}_{N\times K})$ is an immediate consequence of Theorem 5.

Corollary 6 (Upper bound on $\kappa(\mathbf{V}_{N\times K})$ for nodes in the unit disk)**.** *Let* $\boldsymbol{z} := \{z_k\}_{k=1}^K \in \mathbb{C}^K$ *with* $z_k := |z_k|e^{2\pi i\xi_k}$ *be such that* $0 < |z_k| \leqslant 1$, $\xi_k \in [0,1)$, *and*

$$\delta_k^{(w)} := \min_{\substack{1\leqslant \ell \leqslant K\\ \ell \neq k}} \min_{n\in\mathbb{Z}} |\xi_k - \xi_\ell + n| > 0,$$

for all $k \in \{1, 2, \dots, K\}$. *The spectral condition number satisfies*

$$\kappa(\mathbf{V}_{N\times K}) \leqslant \sqrt{\frac{\min\Big\{\mathcal{U}(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)}), \mathcal{U}(N-1, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)})\Big\}}{\mathcal{L}(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)})}}, \tag{52}$$

if for all $k \in \{1, 2, \dots, K\}$,

$$\delta_k^{(w)} > \frac{42}{\pi}\left(\frac{1+|z_k|^{2N}}{\varphi_N(|z_k|)}\right), \tag{53}$$

where $\mathcal{L}(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)})$, $\mathcal{U}(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)})$, *and* $\varphi_N(|z_k|)$ *are defined in* (47), (48), *and* (49), *respectively,* $|\boldsymbol{z}| := \{|z_k|\}_{k=1}^K$, *and* $\boldsymbol{\delta}^{(w)} := \{\delta_k^{(w)}\}_{k=1}^K$. *Moreover, if* $|z_1| = |z_2| = \ldots = |z_K| = A < 1$, *the spectral condition number satisfies*

$$\kappa(\mathbf{V}_{N\times K}) \leqslant A^{-1/\delta^{(w)}} \sqrt{\frac{A^2\big(1 - A^{2(N-1+1/\delta^{(w)})}\big)}{1 - A^{2(N+1/2-1/\delta^{(w)})}}} \tag{54}$$

if $N > 1/\delta^{(w)} - 1/2$.

Proof. See Appendix D. □

First, we note that the upper bounds in (46) and (51) lead to a generalization of the large sieve inequality from the unit circle to the unit disk in the following sense. For $\boldsymbol{z} := \{z_k\}_{k=1}^K \in \mathbb{C}^K$ such that $z_k := |z_k| e^{2\pi i \xi_k}$, $0 < |z_k| \leqslant 1$, $\xi_k \in [0, 1)$, and

$$\delta_k^{(w)} := \min_{\substack{1\leqslant \ell \leqslant K \\ \ell \neq k}} \min_{n\in\mathbb{Z}} |\xi_k - \xi_\ell + n| > 0,$$

and $\boldsymbol{y} := \{y_n\}_{n=0}^{N-1} \in \mathbb{C}^N$, we have

$$\sum_{k=1}^{K} |S_{\boldsymbol{y},N}(z_k)|^2 \leqslant \Delta(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)}) \sum_{n=0}^{N-1} |y_n|^2,$$

where the trigonometric polynomial (10) in (11) is replaced by the polynomial

$$\forall z \in \mathbb{C}, \qquad S_{\boldsymbol{y},N}(z) := \sum_{n=0}^{N-1} y_n \overline{z}^n,$$

the sieve factor $\Delta(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)})$ is given by

$$\Delta(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)}) = \begin{cases} \dfrac{A^{-2/\delta^{(w)}}\left(1 - A^{2(N-1+1/\delta^{(w)})}\right)}{\delta^{(w)}(A^{-2/\delta^{(w)}} - 1)}, & |z_1| = \ldots = |z_K| = A, \\ \min\left\{\mathcal{U}(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)}), \mathcal{U}(N-1, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)})\right\}, \text{ else}, \end{cases}$$

$|\boldsymbol{z}| := \{|z_k|\}_{k=1}^{K}$, $\boldsymbol{\delta}^{(w)} := \{\delta_k^{(w)}\}_{k=1}^{K}$, and $\mathcal{U}(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)})$ is as defined in (48).

We furthermore note that the bound (34), valid for nodes on the unit circle, can be recovered by letting $A \to 1$ in (54). Moreover, (32) and (34) can be improved to

$$\sigma_{\min}^2(\mathbf{V}_{N\times K}) \geqslant N + 1/2 - 1/\delta^{(w)} \tag{55}$$

and

$$\kappa(\mathbf{V}_{N\times K}) \leqslant \sqrt{\frac{N - 1 + 1/\delta^{(w)}}{N + 1/2 - 1/\delta^{(w)}}}, \tag{56}$$

respectively, by letting $A \to 1$ in (50) and (54), respectively, which leads to the announced improvement of the Selberg–Moitra bound. Indeed, $\sigma_{\min}^2(\mathbf{V}_{N\times K})$ and $\kappa(\mathbf{V}_{N\times K})$ are continuous functions of A for $N > 1/\delta^{(w)} - 1/2$. We can therefore establish (55) and (56) by taking the limits

$$\begin{aligned} \frac{1 - A^{2(N+1/2-1/\delta^{(w)})}}{\delta^{(w)}(A^{-2/\delta^{(w)}} - 1)A^2} &= \underbrace{\frac{e^{2(N+1/2-1/\delta^{(w)})\ln(A)} - 1}{2(N + 1/2 - 1/\delta^{(w)})\ln(A)}}_{\xrightarrow[A\to 1]{} 1} \cdot \underbrace{\frac{-(2/\delta^{(w)})\ln(A)}{e^{-(2/\delta^{(w)})\ln(A)} - 1}}_{\xrightarrow[A\to 1]{} 1} \\ &\quad \cdot \underbrace{\frac{N + 1/2 - 1/\delta^{(w)}}{A^2}}_{\xrightarrow[A\to 1]{} N+1/2-\frac{1}{\delta^{(w)}}} \\ &\xrightarrow[A\to 1]{} N + \frac{1}{2} - \frac{1}{\delta^{(w)}} \end{aligned} \tag{57}$$

and

$$\begin{aligned} \frac{A^{-2/\delta^{(w)}}\left(1 - A^{2(N-1+1/\delta^{(w)})}\right)}{\delta^{(w)}(A^{-2/\delta^{(w)}} - 1)A} &= \underbrace{e^{-(2\delta^{(w)})\ln(A)}}_{\xrightarrow[A\to 1]{} 1} \cdot \underbrace{\frac{1 - A^{2(N-1+1/\delta^{(w)})}}{\delta^{(w)}(A^{-2/\delta^{(w)}} - 1)A}}_{\xrightarrow[A\to 1]{} N-1+1/\delta^{(w)}} \\ &\xrightarrow[A\to 1]{} N - 1 + \frac{1}{\delta^{(w)}}, \end{aligned} \tag{58}$$

respectively. We note that (56) holds under the condition $N > 1/\delta^{(w)} - 1/2$. This improvement is interesting as the condition for validity of the bound on $\kappa(\mathbf{V}_{N\times K})$

no longer excludes the case of square Vandermonde matrices, as is the case for the original Selberg–Moitra bound and our first improvement thereof provided in (34). To see this, simply note that thanks to $N > 1/\delta^{(w)} - 1/2 \geqslant K - 1/2$, the square case $N = K$ is now allowed as here $N = K > K - 1/2$. Owing to $\delta^{(w)} \leqslant 1/K$ this comes, however, at the cost of the nodes being almost equally spaced.

We next investigate the qualitative dependence of our bounds on the quantities N, $\delta_k^{(w)}$, and $|z_k|$. To this end, we first show that $\varphi_N(|z_k|)$ is non-decreasing in $|z_k|$, for fixed N, and non-increasing in N, for fixed $|z_k|$. While the latter follows by inspection, to see the former, we write $\varphi_N(|z_k|) = N f(|z_k|^{2N})$ and note that

$$f(x) := \begin{cases} \dfrac{x-1}{\ln(x)}, & x \in (0,1) \\ 1, & x = 1 \end{cases}$$

is non-decreasing. Consequently, the lower bound (45) increases and the upper bound (46) decreases as the nodes $z_k = |z_k| e^{2\pi i \xi_k}$ move closer to the unit circle. Furthermore, $\mathscr{L}(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)})$ and $\mathscr{U}(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)})$ are increasing and decreasing in $\delta^{(w)}$, respectively. This allows us to conclude that the upper bound on $\kappa(\mathbf{V}_{N\times K})$ in (52) decreases as the nodes $z_k = |z_k| e^{2\pi i \xi_k}$ get closer to the unit circle and/or the node frequencies ξ_k are more separated. Indeed, we have

$$\frac{1 + |z_k|^{2N}}{\varphi_N(|z_k|)} = -2 \ln |z_k| \left(\frac{1 + |z_k|^{2N}}{1 - |z_k|^{2N}} \right) =: h(|z_k|)$$

and note that the function $h\colon x \mapsto -2\ln(x)(1 + x^{2N})/(1 - x^{2N})$ is non-increasing. The condition (53) therefore requires that the wrap-around distance $\delta_k^{(w)}$ increase as $|z_k|$ gets smaller. Specifically, (53) is violated if there exists a node z_k with small modulus $|z_k|$ together with another node z_ℓ (of arbitrary modulus $|z_\ell| \leqslant 1$) so that the wrap-around distance between ξ_k and ξ_ℓ, i.e., $\min_{n\in\mathbb{Z}} |\xi_k - \xi_\ell + n|$, is small. This shows that a large minimum distance

$$\sigma_k := \min_{\substack{1 \leqslant \ell \leqslant K \\ \ell \neq k}} |z_k - z_\ell|$$

alone is not enough to guarantee (53). Moreover, condition (53) excludes the case of nodes placed on a ray emanating from the origin, as here the wrap-around distance equals zero. Finally, we emphasize that owing to the large constant $42/\pi$ in (53), which stems from the Montgomery–Vaaler result (35) not being best possible, (53) is quite restrictive as it will be satisfied only for nodes z_k that are very close to the unit circle. For $|z_1| = |z_2| = \ldots = |z_K| = A$, we not only get a much larger range of validity for our upper bound (54) on $\kappa(\mathbf{V}_{N\times K})$ than for the general upper bound (52), but we also obtain sharper bounds on $\sigma^2_{\min}(\mathbf{V}_{N\times K})$ and

