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Diagonalizing the Gabor Frame Operator*

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Abstract — The Gabor expansion is a signal decomposition into time-frequency shifted versions of a window function. Computation of the expansion coefficients requires a "dual" window. This paper discusses fast algorithms for calculating the dual window. We consider situations where the Gabor frame operator can be expressed—either directly or in a transform domain—as a multiplication operator, and hence the dual window can be calculated by pointwise division. In the cases of critical sampling and integer oversampling, the Zak transform allows to do this independently of the original window. In the general case (including the case of rational oversampling), one has to make restrictions about the window's temporal or spectral support. We furthermore obtain expressions for the eigenfunctions, eigenvalues, and frame bounds of the Gabor frame operator, and we derive an efficient algorithm for the construction of tight Gabor-type (Weyl-Heisenberg) frames.

1 Introduction

The *Gabor expansion* [1]-[4] is a decomposition of a signal into time-frequency (TF) shifted versions of an elementary window function. Since the functions into which the signal is expanded do not generally form an orthonormal basis, the question of how to obtain the expansion coefficients (Gabor coefficients) and related questions concerning the existence and uniqueness of the Gabor expansion are nontrivial. The theory of *Weyl-Heisenberg frames* (WHFs) [5]-[8] yields important results on these questions. Frame theory tells us that the calculation of the Gabor coefficients can be based on the so-called *dual window*. The dual window is derived from the original window via the inversion of a linear operator (the *frame operator*), which is a computationally intensive task in general. Therefore, in the last few years there has been growing interest in fast algorithms for the computation of the dual Gabor window [4],[9]-[12].

This paper studies situations where the (discrete-time) Gabor frame operator can be "diagonalized," i.e., expressed as a multiplication operator, so that the frame operator's inversion reduces to a simple pointwise division. We show that two cases have to be distinguished:

- In the cases of critical sampling and integer oversampling, the Zak transform [13]-[15] allows to express the Gabor frame operator as a multiplication operator, independently of the particular Gabor window used.
- In the general case (including the case of rational oversampling), the Gabor frame operator is a multiplication operator if certain restrictions on the temporal or spectral support of the Gabor window are satisfied.

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The paper is organized as follows. In Section 2, we briefly review the Gabor expansion and the theory of WHFs. Section 3 considers the diagonalization of the Gabor frame operator in the cases of critical sampling and integer oversampling. We give explicit expressions for the eigenfunctions and eigenvalues of the frame operator, and we discuss the efficient construction of tight WHFs. Section 4 considers the general case (including rational oversampling) and shows that the Gabor frame operator is a multiplication operator if support restrictions on the Gabor synthesis window are satisfied. Also for this case, the eigenfunctions and eigenvalues are calculated and the efficient construction of tight WHFs is discussed. In Section 5, we briefly address the implementation of the proposed algorithms and present simulation results.

2 Gabor Expansion and Weyl-Heisenberg Frames

2.1 Discrete-Time Gabor Expansion

The *discrete-time Gabor expansion* of a signal $x[n] \in l^2(\mathbb{Z})$ is defined as[1]

$$x[n] = \sum_{l=-\infty}^{\infty} \sum_{m=0}^{M-1} a_{l,m}\, g_{l,m}[n] \qquad \text{with} \quad g_{l,m}[n] = g[n-lL]\, e^{j2\pi \frac{m}{M} n}, \tag{1}$$

where $a_{l,m}$ are the Gabor coefficients, $g[n]$ is a "synthesis window," and the parameters $L, M \in \mathbb{N}$ are the grid constants. The Gabor coefficients can be calculated as

$$a_{l,m} = \langle x, \gamma_{l,m} \rangle = \sum_{n=-\infty}^{\infty} x[n]\, \gamma_{l,m}^*[n] \qquad \text{with} \quad \gamma_{l,m}[n] = \gamma[n-lL]\, e^{j2\pi \frac{m}{M} n} \tag{2}$$

with an "analysis window" $\gamma[n]$. In the cases of *oversampling* ($M > L$) and *critical sampling* ($M = L$), the Gabor expansion (1), (2) exists for arbitrary $x[n] \in l^2(\mathbb{Z})$ if the windows $g[n]$ and $\gamma[n]$ are chosen properly (see below). Oversampling yields better numerical stability at the cost of redundant and non-unique Gabor coefficients. In the case of *undersampling* ($M < L$), the Gabor expansion will not exist for arbitrary signals $x[n] \in l^2(\mathbb{Z})$.

