Impact of Phase Noise on MIMO Channel Measurement Accuracy

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***Abstract*— A widespread design for multiple-input multiple-output (MIMO) radio channel measurement devices (sounders) is based on time-division multiplexed switching of a single transmit/receive radio frequency (RF) chain into the elements of a transmit/receive antenna array. While being cost-effective, such a solution can cause significant measurement errors due to phase noise in the local oscillators. In this paper, we analyze the impact of phase noise on MIMO channel measurement accuracy and show that it can lead to overestimation of channel capacity of up to 100% in practice. Furthermore, we demonstrate that the impact of phase noise is most pronounced if the physical channel has low rank (typical for line-of-sight or low scattering scenarios). The extreme case of a rank-1 physical channel is analyzed in detail.**

I. Introduction

Multiple-input multiple-output (MIMO) wireless communications promises significant improvements over existing systems both in terms of spectral efficiency and link reliability. Accurate measurements of MIMO radio channels are vital for system design, simulation, and performance analysis.

A common and widespread MIMO channel sounder design is based on time-division multiplexing with synchronized switching (TDMS) of a single transmit/receive radio frequency (RF) chain into the individual elements of a transmit/receive antenna array. Such an architecture constitutes a natural extension of single-input single-output (SISO) channel sounders and leads to very cost-effective solutions. An important drawback of this approach, however, results from phase noise related errors in the local oscillators being translated into the spatial domain (through switching of a single transmit or receive RF chain). The resulting measurement errors in terms of estimated channel capacity can be up to around 100% in practice. It is therefore immediately clear that understanding the impact of phase noise on MIMO channel capacity estimation accuracy is of fundamental importance for MIMO channel sounder design.

One may argue that in a wireless communication link the impact of phase noise can simply be absorbed into an effective channel consisting of the physical propagation channel combined with phase noise related impairments. Channel estimation at the receiver for demodulation and decoding or for precoding at the transmitter (through feedback) would then simply work on the combined channel. While this point of view is certainly sensible in a data transmission setup, it does not apply to channel sounding, where it is crucial to separate the physical propagation channel from transmitter/receiver impairments.

Contributions. The goal of this paper is to analyze the impact of phase noise on estimated MIMO channel capacity. Our specific contributions are as follows:

- We present a signal model for correlation-based narrowband MIMO channel sounders taking into account phase noise.
- We identify situations where phase noise has no (or little) impact on MIMO capacity estimates and where it leads to significant capacity estimation errors. In the latter case, we demonstrate that phase noise can turn rank-1 physical channels into full-rank effective channels, and we provide analytical expressions for the resulting capacity estimation error.

Relation to previous work. Ref. [4] analyzes the impact of thermal noise at the receiver and gain imbalance in parallel RF chains on the estimated capacity of a physical rank-1 MIMO channel. In [7], approximations of the MIMO channel capacity estimation error due to additive perturbations of the channel coefficients in an i.i.d. physical channel are derived. For the SISO case, an analysis of the measurement errors due to time-varying physical channels has been reported in [9].

Organization of the paper. The remainder of this paper is organized as follows. In Section II, the architecture of a TDMS-based MIMO channel sounder is introduced and the corresponding channel and signal model are provided. Section III contains a systematic study of the impact of phase noise on MIMO channel capacity estimation errors. We conclude in Section IV.

Notation. $E\{\cdot\}$ denotes the expectation operator. $(f * g)(t)$ stands for the convolution of the functions $f(t)$ and $g(t)$. The Dirac delta function is denoted as $\delta(t)$, and $\delta_i = 1$ for $i = 0$ and 0 otherwise. The superscripts ^{T}, ^{H}, and * denote transposition, conjugate transpose, and elementwise conjugation, respectively. $\mathbf{1}$ and $\mathbf{I}$ stand for an all ones matrix and the identitiy matrix, respectively, of appropriate size. $\mathbf{A} \circ \mathbf{B}$ and $\mathbf{A} \otimes \mathbf{B}$ denote the Hadamard product and the Kronecker product, respectively, of the matrices $\mathbf{A}$ and $\mathbf{B}$, and $f^{\circ}(\mathbf{A})$ stands for the matrix resulting from component-wise application of $f(\cdot)$ to the elements of $\mathbf{A}$. $\|\mathbf{A}\|_{\mathrm{F}}$ is