$\sigma^2_{\max}(\mathbf{V}_{N\times K})$ as the corresponding results are based on (36), which, as pointed out in [53], is best possible. One would hope that the constant $42/\pi$ in (35) could be improved to be closer to the corresponding constant $1/2$ in (25) or that $42/\pi$ could be turned into a smaller constant which would possibly depend on $\min_{1\leqslant k\leqslant K}|z_k|$ and/or $\max_{1\leqslant k\leqslant K}|z_k|$ as in the Graham–Vaaler result (36).

5.3. Comparison to Bazán's bound

We finally compare our bounds (52) and (54) to Bazán's bound [3] on $\kappa(\mathbf{V}_{N\times K})$ and start by reviewing Bazán's bound. It is shown in [3, Thm. 6] that the spectral condition number of $\mathbf{V}_{N\times K}$, $N \geqslant K$, with nodes z_k in the unit disk satisfies

$$\frac{\sigma_{\max}(\mathbf{G}_N)}{A_{\max}} \leqslant \kappa(\mathbf{V}_{N\times K}) \leqslant \frac{1}{2}\left(\eta + \sqrt{\eta^2-4}\right). \tag{59}$$

Here, $A_{\max} := \max\limits_{1\leqslant k\leqslant K}|z_k|$ and $\mathbf{G}_N \in \mathbb{C}^{K\times K}$ is a matrix constructed as follows. Let $\widehat{\boldsymbol{f}}_N \in \mathbb{C}^N$ be the minimum ℓ^2-norm solution of the linear system of equations $(\mathbf{V}_{N\times K})^T\boldsymbol{f} = \boldsymbol{z}_N$, where $\boldsymbol{z}_N := \begin{pmatrix} z_1^N & z_2^N & \dots & z_K^N\end{pmatrix}^T \in \mathbb{C}^K$. Set

$$\mathbf{G}_N := \mathbf{W}^H_{N\times K}\mathbf{\Gamma}_N\mathbf{W}_{N\times K},$$

where

$$\mathbf{W}_{N\times K} := \overline{\mathbf{V}_{N\times K}\left((\mathbf{V}_{N\times K})^H\,\mathbf{V}_{N\times K}\right)^{-1/2}} \in \mathbb{C}^{N\times K},$$

$\mathbf{\Gamma}_N := \begin{pmatrix}\mathbf{e}_2 & \mathbf{e}_3 & \dots & \mathbf{e}_N & \widehat{\boldsymbol{f}}_N\end{pmatrix} \in \mathbb{C}^{N\times N}$, and $\boldsymbol{e}_n \in \mathbb{C}^N$, for $n \in \{2,3,\dots,N\}$, is the nth unit vector whose elements are all zero apart from the nth entry which equals 1. The quantity η is given by

$$\eta := K\left(1+\frac{D_N^2}{(K-1)\sigma^2}\right)^{\frac{K-1}{2}}\left(\frac{\psi_N(A_{\max})}{\psi_N(A_{\min})}\right)^{1/2} - K + 2, \tag{60}$$

where $D_N^2 := \|\mathbf{G}_N\|^2 - \left(|z_1|^2+\ldots+|z_K|^2\right)$ is the so-called departure of $\mathbf{G}_N$ from normality, σ is the minimum distance between the nodes $z_1, z_2, \dots, z_K$ as defined in (8), $\psi_N(x) := \sum_{n=0}^{N-1} x^{2n}$, and $A_{\min} := \min\limits_{1\leqslant k\leqslant K}|z_k|$.

Owing to the complicated and, in particular, implicit nature of Bazán's bound, it appears difficult to draw crisp conclusions therefrom on the behavior of $\kappa(\mathbf{V}_{N\times K})$ as a function of the nodes $z_1, z_2, \dots, z_K$ and N. It is, however, possible to extract a statement of asymptotic (in N, for fixed K) nature from (59). Specifically, it is stated in [3, Lem. 7] that

$$(K-1) + \frac{\prod_{k=1}^K|z_k|^2}{1+\|\widehat{\boldsymbol{f}}_N\|_2^2} - \sum_{k=1}^K|z_k|^2 \leqslant D_N^2 \leqslant (K-1) + \|\widehat{\boldsymbol{f}}_N\|_2^2 + \prod_{k=1}^K|z_k|^2 - \sum_{k=1}^K|z_k|^2,$$

for $N \geqslant K$. Since $\lim_{N\to\infty} \left\| \widehat{\boldsymbol{f}}_N \right\|_2^2 = 0$ [3, Thm. 2], we get

$$\lim_{N\to\infty} D_N^2 = (K-1) + \prod_{k=1}^{K} |z_k|^2 - \sum_{k=1}^{K} |z_k|^2 . \tag{61}$$

Now, since for fixed $u_\ell \in [0,1]$, $\ell \in \{1, 2, \ldots, K\} \backslash \{k\}$, the function $u_k \mapsto \prod_{\ell=1}^{K} u_\ell - \sum_{\ell=1}^{K} u_\ell$ is non-increasing, the limit in (61) increases when the nodes $z_1, z_2, \ldots, z_K$ move closer to the unit circle. Moreover, (61) equals zero when $|z_1| = |z_2| = \ldots = |z_K| = 1$. Based on (61) Bazán obtained the large–N asymptotes of the lower and upper bounds in (59). Specifically, it is shown in [3, Lem. 8, Cor. 9] that the limit $\kappa_a := \lim_{N\to\infty} \kappa(\mathbf{V}_{N\times K})$ exists, and for $A_{\min} \leqslant A_{\max} < 1$ satisfies

$$\frac{1}{A_{\max}} \leqslant \kappa_a \leqslant \frac{1}{2}\left(\eta_a + \sqrt{\eta_a^2 - 4}\right),$$

where

$$\eta_a := K\left(1 + \frac{1}{\sigma^2} + \frac{\prod_{k=1}^{K} |z_k|^2 - \sum_{k=1}^{K} |z_k|^2}{(K-1)\sigma^2}\right)^{\frac{K-1}{2}} \left(\frac{1 - A_{\min}^2}{1 - A_{\max}^2}\right)^{1/2} - K + 2.$$

In addition, it is proven in [3, Cor. 10] that for $|z_1| = |z_2| = \ldots = |z_K| = A < 1$ and $1 - A^2 \leqslant \sigma^2$, one has

$$\kappa_a \leqslant K 2^{\frac{K-1}{2}} - K + 2. \tag{62}$$

For $A_{\min} = A_{\max} = 1$, it follows from (61) that $\lim_{N\to\infty} D_N^2 = 0$, and hence, by (59), that $\kappa_a \leqslant 1$, which, together with $\kappa_a \geqslant 1$, implies $\kappa_a = 1$.

While these results provide insight into the asymptotic behavior of D_N^2 as $N \to \infty$, an analysis of the speed of convergence of D_N^2 to the right-hand side (RHS) of (61) does not seem to be available in the literature. It therefore appears difficult to draw conclusions about the finite–N behavior of D_N^2. In fact, D_N^2 seems as difficult to characterize, in the finite–N regime, as the condition number itself. Moreover, the numerical evaluation of D_N^2 requires the computation of $\widehat{\boldsymbol{f}}_N$, which in turn requires solving the linear system of equations $(\mathbf{V}_{N\times K})^T \boldsymbol{f} = \boldsymbol{z}_N$. When $\kappa(\mathbf{V}_{N\times K})$ is large, the computation of D_N^2 and therefore the numerical evaluation of Bazán's upper bound can become numerically unstable.

Finally, we compare Bazán's upper bound with our results. We start by noting that, owing to the large constant $42/\pi$ in condition (53) needed for our bound (52) to hold, Bazán's bound is valid for more general node configurations. In particular, since our bound (52) holds only for nodes that are very close to the unit circle, a comparison to Bazán's bound in the general case is not particularly meaningful. For the special case $|z_1| = |z_2| = \ldots = |z_K| = A$, however, our bound (54) is based on the Graham–Vaaler result (36) for which the (non-optimal) constant $42/\pi$ does not appear. Specifically, the asymptote (in N, with K fixed) of our bound (54) satisfies

$$\kappa_a \leqslant A^{1/2-1/\delta^{(w)}}. \tag{63}$$

A general comparison of (62) and (63) is difficult as the two bounds do not depend on the same quantities. We can, however, make specific exemplary statements. For example, for $A \geqslant 0.8$ and $\delta^{(w)} = 1/K$, (63) implies $\kappa_a \leqslant 1.25^{K-1/2}$, which improves upon (62) for $K \geqslant 1$. On the other hand, for $A \geqslant 1/2$ and equally spaced nodes, i.e., $\delta^{(w)} = 1/K$, (63) becomes $\kappa_a \leqslant 2^{K-1/2}$, so that Bazán's bound (62), for $K \geqslant 4$, is better in that case. Detailed numerical comparisons between our bound (54) and Bazán's bound for $|z_1| = |z_2| = \ldots = |z_K| = A$ are provided in the next section.