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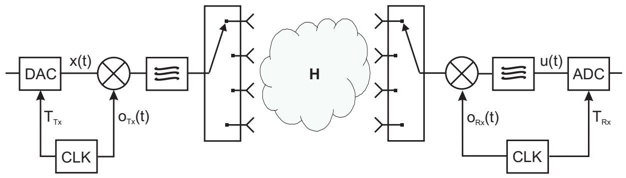


Fig. 1. Architecture of a TDMS-based MIMO channel sounder.

the Frobenius norm of $\mathbf{A}$, $r(\mathbf{A})$ and $\lambda_i(\mathbf{A})$ denote the rank and the ith eigenvalue of $\mathbf{A}$, respectively. For an $m \times n$ matrix $\mathbf{A} = [\,\mathbf{a}_1 \;\mathbf{a}_2 \;\cdots\; \mathbf{a}_n\,]$, we define the $mn \times 1$ vector $\text{vec}(\mathbf{A}) = [\,\mathbf{a}_1^T \;\mathbf{a}_2^T \;\cdots\; \mathbf{a}_n^T\,]^T$. $\mathcal{N}(\mathbf{0}, \mathbf{C})$ and $\mathcal{CN}(\mathbf{0}, \mathbf{C})$ denote a zero-mean real and a zero-mean complex Gaussian random vector, respectively, with covariance matrix $\mathbf{C}$. For two random variables X and Y, $X \sim Y$ and $X \approx Y$ stand for equivalence and approximate equivalence in distribution, respectively.

II. Time-Division Multiplexing MIMO Channel Sounders

In this section, we briefly describe the system architecture of a TDMS MIMO channel sounder and present the corresponding signal model including the effect of phase noise.

A. Channel sounder architecture

The basic architecture of a TDMS MIMO channel sounder is depicted in Fig. 1. A sounding signal $x(t)$ is generated at baseband and mixed to the propagation channel center frequency. At the receiver, downconversion to baseband is followed by extraction of the channel estimate from the signal $u(t)$. Both the transmitter and the receiver employ a multiplexing unit, which is used to step a single RF chain through all transmit/receive antenna elements sequentially in time. Clocks at transmitter and receiver serve as reference for the digital processing rates and RF mixing frequencies.

Another frequently found MIMO channel sounder setup employs a single antenna element (either at the transmitter or the receiver or at both sides of the link) which is physically moved to form a "virtual" array. Such a setup also fits into the framework described in this paper. The impact of phase noise on MIMO channel capacity estimates in a "virtual" array architecture will, however, be much more severe than in the setup shown in Fig. 1. This is due to the fact that the switching time in the "virtual" array case (i.e., the time that passes when moving the antenna element from a given position to the next one) is much longer than with switched antenna elements.

B. SISO signal model

We start by presenting the signal model for a SISO channel in the presence of phase noise. In correlative channel sounders, the sounding signal $x(t)$ is typically the result of an impulse train $\Delta_{T_s}(t) \triangleq \sum_k \delta(t - kT_s)$ passed through a time-invariant transmit filter with impulse response $s(\tau)$. The receiver applies a time-invariant receive filter with impulse response $r(\tau)$, followed by analog-to-digital conversion (ADC). In the following, we assume that the conversion rates of the DAC at the transmitter and the ADC at the receiver are equal. The transmit and receive filters must be designed to satisfy $(s*r)(\tau) \approx \delta(\tau)$. One of the most widely used sounder types, usually referred to as *correlation sounder*, uses the same pseudo-noise sequence (typically consisting of rectangular chip pulses) for $s(\tau)$ and $r(\tau)$. Other sounder types [11] such as the chirp sounder are, however, based on correlation techniques as well and fit into our framework.