We finally note that our upper bound on $\kappa(\mathbf{V}_{N\times K})$ is obtained by combining a lower bound on $\sigma^2_{\min}(\mathbf{V}_{N\times K})$ and an upper bound on $\sigma^2_{\max}(\mathbf{V}_{N\times K})$ in (1). Bazán, on the other hand, directly provides a condition number upper bound and does not report individual bounds on $\sigma^2_{\min}(\mathbf{V}_{N\times K})$ and $\sigma^2_{\max}(\mathbf{V}_{N\times K})$.

6. Numerical results

We consider the case $|z_1| = |z_2| = \ldots = |z_K| = A$ and compare our bound (54) to Bazán's bound by averaging over 500 randomly selected node configurations. Specifically, for each $d \in (0, 0.5]$, we construct an $N \times K$ Vandermonde matrix with nodes $z_k := Ae^{2\pi i \xi_k}$, for $k \in \{1, 2, \ldots, K\}$, where $N = 100$, A varies from 0.1 to 1, K is chosen uniformly at random in the set $\{2, 3, \ldots, \lfloor 1/d \rfloor\}$, $\xi_k = k/K + r_k$, and r_k is chosen uniformly at random in the interval $[0, 1/K - d]$. The minimum wrap-around distance $\delta^{(w)}$ between the ξ_k is therefore guaranteed to satisfy $\delta^{(w)} \geqslant d$. The results are depicted in Figures 1 and 2. We observe that for $d \leqslant 0.1$, our bound is much tighter than Bazán's bound. For $d = 0.18$ (Figure 2d), Bazán's bound is tighter than our bound for small values of A. For $d \geqslant 0.2$ (not depicted), Bazán's bound is slightly tighter than our bound, and this for all values of A. However, as $\delta^{(w)} \leqslant 1/K$ and our construction guarantees that $\delta^{(w)} \geqslant d$, $d \geqslant 0.2$ implies $K \leqslant 5$, which is small compared to $N = 100$; the assumption $d \geqslant 0.2$ leading to Bazán's bound being tighter than our bound (54) is therefore quite restrictive. We finally note that the curve corresponding to Bazán's bound in the case $d = 0.05$ (Figure 2a) is wiggly for small values of A. This is because numerical evaluation of Bazán's bound involves solving the linear system $(\mathbf{V}_{N\times K})^T \boldsymbol{f} = \boldsymbol{z}_N$ and $\mathbf{V}_{N\times K}$ is ill-conditioned in these cases, leading to numerical instability.

Appendix A. Proof of Proposition 1

We start by showing ii) $\Rightarrow$ i), which is accomplished using a scaling argument that generalizes the scaling argument in the proof of [51, Thm. 2]. To this end,

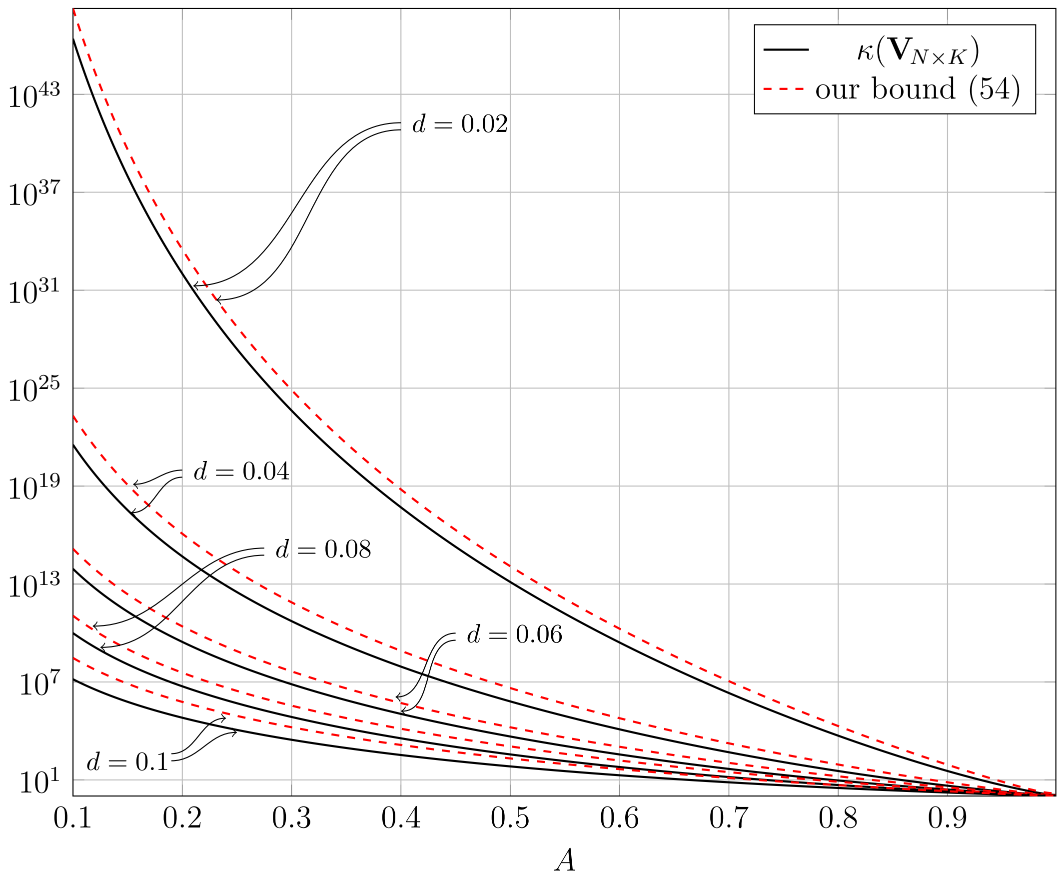
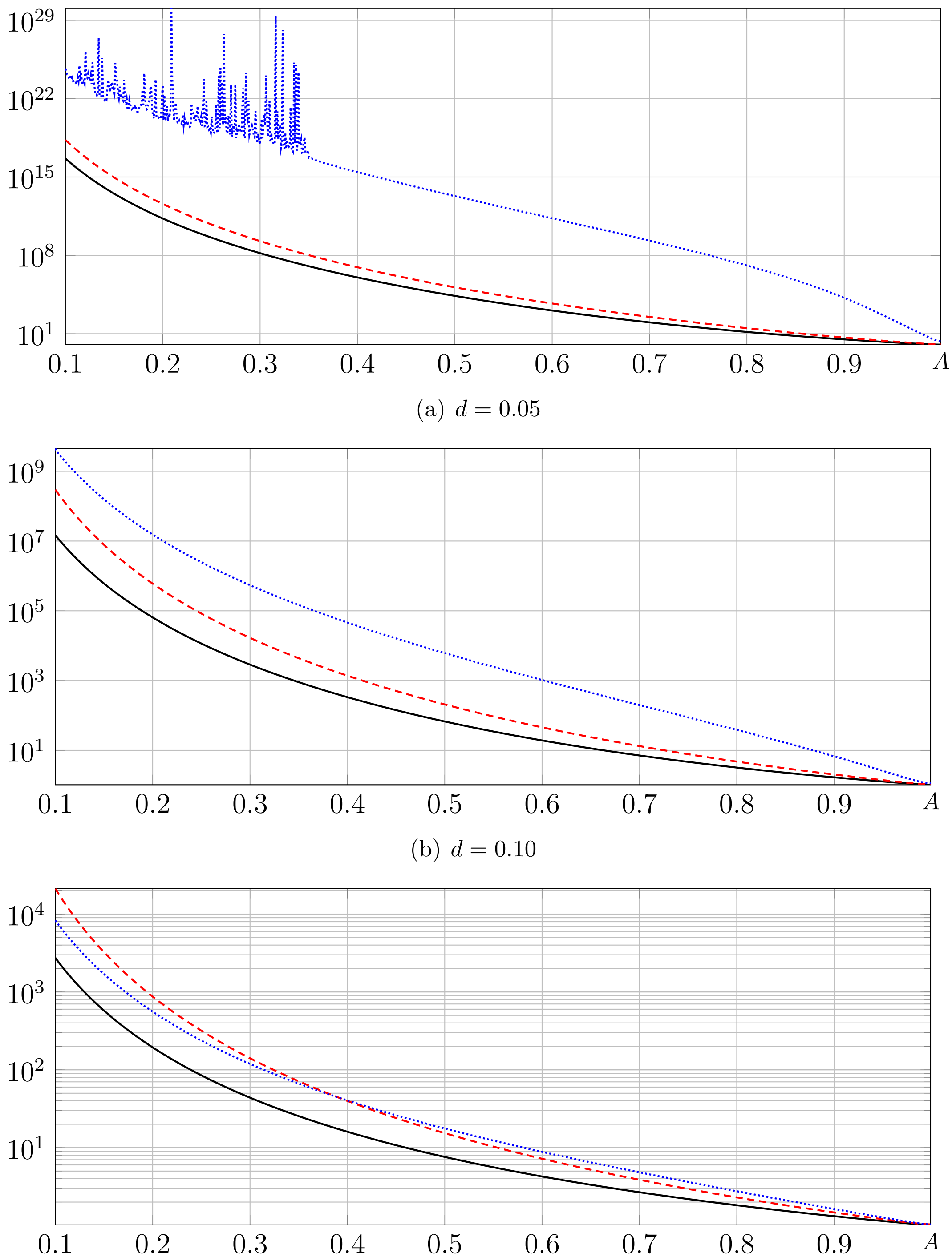


Figure 1: Comparison of our upper bound (54) to the true (average) condition number.