Phase noise. Our focus is on the measurement error due to phase noise in the local oscillators (LOs) with output signals[1]

$$o_{\text{Tx}}(t) = e^{j\left(2\pi f_{\text{LO}} t + \varphi_{\text{Tx}}(t)\right)} \quad o_{\text{Rx}}(t) = e^{j\left(2\pi f_{\text{LO}} t + \varphi_{\text{Rx}}(t)\right)}$$

where the subscripts $_{\text{Tx}}$ and $_{\text{Rx}}$ stand for transmitter and receiver, respectively, f_{LO} is the desired LO frequency (assumed equal at transmitter and receiver), and $\varphi_{\text{Tx}}(t)$ and $\varphi_{\text{Rx}}(t)$ represent additive noise on the LO phase. In the following, we define $\theta_{\text{Tx}}(t) = \exp\left(j\varphi_{\text{Tx}}(t)\right)$ and $\theta_{\text{Rx}}(t) = \exp\left(j\varphi_{\text{Rx}}(t)\right)$. In practice, one often employs a linear approximation according to $\exp(j\varphi) \approx 1 + j\varphi$. In the remainder of this paper, we assume that phase noise is modeled as a $\mathcal{N}(0, \sigma_\Phi^2)$ wide-sense stationary random process [16].

Narrowband signal model. Throughout the paper, we assume that the physical channel to be measured is narrowband and static during the snapshot measurement period. Noting that $o_{\text{Tx}}(t)\, o_{\text{Rx}}^*(t) = \exp\left(j\left[\varphi_{\text{Tx}}(t) - \varphi_{\text{Rx}}(t)\right]\right) = \exp\left(j\varphi_{\text{C}}(t)\right)$ with $\varphi_{\text{C}}(t)$ again being a Gaussian wide-sense stationary random process, a simple calculation shows that

$$u(t) = \sum_{k=-\infty}^{\infty} h_k \int_{\Gamma} e^{j\varphi_{\text{C}}(t-\tau)}\, s(t - \tau - kT_s)\, r(\tau)\; d\tau$$

where h_k denotes the complex-valued scalar channel gain during the time period $[kT_s, (k+1)T_s]$ and Γ is the supporting interval of $r(\tau)$. Now, assuming that phase noise changes slowly compared to the length of Γ and using $(s*r)(\tau) = \delta(\tau)$, we obtain

$$u(t) = \sum_{k=-\infty}^{\infty} h_k\, e^{j\varphi_k}\, \delta(t - kT_s).$$

In summary, we can conclude that the phase noise process "modulates" the physical channel h_k, which results in an effective channel $\hat{h}_k$ with a time-varying complex-valued channel gain given by

$$\hat{h}_k = h_k\, e^{j\varphi_k}. \tag{1}$$

C. MIMO signal model

In the MIMO case, assuming M_{T} transmit and M_{R} receive antennas, the effective scalar subchannel between the mth $(m = 1, 2, \ldots, M_{\text{T}})$ transmit and the nth $(n = 1, 2, \ldots, M_{\text{R}})$ receive antenna in the presence of phase noise is given by $\hat{h}_{n,m} = h_{n,m}\, \exp\left(j\varphi_{n,m}\right)$. This relation can be expressed in more compact notation as

$$\hat{\mathbf{H}} = \mathbf{H} \circ \exp^{\circ}(j\,\boldsymbol{\Phi}) = \mathbf{H} \circ \boldsymbol{\Theta} \tag{2}$$

[1]We ignore the impact of frequency offset discussed in detail in [1].

where $[\mathbf{\Phi}]_{n,m} = \varphi_{n,m}$, $\mathbf{H}$ denotes the physical MIMO channel matrix, and $\hat{\mathbf{H}}$ is the effective MIMO channel matrix. The correlation properties of the entries in the phase noise matrix $\mathbf{\Phi}$ are obtained from the power spectral density of the phase noise process and furthermore depend on the antenna switching order as well as the switching time (i.e., the time difference between two specific subchannel measurements). The correlation properties of the multiplicative phase noise matrix $\mathbf{\Theta}$ depend both on the correlation of the entries in $\mathbf{\Phi}$ as well as σ_Φ^2; a detailed discussion can be found in [1]. In the following, we shall often use the linear approximation[2] $\mathbf{\Theta} \approx \mathbf{1} + j\,\mathbf{\Phi}$.

III. Effect of Phase Noise on Estimated Mutual Information

We analyze a MIMO channel with input-output relation

$$\mathbf{r} = \mathbf{H}\mathbf{s} + \mathbf{n}$$

where $\mathbf{s}$ is the $M_\mathrm{T} \times 1$ transmit vector, $\mathbf{r}$ is the $M_\mathrm{R} \times 1$ receive vector, and $\mathbf{n}$ is a $M_\mathrm{R} \times 1$ noise vector distributed as $\mathcal{CN}(\mathbf{0}, \sigma_N^2 \mathbf{I}_{M_\mathrm{R}})$. Assuming no channel state information (CSI) at the transmitter and perfect CSI at the receiver, the mutual information (in bits/s/Hz) of this channel is given by [14]

$$I = \log_2 \det\left(\mathbf{I}_{M_\mathrm{R}} + \frac{\rho}{M_\mathrm{T}} \mathbf{H}\mathbf{H}^H\right) \tag{3}$$

where ρ is the average SNR at each of the receive antennas, and the input signal vector was assumed to be circularly symmetric complex Gaussian with covariance matrix $(\rho/M_\mathrm{T})\mathbf{I}_{M_\mathrm{T}}$.

In this section, we investigate how phase noise impacts the mutual information I, or more specifically, we study how I changes when $\mathbf{H}$ in (3) is replaced by $\hat{\mathbf{H}}$ in (2). We shall consider both deterministic and random physical channels $\mathbf{H}$. For deterministic $\mathbf{H}$, the presence of phase noise induces randomness and hence makes the static channel appear fading. In the case of random $\mathbf{H}$, phase noise alters the statistics of the channel. The quantities we shall be interested in are ergodic capacity[3] $C = E\{I\}$ and the variance $\sigma_I^2 = E\{(I - C)^2\}$ of mutual information. The spread of the distribution of I and hence σ_I^2 quantifies the amount of fading in the channel [15]. In the SIMO/MISO and the low-SNR MIMO case, σ_I^2 is directly related to the amount of fading as introduced in [13]. For a discussion of the significance of σ_I^2 in the high-SNR MIMO case, the reader is referred to [10].

A. Random physical channel

If the entries of the physical channel matrix $\mathbf{H}$ are i.i.d. $\mathcal{CN}(0,1)$, the joint distribution of $\hat{\mathbf{H}}\hat{\mathbf{H}}^H$ equals the joint distribution of $\mathbf{H}\mathbf{H}^H$ and hence phase noise has no impact on the distribution of I. A detailed explanation of this observation can be found in [1], where it is also shown that for correlated Rayleigh fading and for Ricean fading, phase noise can have a significant impact on I. The reason for this behavior lies in the fact that in the correlated Rayleigh fading case phase noise alters the correlation structure of the elements in $\mathbf{H}$; in the Ricean case phase noise additionally "randomizes" the Ricean component and hence leads to a modification of the MIMO channel statistics. In the low-SNR regime, however, it can be shown that phase noise has no impact on I, irrespectively of the channel statistics. This result follows easily by noting that in the low-SNR case

$$I \approx \log_2\left(1 + \frac{\rho}{M_\mathrm{T}} \|\mathbf{H}\|_\mathrm{F}^2\right)$$

and that the pdf of $\|\mathbf{H}\|_\mathrm{F}^2$, and hence the pdf of I, is invariant to multiplicative unit-modulus modifications of the elements of $\mathbf{H}$, even for Ricean fading and for a correlated Rayleigh fading component. In general, we can conclude that the impact of phase noise on mutual information is more pronounced for higher SNR. Finally, we note that a diagonal channel $\mathbf{H}$ is not affected by phase noise, irrespectively of whether the elements on the diagonal of $\mathbf{H}$ are (potentially correlated) Rayleigh or Ricean distributed.