(a) $d = 0.05$

(b) $d = 0.10$

(c) $d = 0.18$

Figure 2: Comparison of our upper bound (54) (dashed) and Bazán's upper bound [3, Thm. 6] (dotted) with the condition number $\kappa(\mathbf{V}_{N\times K})$ (solid).

let $M \in \mathbb{N}\backslash\{0\}$, $\boldsymbol{\alpha} := \{\alpha_k\}_{k=1}^M \in \mathbb{C}^M$, consider $\boldsymbol{\rho} := \{\rho_k\}_{k=1}^M \in \mathbb{C}^M$ with $\rho_k := \lambda_k + 2\pi i u_k$, $\lambda_k > 0$, and $u_k \in \mathbb{R}$ such that

$$\delta_k := \min_{\substack{1 \leqslant \ell \leqslant M \\ \ell \neq k}} |u_k - u_\ell| > 0, \tag{A.1}$$

for all $k \in \{1, 2, \ldots, M\}$, and take ε satisfying

$$0 < \varepsilon < \frac{1}{2 \max\limits_{1 \leqslant k, \ell \leqslant M} |u_k - u_\ell|}. \tag{A.2}$$

We apply ii) with $d_k := \varepsilon\lambda_k$, $\xi_k := \varepsilon u_k$, $r_k := \varepsilon\rho_k = d_k + 2\pi i \xi_k$, and $a_k := \alpha_k$, for all $k \in \{1, 2, \ldots, M\}$. As $\min_{n \in \mathbb{Z}} |\varepsilon(u_k - u_\ell) + n|$ is the distance between $\varepsilon(u_k - u_\ell)$ and its nearest integer and, by (A.2), $0 \leqslant \varepsilon|u_k - u_\ell| < 1/2$, we have $\min_{n \in \mathbb{Z}} |\varepsilon(u_k - u_\ell) + n| = \varepsilon|u_k - u_\ell|$, and hence

$$\delta_k^{(w)} := \min_{\substack{1 \leqslant \ell \leqslant M \\ \ell \neq k}} \min_{n \in \mathbb{Z}} |\varepsilon(u_k - u_\ell) + n| = \varepsilon \min_{\substack{1 \leqslant \ell \leqslant M \\ \ell \neq k}} |u_k - u_\ell| = \varepsilon\delta_k, \tag{A.3}$$

for $k \in \{1, 2, \ldots, M\}$, and

$$\delta^{(w)} = \min_{1 \leqslant k \leqslant M} \delta_k^{(w)} = \min_{1 \leqslant k \leqslant M} \varepsilon\delta_k = \varepsilon\delta.$$

It follows from (A.1), (A.2), and (A.3) that $\delta_k^{(w)} > 0$, for $k \in \{1, 2, \ldots, K\}$. Moreover, as $\lambda_k > 0$, by assumption, it follows from $d_k = \varepsilon\lambda_k$ and (A.2) that $d_k > 0$, for $k \in \{1, 2, \ldots, M\}$. The conditions for application of ii) are therefore met, and (40) yields

$$\sum_{k=1}^M 2\varepsilon A(\varepsilon\lambda_k, \varepsilon\delta_k, \varepsilon\delta)\, |a_k|^2 \leqslant \sum_{k,\ell=1}^M \frac{\varepsilon a_k \overline{a_\ell}}{\sinh(\varepsilon(\rho_k + \overline{\rho_\ell})/2)} \leqslant \sum_{k=1}^M 2\varepsilon B(\varepsilon\lambda_k, \varepsilon\delta_k, \varepsilon\delta)\, |a_k|^2.$$

As $\varepsilon A(\varepsilon\lambda_k, \varepsilon\delta_k, \varepsilon\delta) = A(\lambda_k, \delta_k, \delta)$ and $\varepsilon B(\varepsilon\lambda_k, \varepsilon\delta_k, \varepsilon\delta) = B(\lambda_k, \delta_k, \delta)$, for all $k \in \{1, 2, \ldots, M\}$, both by assumption, we have

$$\sum_{k=1}^M A(\lambda_k, \delta_k, \delta)\, |a_k|^2 \leqslant \sum_{k,\ell=1}^M \frac{\varepsilon a_k \overline{a_\ell}}{2\sinh(\varepsilon(\rho_k + \overline{\rho_\ell})/2)} \leqslant \sum_{k=1}^M B(\lambda_k, \delta_k, \delta)\, |a_k|^2. \tag{A.4}$$

Taking the limit $\varepsilon \to 0$ in (A.4) and noting that $\lim_{z \to 0} \sinh(z)/z = 1$ and hence

$$\frac{1}{\rho_k + \overline{\rho_\ell}} = \lim_{\varepsilon \to 0} \frac{\varepsilon}{2\sinh(\varepsilon(\rho_k + \overline{\rho_\ell})/2)},$$

we get (39).

The proof of i) $\Rightarrow$ ii) is accomplished by generalizing the proof of Theorem 1b) in [55, Prop. LS1.3]. Let $K \in \mathbb{N}\setminus\{0\}$, $\boldsymbol{a} := \{a_k\}_{k=1}^K \in \mathbb{C}^K$, and consider $\boldsymbol{r} := \{r_k\}_{k=1}^K \in \mathbb{C}^K$ with $r_k := d_k + 2\pi i \xi_k$, where $d_k > 0$ and $\xi_k \in \mathbb{R}$ is such that

$$\delta_k^{(w)} := \min_{\substack{1\leqslant \ell \leqslant K\\ \ell \neq k}} \min_{n\in\mathbb{Z}} |\xi_k - \xi_\ell + n| > 0, \tag{A.5}$$

for all $k \in \{1,2,\ldots,K\}$. For $k \in \{1,2,\ldots,K\}$ and $m \in \{1,2,\ldots,N\}$, define $\lambda_{k,m} := d_k$, $u_{k,m} := \xi_k + m$, $\rho_{k,m} := r_k + 2\pi i m = d_k + 2\pi i(\xi_k + m)$, and $\alpha_{k,m} := (-1)^m a_k$, and apply i) with $M = KN$, $\boldsymbol{\rho} := \mathrm{vec}(\mathbf{P}) \in \mathbb{C}^M$, and $\boldsymbol{\alpha} := \mathrm{vec}(\mathbf{A}) \in \mathbb{C}^M$, where

$$\mathbf{P} := \{\rho_{k,m}\}_{\substack{1\leqslant k\leqslant K\\ 1\leqslant m\leqslant N}} \in \mathbb{C}^{K\times N} \qquad \text{and} \qquad \mathbf{A} := \{\alpha_{k,m}\}_{\substack{1\leqslant k\leqslant K\\ 1\leqslant m\leqslant N}} \in \mathbb{C}^{K\times N}.$$

The conditions for application of i) are met, as for all $k \in \{1,2,\ldots,K\}$ and all $m \in \{1,2,\ldots,N\}$,

$$\begin{aligned}\delta_{k,m} := \min_{\substack{1\leqslant \ell\leqslant K\\ 1\leqslant n\leqslant N\\ (\ell,n)\neq(k,m)}} |u_{k,m} - u_{\ell,n}| &= \min_{\substack{1\leqslant \ell\leqslant K\\ 1\leqslant n\leqslant N\\ (\ell,n)\neq(k,m)}} |(\xi_k + m) - (\xi_\ell + n)| \\ &= \min_{\substack{1\leqslant \ell \leqslant K\\ \ell \neq k}} \min_{n\in\mathbb{Z}} |\xi_k - \xi_\ell + n| = \delta_k^{(w)} > 0,\end{aligned}$$

where the inequality follows by (A.5). We therefore get

$$\begin{aligned}N\sum_{k=1}^{K} A(d_k, \delta_k^{(w)}, \delta^{(w)})|a_k|^2 &\leqslant \sum_{k,\ell=1}^{K}\sum_{m,n=1}^{N} \frac{(-1)^{m+n} a_k \overline{a_\ell}}{r_k + \overline{r_\ell} + 2\pi i(m-n)} \\ &\leqslant N\sum_{k=1}^{K} B(d_k, \delta_k^{(w)}, \delta^{(w)})|a_k|^2,\end{aligned} \tag{A.6}$$

where $\delta^{(w)} := \min\limits_{1\leqslant k\leqslant K} \delta_k^{(w)}$. Next, we find an alternative expression for the center term in (A.6). To this end, let $z \in \mathbb{C}\setminus 2\pi i\mathbb{Z}$, and note that

$$\begin{aligned}\sum_{m,n=1}^{N} \frac{(-1)^{m+n}}{z + 2\pi i(m-n)} &= \frac{N}{z} + \sum_{q=1}^{N-1}\sum_{m=1}^{N-q} \frac{(-1)^q}{z - 2\pi i q} + \sum_{q=1}^{N-1}\sum_{m=1}^{N-q} \frac{(-1)^q}{z + 2\pi i q} \\ &= \frac{N}{z} + \sum_{q=1}^{N-1} (N-q)\frac{(-1)^q}{z - 2\pi i q} + \sum_{q=1}^{N-1}(N-q)\frac{(-1)^q}{z + 2\pi i q} \\ &= \frac{N}{z} + \sum_{q=-N}^{-1} (N-|q|)\frac{(-1)^q}{z + 2\pi i q} + \sum_{q=1}^{N}(N-|q|)\frac{(-1)^q}{z + 2\pi i q}\end{aligned}$$

$$= \sum_{q=-N}^{N} (N-|q|)\frac{(-1)^q}{z+2\pi i q}. \tag{A.7}$$

Setting $z = r_k + \overline{r_\ell}$, $k, \ell \in \{1, 2, \ldots, K\}$, in (A.7) to recover the center term in (A.6) yields

$$\sum_{k,\ell=1}^{K} \sum_{m,n=1}^{N} \frac{(-1)^{m+n} a_k \overline{a_\ell}}{r_k + \overline{r_\ell} + 2\pi i(m-n)} = \sum_{k,\ell=1}^{K} \sum_{q=-N}^{N} (N-|q|) \frac{(-1)^q a_k \overline{a_\ell}}{r_k + \overline{r_\ell} + 2\pi i q}.$$

We next establish that

$$\lim_{N\to\infty} \sum_{q=-N}^{N} \left(1 - \frac{|q|}{N}\right) \frac{(-1)^q}{\rho + 2\pi i q} = \frac{1}{2\sinh(\rho/2)}, \tag{A.8}$$

for $\rho \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$. To this end, take $\rho \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$ and let $\varphi \colon \mathbb{R} \to \mathbb{C}$ be the 1-periodic function defined by $\varphi(t) := e^{-\rho t}$, for $t \in [-1/2, 1/2)$. The q-th Fourier series coefficient of φ is

$$\int_{-1/2}^{1/2} \varphi(t) e^{-2\pi i q t} \mathrm{d}t = \int_{-1/2}^{1/2} e^{-(\rho + 2\pi i q)t} \mathrm{d}t = \frac{2\sinh(\rho/2)(-1)^q}{\rho + 2\pi i q}.$$

As φ is continuous on $(-1/2, 1/2)$, according to Fejér's theorem [56, Thm. III.3.4], the sequence $\{\sigma_N\}_{N\in\mathbb{N}}$ of functions $\sigma_N \colon \mathbb{R} \to \mathbb{C}$ defined as

$$\forall t \in \mathbb{R}, \qquad \sigma_N(t) := 2\sinh(\rho/2) \sum_{q=-N}^{N} \left(1 - \frac{|q|}{N}\right) \frac{(-1)^q}{\rho + 2\pi i q} e^{2\pi i q t},$$

converges pointwise to φ on $(-1/2, 1/2)$, that is,

$$e^{-\rho t} = 2\sinh(\rho/2) \lim_{N\to\infty} \sum_{n=-N}^{N} \left(1 - \frac{|q|}{N}\right) \frac{(-1)^q}{\rho + 2\pi i q} e^{2\pi i q t}, \tag{A.9}$$

for $t \in (-1/2, 1/2)$. Evaluating (A.9) at $t = 0$, we obtain (A.8) as desired and hence

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \sum_{k,\ell=1}^{K} \sum_{m,n=1}^{N} \frac{(-1)^{m+n} a_k \overline{a_\ell}}{r_k + \overline{r_\ell} + 2\pi i(m-n)} &= \sum_{k,\ell=1}^{K} \lim_{N\to\infty} \sum_{q=-N}^{N} \left(1 - \frac{|q|}{N}\right) \frac{(-1)^q a_k \overline{a_\ell}}{r_k + \overline{r_\ell} + 2\pi i q} \\ &= \sum_{k,\ell=1}^{K} \frac{a_k \overline{a_\ell}}{2\sinh((r_k + \overline{r_\ell})/2)}. \end{aligned} \tag{A.10}$$

Dividing (A.6) by N and letting $N \to \infty$, it follows from (A.10) that

$$\sum_{k=1}^{K} 2A(d_k, \delta_k^{(w)}, \delta^{(w)})\, |a_k|^2 \leqslant \sum_{k,\ell=1}^{K} \frac{a_k \overline{a_\ell}}{\sinh((r_k + \overline{r_\ell})/2)} \leqslant \sum_{k=1}^{K} 2B(d_k, \delta_k^{(w)}, \delta^{(w)})\, |a_k|^2 \,,$$

which completes the proof.

Appendix B. Proof of Theorem 3

According to Theorem 2, for all $K \in \mathbb{N} \setminus \{0\}$, $\boldsymbol{\alpha} := \{\alpha_k\}_{k=1}^K \in \mathbb{C}^K$, and $\boldsymbol{\rho} := \{\rho_k\}_{k=1}^K \in \mathbb{C}^K$ with $\rho_k := \lambda_k + 2\pi i u_k$, where $\lambda_k > 0$ and $u_k \in \mathbb{R}$ is such that

$$\delta_k := \min_{\substack{1 \leqslant \ell \leqslant K \\ \ell \neq k}} |u_k - u_\ell| > 0,$$

for all $k \in \{1, 2, \ldots, K\}$, (35) holds so that adding $\sum_{k=1}^K |a_k|^2/(2\lambda_k)$ to (35) yields

$$\sum_{k=1}^{K} \left(\frac{1}{2\lambda_k} - \frac{42}{\pi\delta_k} \right) |\alpha_k|^2 \leqslant \sum_{k,\ell=1}^{K} \frac{\alpha_k \overline{\alpha_\ell}}{\rho_k + \overline{\rho_\ell}} \leqslant \sum_{k=1}^{K} \left(\frac{1}{2\lambda_k} + \frac{42}{\pi\delta_k} \right) |\alpha_k|^2 .$$

Application of Proposition 1, with

$$A(x,y,z) = 1/x - 42/(\pi y)$$
$$B(x,y,z) = 1/x + 42/(\pi y),$$

for $x > 0$, $y > 0$, and $z > 0$, to (35) now yields (41). In the case $d_1 = d_2 = \ldots = d_K = d$, (35) is refined to (36). Application of Proposition 1, with

$$A(x,y,z) = 1/(z(e^{2x/z} - 1))$$
$$B(x,y,z) = e^{2x/z}/(z(e^{2x/z} - 1)),$$

for $x > 0$, $y > 0$, and $z > 0$, to (36) then yields (42).

Appendix C. Proof of Theorem 5

The proof strategy is as follows. We first establish a lower bound on $\sigma^2_{\min}(\mathbf{V}_{N\times K})$ and an upper bound on $\sigma^2_{\max}(\mathbf{V}_{N\times K})$, both valid for nodes $z_1, z_2, \ldots, z_K$ strictly inside the unit circle, i.e., $|z_k| < 1$, $k \in \{1, 2, \ldots, K\}$, and then use a limiting argument to extend these bounds, stated in Lemma 7, to the case where one or more of the nodes lie on the unit circle. Finally, we refine the resulting upper bound on $\sigma^2_{\max}(\mathbf{V}_{N\times K})$ through Cohen's dilatation trick.

Lemma 7 (Lower bound on $\sigma^2_{\min}(\mathbf{V}_{N\times K})$ and upper bound on $\sigma^2_{\max}(\mathbf{V}_{N\times K})$ for nodes strictly inside the unit circle)**.** *Let* $\boldsymbol{z} := \{z_k\}_{k=1}^K \in \mathbb{C}^K$ *with* $z_k := |z_k|e^{2\pi i \xi_k}$ *be such that* $0 < |z_k| < 1$*,* $\xi_k \in [0,1)$*, and*

$$\delta_k^{(w)} := \min_{\substack{1\leqslant \ell \leqslant K\\ \ell\neq k}} \min_{n\in\mathbb{Z}} |\xi_k - \xi_\ell + n| > 0, \tag{C.1}$$

for all $k \in \{1, 2, \ldots, K\}$*. The extremal singular values of the Vandermonde matrix* $\mathbf{V}_{N\times K}$ *with nodes* $z_1, z_2, \ldots, z_K$ *satisfy*

$$\sigma^2_{\min}(\mathbf{V}_{N\times K}) \geqslant \min_{1\leqslant k\leqslant K}\left\{\frac{1}{2|z_k|}\left[\frac{1}{-\ln|z_k|}\left(1-|z_k|^{2N}\right) - \frac{84}{\pi\delta_k^{(w)}}\left(1+|z_k|^{2N}\right)\right]\right\} \tag{C.2}$$

$$\sigma^2_{\max}(\mathbf{V}_{N\times K}) \leqslant \max_{1\leqslant k\leqslant K}\left\{\frac{1}{2|z_k|}\left[\frac{1}{-\ln|z_k|}\left(1-|z_k|^{2N}\right) + \frac{84}{\pi\delta_k^{(w)}}\left(1+|z_k|^{2N}\right)\right]\right\}. \tag{C.3}$$