A numerical result concludes our discussion of the random $\mathbf{H}$ case. We examine an 8×8 physical channel characterized by $\mathbf{H} = \mathbf{R}^{1/2}\mathbf{H}_\mathrm{w}$, where $\mathbf{R} = \mathbf{R}^{1/2}\mathbf{R}^{1/2}$ denotes the receive correlation matrix [2] and the entries of $\mathbf{H}_\mathrm{w}$ are i.i.d. $\mathcal{CN}(0,1)$. We choose $\mathbf{R}$ such that $\lambda_i(\mathbf{R}) = M_\mathrm{R}/r(\mathbf{R})$ for $i = 1, 2, \ldots, r(\mathbf{R})$, and $\lambda_i(\mathbf{R}) = 0$ otherwise. Fig. 2 shows the ergodic capacity as a function of SNR for varying rank of $\mathbf{R}$ and varying amount of σ_Φ^2 with uncorrelated phase noise matrix $\mathbf{\Phi}$, i.e., $E\{\mathrm{vec}(\mathbf{\Phi})\mathrm{vec}(\mathbf{\Phi})^T\} = \sigma_\Phi^2 \mathbf{I}_{M_\mathrm{T} M_\mathrm{R}}$. We have chosen $3.5°$ and $7°$ rms phase noise as typical and worst case values, respectively. In practice, phase noise variance and correlation properties depend on LO characteristics, switching speed, and number of transmit and receive antennas. An uncorrelated phase noise matrix $\mathbf{\Phi}$ yields the most severe pertubations and is obtained for very large switching times. In Fig. 2 we can see that phase noise never results in a reduction

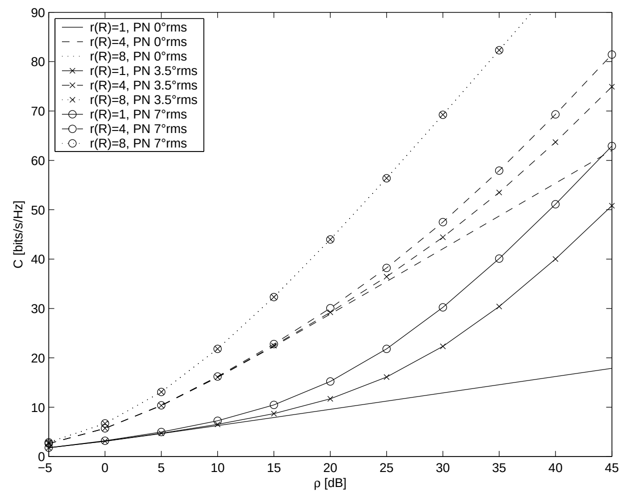


Fig. 2. Ergodic capacity of an 8×8 physical channel subject to varying amounts of phase noise.

[2]This linear approximation does not preserve the Frobenius norm of $\mathbf{\Theta}$. For small σ_Φ^2 though, the increase in $\|\mathbf{\Theta}\|_\mathrm{F}$ is negligible.

[3]We assume that the phase noise process is ergodic.