Moreover, if $|z_1| = |z_2| = \ldots = |z_K| = A$*,* (C.2) *and* (C.3) *can be refined to*

$$\sigma^2_{\min}(\mathbf{V}_{N\times K}) \geqslant \frac{1 - A^{2(N+1/2-1/\delta^{(w)})}}{\delta^{(w)}(A^{-2/\delta^{(w)}}-1)A^2} \tag{C.4}$$

and

$$\sigma^2_{\max}(\mathbf{V}_{N\times K}) \leqslant \frac{A^{-2/\delta^{(w)}}\left(1 - A^{2(N-1/2+1/\delta^{(w)})}\right)}{\delta^{(w)}(A^{-2/\delta^{(w)}}-1)A}, \tag{C.5}$$

respectively, with $\delta^{(w)} := \min_{1\leqslant k\leqslant K} \delta_k^{(w)}$*.*

Proof. Let $\boldsymbol{x} := (x_1\ x_2\ \ldots\ x_K)^T \in \mathbb{C}^K$. We have

$$\begin{aligned}
\|\mathbf{V}_{N\times K}\boldsymbol{x}\|_2^2 &= \sum_{n=0}^{N-1}\left|\sum_{k=1}^K x_k z_k^n\right|^2 = \sum_{n=0}^{N-1}\sum_{k,\ell=1}^K \overline{x_k}x_\ell(\overline{z_k}z_\ell)^n \\
&= \sum_{k,\ell=1}^K \overline{x_k}x_\ell \frac{1-(\overline{z_k}z_\ell)^N}{1-\overline{z_k}z_\ell} && \text{(C.6)}\\
&= \sum_{k,\ell=1}^K \overline{x_k}x_\ell(\overline{z}_k z_\ell)^{-1/2}\frac{1-(\overline{z_k}z_\ell)^N}{(\overline{z_k}z_\ell)^{-1/2}-(\overline{z_k}z_\ell)^{1/2}} \\
&= \sum_{k,\ell=1}^K \overline{x_k}x_\ell(\overline{z_k}z_\ell)^{-1/2}\frac{1-(\overline{z_k}z_\ell)^N}{e^{(r_k+\overline{r_\ell})/2}-e^{-(r_k+\overline{r_\ell})/2}} && \text{(C.7)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k,\ell=1}^{K} \overline{x_k} x_\ell (\overline{z_k} z_\ell)^{-1/2} \frac{1-(\overline{z_k} z_\ell)^N}{2\sinh((r_k+\overline{r_\ell})/2)} \\
&= \underbrace{\sum_{k,\ell=1}^{K} \frac{\overline{x_k} x_\ell (\overline{z_k} z_\ell)^{-1/2}}{2\sinh((r_k+\overline{r_\ell})/2)}}_{=:\, X_1} - \underbrace{\sum_{k,\ell=1}^{K} \frac{\overline{x_k} x_\ell (\overline{z_k} z_\ell)^{N-1/2}}{2\sinh((r_k+\overline{r_\ell})/2)}}_{=:\, X_2}, \qquad \text{(C.8)}
\end{aligned}
$$

where in (C.7) we set $r_k := d_k + 2\pi i \xi_k$ with $d_k := -\ln|z_k|$, for $k \in \{1, 2, \dots, K\}$. To get (C.6), we used $|z_k| < 1$, for $k \in \{1, 2, \dots, K\}$, which is by assumption and ensures that $\overline{z_k} z_\ell \neq 1$, for all $k, \ell \in \{1, 2, \dots, K\}$. We proceed to derive lower and upper bounds on the terms X_1 and X_2 in (C.8). To this end, we first note that, by assumption, $0 < |z_k| < 1$, and hence $d_k > 0$, for all $k \in \{1, 2, \dots, K\}$. We can therefore apply (41) in Theorem 3 first with $a_k := \overline{x_k}(\overline{z_k})^{-1/2}$, $k \in \{1, 2, \dots, K\}$, to get

$$
\sum_{k=1}^{K} \left(\frac{1}{-\ln|z_k|} - \frac{84}{\pi \delta_k^{(w)}} \right) \frac{|x_k|^2}{2|z_k|} \leqslant X_1 \leqslant \sum_{k=1}^{K} \left(\frac{1}{-\ln|z_k|} + \frac{84}{\pi \delta_k^{(w)}} \right) \frac{|x_k|^2}{2|z_k|}, \qquad \text{(C.9)}
$$

and then with $a_k := \overline{x_k}(\overline{z_k})^{N-1/2}$, $k \in \{1, 2, \dots, K\}$, to conclude that

$$
\sum_{k=1}^{K} \left(\frac{1}{-\ln|z_k|} - \frac{84}{\pi \delta_k^{(w)}} \right) \frac{|x_k|^2 \, |z_k|^{2N}}{2|z_k|} \leqslant X_2 \leqslant \sum_{k=1}^{K} \left(\frac{1}{-\ln|z_k|} + \frac{84}{\pi \delta_k^{(w)}} \right) \frac{|x_k|^2 \, |z_k|^{2N}}{2|z_k|}. \qquad \text{(C.10)}
$$

With the left-hand side (LHS) of (C.9) and the RHS of (C.10), we get

$$
\begin{aligned}
\|\mathbf{V}_{N\times K}\boldsymbol{x}\|_2^2 &\geqslant \sum_{k=1}^{K} \left[\frac{1}{-\ln|z_k|}\left(1-|z_k|^{2N}\right) - \frac{84}{\pi \delta_k^{(w)}}\left(1+|z_k|^{2N}\right) \right] \frac{|x_k|^2}{2|z_k|} \qquad \text{(C.11)} \\
&\geqslant \min_{1\leqslant k\leqslant K} \left\{ \frac{1}{2|z_k|} \left[\frac{1}{-\ln|z_k|}\left(1-|z_k|^{2N}\right) - \frac{84}{\pi \delta_k^{(w)}}\left(1+|z_k|^{2N}\right) \right] \right\} \|\boldsymbol{x}\|_2^2 ,
\end{aligned}
$$

which implies (C.2). Combining the RHS of (C.9) and the LHS of (C.10), we obtain

$$
\begin{aligned}
\|\mathbf{V}_{N\times K}\boldsymbol{x}\|_2^2 &\leqslant \sum_{k=1}^{K} \left[\frac{1}{-\ln|z_k|}\left(1-|z_k|^{2N}\right) + \frac{84}{\pi \delta_k^{(w)}}\left(1+|z_k|^{2N}\right) \right] \frac{|x_k|^2}{2|z_k|} \\
&\leqslant \max_{1\leqslant k\leqslant K} \left\{ \frac{1}{2|z_k|} \left[\frac{1}{-\ln|z_k|}\left(1-|z_k|^{2N}\right) + \frac{84}{\pi \delta_k^{(w)}}\left(1+|z_k|^{2N}\right) \right] \right\} \|\boldsymbol{x}\|_2^2 ,
\end{aligned}
$$

which proves (C.3). For the refinements (C.4) and (C.5), we derive specialized lower and upper bounds on the terms X_1 and X_2 in (C.8). To this end, we first note that in the case $|z_1| = |z_2| = \ldots = |z_K| = A$, we have $d_1 = d_2 = \ldots = d_K = -\ln(A)$ and hence $r_k = -\ln(A) + 2\pi i \xi_k$, for $k \in \{1, 2, \ldots, K\}$, in X_1 and X_2. The proof of (C.4) and (C.5) is effected by employing (43) in Corollary 4 first with $a_k := \overline{x_k}(\overline{z_k})^{-1/2}$, $k \in \{1, 2, \ldots, K\}$, to get

$$\frac{1}{A\delta^{(w)}(A^{-2/\delta^{(w)}} - 1)} \sum_{k=1}^{K} \frac{|x_k|^2}{A} \leqslant X_1 \leqslant \frac{A^{-2/\delta^{(w)}}}{\delta^{(w)}(A^{-2/\delta^{(w)}} - 1)} \sum_{k=1}^{K} \frac{|x_k|^2}{A}, \tag{C.12}$$

and then with $a_k := \overline{x_k}(\overline{z_k})^{N-1/2}$, $k \in \{1, 2, \ldots, K\}$, to conclude that

$$\frac{1}{A\delta^{(w)}(A^{-2/\delta^{(w)}} - 1)} \sum_{k=1}^{K} \frac{|x_k|^2 A^{2N}}{A} \leqslant X_2 \leqslant \frac{A^{-2/\delta^{(w)}}}{\delta^{(w)}(A^{-2/\delta^{(w)}} - 1)} \sum_{k=1}^{K} \frac{|x_k|^2 A^{2N}}{A}. \tag{C.13}$$

With the LHS of (C.12) and the RHS of (C.13), we have

$$\|\mathbf{V}_{N\times K}\boldsymbol{x}\|_2^2 \geqslant \frac{1 - A^{2(N+1/2-1/\delta^{(w)})}}{\delta^{(w)}(A^{-2/\delta^{(w)}} - 1)A^2} \|\boldsymbol{x}\|_2^2 .$$

Finally, combining the RHS of (C.12) and the LHS of (C.13), we obtain

$$\|\mathbf{V}_{N\times K}\boldsymbol{x}\|_2^2 \leqslant \frac{A^{-2/\delta^{(w)}}\left(1 - A^{2(N-1/2+1/\delta^{(w)})}\right)}{\delta^{(w)}(A^{-2/\delta^{(w)}} - 1)A} \|\boldsymbol{x}\|_2^2 .$$

□

An immediate consequence of Lemma 7 is the following bound on the condition number $\kappa(\mathbf{V}_{N\times K})$.

Lemma 8 (Upper bound on $\kappa(\mathbf{V}_{N\times K})$ for nodes strictly inside the unit circle)**.** *Let $\boldsymbol{z} := \{z_k\}_{k=1}^K \in \mathbb{C}^K$ with $z_k := |z_k|e^{2\pi i \xi_k}$ be such that $0 < |z_k| < 1$, $\xi_k \in [0, 1)$, and*

$$\delta_k^{(w)} := \min_{\substack{1\leqslant \ell \leqslant K \\ \ell \neq k}} \min_{n\in\mathbb{Z}} |\xi_k - \xi_\ell + n| > 0, \tag{C.14}$$

for all $k \in \{1, 2, \ldots, K\}$. The spectral condition number of $\mathbf{V}_{N\times K}$ satisfies

$$\kappa(\mathbf{V}_{N\times K}) \leqslant \left(\frac{\max\limits_{1\leqslant k\leqslant K} \left\{ \frac{1}{|z_k|} \left[\frac{1}{-\ln|z_k|}\left(1 - |z_k|^{2N}\right) + \frac{84}{\pi\delta_k^{(w)}}\left(1 + |z_k|^{2N}\right) \right] \right\}}{\min\limits_{1\leqslant k\leqslant K} \left\{ \frac{1}{|z_k|} \left[\frac{1}{-\ln|z_k|}\left(1 - |z_k|^{2N}\right) - \frac{84}{\pi\delta_k^{(w)}}\left(1 + |z_k|^{2N}\right) \right] \right\}} \right)^{1/2} \tag{C.15}$$

if for all $k \in \{1, 2, \ldots, K\}$,

$$\delta_k^{(w)} > -\ln|z_k| \, \frac{84}{\pi} \left(\frac{1 + |z_k|^{2N}}{1 - |z_k|^{2N}} \right). \tag{C.16}$$

Moreover, if $|z_1| = |z_2| = \ldots = |z_K| = A$, *we have*

$$\kappa(\mathbf{V}_{N\times K}) \leqslant A^{-1/\delta^{(w)}} \sqrt{\frac{A\big(1 - A^{2(N-1/2+1/\delta^{(w)})}\big)}{1 - A^{2(N+1/2-1/\delta^{(w)})}}} \tag{C.17}$$

under the condition $N > 1/\delta^{(w)} - 1/2$.

Proof. Using (C.2) and (C.3) in (1) yields (C.15). Condition (C.16) ensures that the lower bound in (C.2) is strictly positive, which enables division in (C.15). The refinement (C.17) is obtained by using (C.4) and (C.5) in (1). The condition $N > 1/\delta^{(w)} - 1/2$ ensures that the lower bound in (C.4) is strictly positive, which, again, enables division in (C.17). □

We are now ready to prove Theorem 5 proper. This will be accomplished by first showing that (C.2), (C.3), and (C.15) can be extended (through a limiting argument) to the case where one or more of the nodes satisfy $|z_k| = 1$, and second by refining the resulting upper bound on $\sigma^2_{\max}(\mathbf{V}_{N\times K})$ and hence the upper bound on $\kappa(\mathbf{V}_{N\times K})$ via Cohen's dilatation trick.

The basic idea of the proof is to construct a sequence of Vandermonde matrices parametrized by $M \in \mathbb{N}\backslash\{0\}$, with nodes strictly inside the unit circle and approaching the unit circle as $M \to \infty$.

Specifically, let $M \in \mathbb{N}\backslash\{0\}$ and $\mathbf{V}^{(M)}_{N\times K}$ be the Vandermonde matrix with nodes $z_1^{(M)}, z_2^{(M)}, \ldots, z_K^{(M)}$ such that

$$z_k^{(M)} = \begin{cases} |z_k| e^{2\pi i \xi_k}, & |z_k| < 1 \\ \big(1 - \frac{1}{M}\big) e^{2\pi i \xi_k}, & |z_k| = 1. \end{cases}$$

The nodes $z_1^{(M)}, z_2^{(M)}, \ldots, z_K^{(M)}$ are all strictly inside the unit circle, that is, $\left|z_k^{(M)}\right| < 1$, for all $k \in \{1, 2, \ldots, K\}$, and we can therefore apply results from the proof of Lemma 7 to obtain bounds on the extremal singular values of $\mathbf{V}^{(M)}_{N\times K}$. Specifically, let $\boldsymbol{x} := \{x_k\}_{k=1}^K \in \mathbb{C}^K$ and evaluate the lower bound in (C.11) for the nodes $z_1^{(M)}, z_2^{(M)}, \ldots, z_K^{(M)}$ to get

$$\left\|\mathbf{V}^{(M)}_{N\times K}\boldsymbol{x}\right\|_2^2 \geqslant \sum_{k=1}^{K} \left[\frac{1}{-\ln\left|z_k^{(M)}\right|} \left(1 - \left|z_k^{(M)}\right|^{2N}\right) - \frac{84}{\pi\delta_k^{(w)}} \left(1 + \left|z_k^{(M)}\right|^{2N}\right) \right] \frac{|x_k|^2}{2\left|z_k^{(M)}\right|}$$

$$= \sum_{\substack{k=1\\|z_k|=1}}^{K} \left[\frac{1}{-\ln\left(1-\frac{1}{M}\right)}\left(1-\left(1-\frac{1}{M}\right)^{2N}\right) - \frac{84}{\pi\delta_k^{(w)}}\left(1+\left(1-\frac{1}{M}\right)^{2N}\right)\right]\frac{|x_k|^2}{2\left(1-\frac{1}{M}\right)}$$

$$+ \sum_{\substack{k=1\\|z_k|<1}}^{K} \left[\frac{1}{-\ln|z_k|}\left(1-|z_k|^{2N}\right) - \frac{84}{\pi\delta_k^{(w)}}\left(1+|z_k|^{2N}\right)\right]\frac{|x_k|^2}{2|z_k|}.$$

As $\lim_{M\to\infty} z_k^{(M)} = z_k$, for $k \in \{1, 2, \ldots, K\}$, it follows that

$$\begin{aligned}
\|\mathbf{V}_{N\times K}\boldsymbol{x}\|_2^2 &= \lim_{M\to\infty}\left\|\mathbf{V}_{N\times K}^{(M)}\boldsymbol{x}\right\|_2^2 \\
&\geqslant \sum_{\substack{k=1\\|z_k|<1}}^{K}\left[\frac{1}{-\ln|z_k|}\left(1-|z_k|^{2N}\right) - \frac{84}{\pi\delta_k^{(w)}}\left(1+|z_k|^{2N}\right)\right]\frac{|x_k|^2}{2|z_k|} \\
&\qquad + \sum_{\substack{k=1\\|z_k|=1}}^{K}\left(N - \frac{42}{\pi\delta_k}\right)|x_k|^2 \\
&= \sum_{k=1}^{K}\left[\varphi_N(|z_k|) - \frac{42}{\pi\delta_k^{(w)}}\left(1+|z_k|^{2N}\right)\right]\frac{|x_k|^2}{|z_k|} \\
&\geqslant \mathscr{L}(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)})\|\boldsymbol{x}\|_2^2,
\end{aligned}$$

for all $\boldsymbol{x} = \{x_k\}_{k=1}^K \in \mathbb{C}^K$. This implies

$$\sigma_{\min}^2(\mathbf{V}_{N\times K}) \geqslant \mathscr{L}(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)}), \tag{C.18}$$

and thereby establishes (45). We can show similarly that

$$\sigma_{\max}^2(\mathbf{V}_{N\times K}) \leqslant \mathscr{U}(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)}). \tag{C.19}$$

To get (46), we refine (C.19) using Cohen's dilatation trick as follows. Let $\boldsymbol{y} := \{y_n\}_{n=0}^{N-1}$, and set

$$\forall(a, \omega) \in [0, \infty) \times \mathbb{R}, \qquad U_{\boldsymbol{y},N}(a, \xi) := \sum_{n=0}^{N-1} y_n a^n e^{-2\pi i \xi n}. \tag{C.20}$$

It follows from (C.19) that

$$\sum_{k=1}^{K} |U_{\boldsymbol{y},N}(|z_k|, \xi_k)|^2 = \left\|(\mathbf{V}_{N\times K})^H \boldsymbol{y}\right\|_2^2 \leqslant \mathscr{U}(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)})\|\boldsymbol{y}\|_2^2. \tag{C.21}$$

Now, we take $R \in \mathbb{N}\backslash\{0\}$ and apply Cohen's dilatation trick with respect to the variable ξ in (C.20). We start by setting

$$\forall (a,\omega) \in [0,\infty) \times \mathbb{R}, \qquad V_{\boldsymbol{y},N,R}(a,\xi) := U_{\boldsymbol{y},N}(a, R\xi) = \sum_{n=0}^{N-1} y_n a^n e^{-2\pi i R\xi n}.$$

With $\boldsymbol{\gamma} := \{\gamma_n\}_{n=0}^{(N-1)R}$ defined as

$$\gamma_n := \begin{cases} y_{n/R}, & \text{if } n \equiv 0 \pmod{R} \\ 0, & \text{otherwise,} \end{cases}$$

$V_{\boldsymbol{y},N,R}$ can be written as

$$\begin{aligned} \forall (a,\omega) \in [0,\infty) \times \mathbb{R}, \quad V_{\boldsymbol{y},N,R}(a,\xi) &= \sum_{n=0}^{(N-1)R} \gamma_n a^{n/R} e^{-2\pi i \xi n} \\ &= U_{\boldsymbol{\gamma},(N-1)R+1}(a^{1/R}, \xi). \end{aligned} \tag{C.22}$$