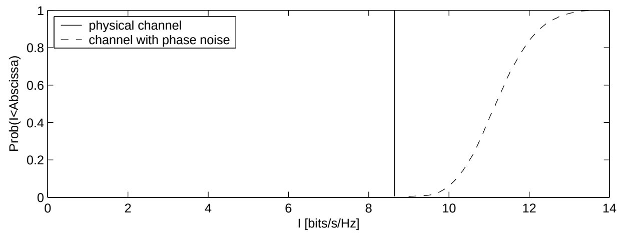


Fig. 3. Cumulative distribution of mutual information at $\rho = 20\,\text{dB}$ of a 4×4 rank-1 physical channel $\mathbf{H} = \mathbf{1}$ subject to 7° rms phase noise.

of ergodic capacity. Moreover, we observe, in agreement with what was said above, that in the full-rank channel case (i.e., $r(\mathbf{R}) = 8$) phase noise has no impact on ergodic capacity. In the $r(\mathbf{R}) = 1$ case, at $\rho = 35\,\text{dB}$, the error in ergodic capacity due to phase noise is more than 100% even for medium phase noise variance. Analytical estimates for the ergodic capacity estimation error due to phase noise in Kronecker-correlated MIMO channels [12] can be found in [1].

B. Deterministic physical channel

When the physical channel is assumed deterministic, the impact of phase noise is a "randomization" of the channel, which leads to a nonzero amount of fading as well as errors in terms of ergodic capacity. The impact of phase noise is more pronounced if the physical MIMO channel has low rank. In the following, we shall therefore analyze the extreme case of a rank-1 physical channel in detail.

In order to motivate our analysis, we start by showing in Fig. 3 the outage capacity for a 4×4 physical channel with $\mathbf{H} = \mathbf{1}$, and uncorrelated phase noise matrix $\mathbf{\Phi}$. We can observe that the presence of phase noise results in a 30% increase in ergodic capacity and $\sigma_I^2 = 0.6$ (note that $\sigma_I^2 = 0$ in the deterministic case).

In the following, for the sake of simplicity, we state all results for $M_\text{R} \leq M_\text{T}$ and furthermore omit most of the proofs. The case of general M_T and M_R as well as the proofs can be found in [1].

Lemma 1. The Hadamard product of a rank-1 matrix $\mathbf{H} = \mathbf{h}_\text{Rx}\mathbf{h}_\text{Tx}^T$ and an arbitrary matrix $\mathbf{\Theta}$ can be written as a matrix product

$$(\mathbf{h}_\text{Rx}\mathbf{h}_\text{Tx}^T) \circ \mathbf{\Theta} = \text{diag}(\mathbf{h}_\text{Rx})\,\mathbf{\Theta}\,\text{diag}(\mathbf{h}_\text{Tx}) \tag{4}$$

where $\text{diag}(\mathbf{x})$ is a diagonal matrix with the elements of $\mathbf{x}$ on its main diagonal.

Proposition 2. For a rank-1 physical channel $\mathbf{H} = \mathbf{h}_\text{Rx}\mathbf{h}_\text{Tx}^T$ with $M_\text{T} = M_\text{R}$ and a multiplicative phase noise matrix $\mathbf{\Theta}$ that has full rank with probability 1, we have

$$\det[\hat{\mathbf{H}}\hat{\mathbf{H}}^H] = \prod_{i=1}^{M_\text{T}} |h_{\text{Tx},i}|^2 \prod_{i=1}^{M_\text{R}} |h_{\text{Rx},i}|^2 \, \det[\mathbf{\Theta}\mathbf{\Theta}^H] \tag{5}$$

where $h_{\text{Tx},i}$ and $h_{\text{Rx},i}$ denote the ith element of h_Tx and h_Rx, respectively.

The proof of Proposition 2 follows in a straightforward fashion from Lemma 1. Proposition 2 says that a physical channel with rank 1 subject to severe phase noise results in a full-rank effective MIMO channel (provided that $h_{\text{Tx},m} \neq 0$ and $h_{\text{Rx},n} \neq 0$ for all m, n). Equivalently, we can conclude that a rank-1 deterministic physical MIMO channel, when subjected to severe phase noise, results in a random channel with full spatial multiplexing gain [10]. A full-rank phase noise matrix $\mathbf{\Phi}$ typically occurs for large M_T and M_R, long sounding sequences, high channel center frequency, or other circumstances that increase or decorrelate the phase noise.