We then have

$$\begin{aligned} R\sum_{k=1}^{K} |U_{\boldsymbol{y},N}(|z_k|,\xi_k)|^2 &= \sum_{r=1}^{R}\sum_{k=1}^{K} |U_{\boldsymbol{y},N}(|z_k|,\xi_k + r)|^2 && \text{(C.23)} \\ &= \sum_{r=1}^{R}\sum_{k=1}^{K} \left| V_{\boldsymbol{y},N,R}\left(|z_k|, \frac{\xi_k + r}{R}\right)\right|^2 && \text{(C.24)} \\ &= \sum_{r=1}^{R}\sum_{k=1}^{K} \left| U_{\boldsymbol{\gamma},(N-1)R+1}\left(|z_k|^{1/R}, \frac{\xi_k + r}{R}\right)\right|^2, && \text{(C.25)} \end{aligned}$$

where (C.23) holds as $\xi \mapsto U_{\boldsymbol{y},N}(a,\xi)$ is 1-periodic, (C.24) is by definition of $V_{\boldsymbol{y},N,R}$, and (C.25) follows from (C.22). We have

$$\min_{\substack{1\leqslant \ell \leqslant K \\ 1 \leqslant s \leqslant R \\ (\ell,s)\neq(k,r)}} \min_{n\in\mathbb{Z}} \left| \frac{\xi_k + r}{R} - \frac{\xi_\ell + s}{R} + \frac{n}{R} \right| = \frac{\delta_k^{(w)}}{R} > 0$$

and can therefore apply (C.21) to $U_{\boldsymbol{\gamma},(N-1)R+1}$ with the substitutions

$$\begin{array}{lcl} N & \longleftarrow & (N-1)R+1 \\ \boldsymbol{y} = \{y_n\}_{n=0}^{N-1} & \longleftarrow & \boldsymbol{\gamma} = \{\gamma_n\}_{n=0}^{(N-1)R} \\ \{(|z_k|,\xi_k)\}_{k=1}^{K} & \longleftarrow & \left\{\left(|z_k|^{1/R}, \dfrac{\xi_k+r}{R}\right)\right\}_{\substack{1\leqslant k\leqslant K \\ 1\leqslant r\leqslant R}} \\ |z_k| & \longleftarrow & |z_k|^{1/R} \\ \delta_k^{(w)} & \longleftarrow & \delta_k^{(w)}/R, \end{array}$$

i.e., we apply (C.21) to the 2-D sequence $\left\{\left(|z_k|^{1/R}, \dfrac{\xi_k+r}{R}\right)\right\}_{\substack{1\leqslant k\leqslant K \\ 1\leqslant r\leqslant R}}$ with the corresponding replacements for N, $\boldsymbol{y}$, and $\delta^{(w)}$. This yields

$$\sum_{r=1}^{R}\sum_{k=1}^{K}\left|U_{\boldsymbol{\gamma},(N-1)R+1}\left(|z_k|^{1/R}, \frac{\xi_k+r}{R}\right)\right|^2 \leqslant \mathcal{U}\left((N-1)R+1, |\boldsymbol{z}|^{1/R}, \frac{\boldsymbol{\delta}^{(w)}}{R}\right)\|\boldsymbol{\gamma}\|_2^2, \tag{C.26}$$

where $|\boldsymbol{z}|^{1/R} := \left\{|z_k|^{1/R}\right\}_{k=1}^{K} \in \mathbb{C}^K$. Since $\|\boldsymbol{y}\|_2 = \|\boldsymbol{\gamma}\|_2$, it follows from (C.25) that

$$R\sum_{k=1}^{K}|U_{\boldsymbol{y},N}(|z_k|,\xi_k)|^2 \leqslant \mathcal{U}\left((N-1)R+1, |\boldsymbol{z}|^{1/R}, \frac{\boldsymbol{\delta}^{(w)}}{R}\right)\|\boldsymbol{y}\|_2^2. \tag{C.27}$$

Thanks to (48), we have

$$\frac{1}{R}\mathcal{U}\left((N-1)R+1, |\boldsymbol{z}|^{1/R}, \frac{\boldsymbol{\delta}^{(w)}}{R}\right)$$
$$= \max_{1\leqslant k\leqslant K}\left\{\frac{1}{|z_k|^{1/R}}\left[\frac{\varphi_{(N-1)R+1}\left(|z_k|^{1/R}\right)}{R} + \frac{42}{\pi\delta_k^{(w)}}\left(1+|z_k|^{2(N-1)+2/R}\right)\right]\right\}.$$

Since

$$\frac{\varphi_{(N-1)R+1}\left(|z_k|^{1/R}\right)}{R} = \begin{cases} \dfrac{|z_k|^{2(N-1)+2/R}-1}{2\ln|z_k|}, & |z_k|<1 \\ (N-1)+1/R, & |z_k|=1, \end{cases}$$

we get

$$\lim_{R\to\infty}\frac{\varphi_{(N-1)R+1}\left(|z_k|^{1/R}\right)}{R} = \varphi_{N-1}(|z_k|),$$

and hence

$$\lim_{R\to\infty}\frac{1}{R}\mathcal{U}\left((N-1)R+1,|\boldsymbol{z}|^{1/R},\frac{\boldsymbol{\delta}^{(w)}}{R}\right)=\mathcal{U}(N-1,|\boldsymbol{z}|,\boldsymbol{\delta}^{(w)}).$$

Dividing (C.27) by $R>0$ and letting $R\to\infty$ yields

$$\left\|(\mathbf{V}_{N\times K})^H\boldsymbol{y}\right\|_2^2=\sum_{k=1}^{K}|U_{\boldsymbol{y},N}(|z_k|,\xi_k)|^2\leqslant\mathcal{U}(N-1,|\boldsymbol{z}|,\boldsymbol{\delta}^{(w)})\|\boldsymbol{y}\|_2^2. \qquad \text{(C.28)}$$

Since (C.28) holds for all $\boldsymbol{y}\in\mathbb{C}^N$, this implies

$$\sigma_{\max}^2(\mathbf{V}_{N\times K})=\sigma_{\max}^2((\mathbf{V}_{N\times K})^H)\leqslant\mathcal{U}(N-1,|\boldsymbol{z}|,\boldsymbol{\delta}^{(w)}). \qquad \text{(C.29)}$$

Neither of the upper bounds in (C.19) and (C.29) is consistently smaller than the other one so that in summary

$$\sigma_{\max}^2(\mathbf{V}_{N\times K})\leqslant\min\left\{\mathcal{U}(N,|\boldsymbol{z}|,\boldsymbol{\delta}^{(w)}),\mathcal{U}(N-1,|\boldsymbol{z}|,\boldsymbol{\delta}^{(w)})\right\}. \qquad \text{(C.30)}$$

This concludes the proof of (46).

It remains to establish (51). To this end, we first note that by (C.26)

$$\sum_{r=1}^{R}\sum_{k=1}^{K}\left|U_{\boldsymbol{\gamma},(N-1)R+1}\left(|z_k|^{1/R},\frac{\xi_k+r}{R}\right)\right|^2 \leqslant\frac{RA^{-2/\delta^{(w)}}(1-A^{2(N-1+3/(2R)+1/\delta^{(w)})})}{\delta^{(w)}(A^{-2/\delta^{(w)}}-1)A^{1/R}}\|\boldsymbol{\gamma}\|_2^2.$$

Since $\|\boldsymbol{y}\|_2=\|\boldsymbol{\gamma}\|_2$, it follows from the equality between (C.23) and (C.25) that

$$\sum_{k=1}^{K}|U_{\boldsymbol{y},N}(|z_k|,\xi_k)|^2\leqslant\frac{A^{-2/\delta^{(w)}}(1-A^{2(N-1+3/(2R)+1/\delta^{(w)})})}{\delta^{(w)}(A^{-2/\delta^{(w)}}-1)A^{1/R}}\|\boldsymbol{y}\|_2^2. \qquad \text{(C.31)}$$

Letting $R\to\infty$ in (C.31) yields

$$\left\|(\mathbf{V}_{N\times K})^H\boldsymbol{y}\right\|_2^2=\sum_{k=1}^{K}|U_{\boldsymbol{y},N}(|z_k|,\xi_k)|^2\leqslant\frac{A^{-2/\delta^{(w)}}(1-A^{2(N-1+1/\delta^{(w)})})}{\delta^{(w)}(A^{-2/\delta^{(w)}}-1)}\|\boldsymbol{y}\|_2^2.$$

Since this holds for all $\boldsymbol{y}\in\mathbb{C}^N$, we get

$$\sigma_{\max}^2(\mathbf{V}_{N\times K})=\sigma_{\max}^2((\mathbf{V}_{N\times K})^H)\leqslant\frac{A^{-2/\delta^{(w)}}(1-A^{2(N-1+1/\delta^{(w)})})}{\delta^{(w)}(A^{-2/\delta^{(w)}}-1)}, \qquad \text{(C.32)}$$

which is (51). This completes the proof.

Appendix D. Proof of Corollary 6

Using (C.18) and (C.30) in (1) yields (52). Condition (53) ensures that $\mathscr{L}(N, |\boldsymbol{z}|, \boldsymbol{\delta}^{(w)}) > 0$, which enables division in (52). The refinement (54) is obtained by employing (50) and (C.32) in (1), and the condition $N > 1/\delta^{(w)} - 1/2$ ensures that the lower bound in (50) is positive, which, again, enables division in (54).

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