In the case of array steering vectors h_Tx and h_Rx with unit modulus entries representative for line-of-sight propagation, we can refine the result in Proposition 2.

Proposition 3. For a rank-1 physical channel matrix $\mathbf{H} = \mathbf{h}_\text{Rx}\mathbf{h}_\text{Tx}^T$ with $M_\text{R} \leq M_\text{T}$, where $\mathbf{h}_\text{Tx}$ and $\mathbf{h}_\text{Rx}$ are such that $|\mathbf{h}_{\text{Tx},i}| = 1$ ($i = 1, 2, \ldots, M_\text{T}$) and $|\mathbf{h}_{\text{Rx},i}| = 1$ ($i = 1, 2, \ldots, M_\text{R}$), we have

$$\lambda_i(\hat{\mathbf{H}}\hat{\mathbf{H}}^H) = \lambda_i(\mathbf{\Theta}\mathbf{\Theta}^H), \quad i = 1, 2, \ldots, M_\text{R}. \tag{6}$$

Proposition 3 shows that for unit-modulus steering vectors, the rank of $\hat{\mathbf{H}}$ equals the rank of the multiplicative phase noise matrix $\mathbf{\Theta}$ and moreover the eigenvalues of the effective MIMO channel (more specifically of $\hat{\mathbf{H}}\hat{\mathbf{H}}^H$) are exactly equal to the eigenvalues of the multiplicative phase noise matrix.

Proposition 4. For the linear approximation $\mathbf{\Theta} \approx \mathbf{1} + j\mathbf{\Phi}$, assuming that $\text{vec}(\mathbf{\Phi}) \sim \mathcal{N}(\mathbf{0}, \sigma_\Phi^2 \mathbf{I}_{M_\text{T}M_\text{R}})$, we have

$$\det(\mathbf{\Theta}\mathbf{\Theta}^H) \approx \left(\chi^2_{M_\text{T}} + M_\text{T}M_\text{R}\right) \prod_{i=1}^{M_\text{R}-1} \chi^2_{M_\text{T}-i} \tag{7}$$

where χ_n^2 is a chi-square distributed random variable with n degrees of freedom and variance $2n\sigma_\Phi^4$.

We are now ready to state an analytical lower bound on the ergodic capacity of a rank-1 physical channel subject to phase noise.

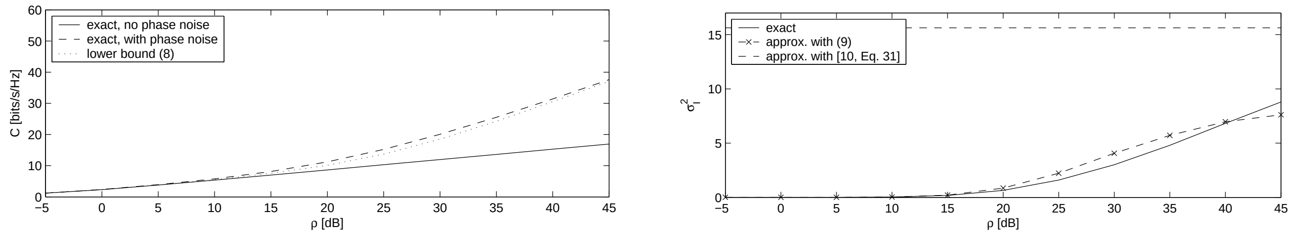
Proposition 5. For the linear approximation $\mathbf{\Theta} \approx \mathbf{1} + j\mathbf{\Phi}$, with $\text{vec}(\mathbf{\Phi}) \sim \mathcal{N}(\mathbf{0}, \sigma_\Phi^2 \mathbf{I}_{M_\text{T}M_\text{R}})$, assuming that the conditions of Proposition 3 are satisfied, the ergodic capacity of the effective channel $\hat{\mathbf{H}}$ satisfies

$$C \geq \sum_{i=0}^{M_\text{R}-1} \log_2\Bigg(1 + \frac{\rho}{M_\text{T}}\Big[M_\text{T}M_\text{R}\delta_i + 2\sigma_\Phi^2 + e^{\psi_0\left(\frac{M_\text{T}-i}{2}\right)}\Big]\Bigg) \tag{8}$$

where $\psi_0(x)$ denotes the digamma function [17].

The digamma function at positive integer multiples of $\frac{1}{2}$ can be expressed as $\psi_0(n) = -\gamma + \sum_{p=1}^{n-1}\frac{1}{p}$, and $\psi_0\big(n - \frac{1}{2}\big) = -\gamma - 2\ln(2) + \sum_{p=1}^{n-1}\frac{2}{2p-1}$, $n = 1, 2, \ldots$, and $\gamma \approx 0.5772$ is Euler's constant. The proof of Proposition 5 is based on a slightly modified version of the lower bound [10, Eq. 14]. Proposition 5 further quantifies the observation discussed as a consequence of Proposition 2, namely that the effective channel corresponding to a rank-1 physical channel subject to severe phase noise exhibits full spatial multiplexing gain.

Impact on the amount of fading. The increase in the amount of fading due to the presence of phase noise is



(a) ergodic capacity

(b) variance of mutual information

Fig. 4. Ergodic capacity and σ_I^2 in a 4×4 rank-1 physical channel $\mathbf{H} = \mathbf{1}$ subject to 7° rms phase noise.

quantified through the increase in the variance of I. Finding exact expressions for σ_I^2 seems very difficult. We can, however, state

Proposition 6. Based on the same assumptions as in Proposition 5, the variance of mutual information of the effective channel $\hat{\mathbf{H}}$ can be approximated as

$$\sigma_I^2 \approx \sum_{i=1}^{M_\mathrm{R}-1} Var\left\{\log_2\left(1 + \frac{\rho}{M_\mathrm{T}}\ \chi^2_{2\lceil (M_\mathrm{T}-i)/2\rceil}\right)\right\}. \qquad (9)$$

Note that the approximation (9) considers only even order terms and the contribution of the highest order chi-square random variable (first term in the product on the RHS of (7)) has been omitted (its influence on the variance can be shown to be negligible [1]). This allows the variance to be evaluated in terms of first and second moments of mutual information of an i.i.d. complex Gaussian physical MIMO channel which were reported, for example, in [6].

Numerical Results. Fig. 4 provides numerical results assessing the quality of the bounds and approximations provided above. For a 4×4 physical channel with $\mathbf{H} = \mathbf{1}$ and uncorrelated phase noise matrix $\mathbf{\Phi}$, Fig. 4 a) shows that the ergodic capacity of the effective channel starts deviating from the capacity of the rank-1 physical MIMO channel at $\rho \approx 15\,\mathrm{dB}$, and that significant estimation errors (up to around 100%) occur in the high-SNR regime. This behavior is consistent with our observation that the low-SNR capacity is not influenced by phase noise. The impact of phase noise on σ_I^2 is quantified in Fig. 4 b) (recall that $\sigma_I^2 = 0$ for a deterministic channel). We can see that the empirical variance obtained through Monte-Carlo methods is reasonably well approximated using (9). The approximation is more accurate for lower SNR.

IV. Conclusion

We showed that phase noise in time-division multiplexed MIMO radio channel sounding systems can lead to significant estimation errors in terms of MIMO channel capacity. The impact of phase noise is most pronounced for low-rank physical channels. An analysis of the rank-1 physical MIMO channel revealed that phase noise can lead to a complete decorrelation of the channel matrix and yield a full-rank effective channel. In the light of the results presented in this paper, MIMO channel measurement campaigns using time-division multiplexing architectures or "virtual array" setups and reporting high-rank MIMO channels and the absence of rank-1 (a.k.a. pin-hole or key-hole channels [3], [5]) should be interpreted with care.

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