2.2 Weyl-Heisenberg Frames

The theory of *Weyl-Heisenberg frames* (WHFs) [5]-[8] yields important results about the Gabor expansion. For $M \geq L$, a set of functions $g_{l,m}[n] = g[n-lL]\, e^{j2\pi \frac{m}{M} n}$ with $-\infty < l < \infty$ and $0 \leq m \leq M-1$ is said to be a WHF for $l^2(\mathbb{Z})$ if for all $x[n] \in l^2(\mathbb{Z})$

$$A\, \|x\|^2 \leq \sum_{l=-\infty}^{\infty} \sum_{m=0}^{M-1} |\langle x, g_{l,m} \rangle|^2 \leq B\, \|x\|^2 \qquad \text{with} \quad 0 < A \leq B < \infty\,, \tag{3}$$

where $\|x\|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$. The constants $A > 0$ and $B < \infty$ are called *frame bounds*. For synthesis window $g[n]$ such that $\{g_{l,m}[n]\}$ is a WHF, the Gabor expansion (1), (2) exists for all $x[n] \in l^2(\mathbb{Z})$, and an analysis window (or "dual" window) can be derived from $g[n]$ as

$$\gamma[n] = (\mathbf{S}^{-1} g)[n]\,. \tag{4}$$

Here, $\mathbf{S}^{-1}$ is the inverse of the *frame operator* $\mathbf{S}$ defined as

$$(\mathbf{S}x)[n] = \sum_{l=-\infty}^{\infty} \sum_{m=0}^{M-1} \langle x, g_{l,m} \rangle\, g_{l,m}[n]\,.$$

The frame operator is a linear, positive definite operator (corresponding to a matrix of infinite size) mapping $l^2(\mathbb{Z})$ onto $l^2(\mathbb{Z})$.

[1] $l^2(\mathbb{Z})$ denotes the space of square-summable discrete-time signals, i.e., $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$ for $x[n] \in l^2(\mathbb{Z})$.

If $\{g_{l,m}[n]\}$ is a WHF, then $\{\gamma_{l,m}[n]\}$ is a WHF as well (the "dual" frame). The frame bounds of $\{\gamma_{l,m}[n]\}$ are $A' = 1/B$ and $B' = 1/A$ where A, B are the frame bounds of $\{g_{l,m}[n]\}$ [6]-[8]. The numerical properties of the Gabor expansion will be better for closer frame bounds A and B. A WHF is called *snug* if $A \approx B$ and *tight* if $A = B$. For a tight WHF, $\mathbf{S} = A\mathbf{I}$ where $\mathbf{I}$ is the identity operator on $l^2(\mathbb{Z})$, and hence there is simply $\gamma[n] = \frac{1}{A}\, g[n]$.

In the case of critical sampling, a WHF $\{g_{l,m}[n]\}$ is known to be an *exact* frame, which means that the $g_{l,m}[n]$ are linearly independent [6]-[8]. As a consequence, both the analysis window $\gamma[n]$ and the Gabor coefficients $a_{l,m}$ are uniquely determined.

3 Diagonalization of the Gabor Frame Operator using the Zak Transform

In general, the operator inversion required for calculating the dual window $\gamma[n]$ according to (4) is computationally intensive. Therefore, it is of interest to identify situations where the Gabor frame operator reduces to, or can be converted to, simple pointwise multiplication. In this section, we consider the cases of *critical sampling* and *integer oversampling*, i.e., $M = KL \geq L$ with the *oversampling factor* $K = M/L \in \mathbb{N}$ (note that $K = 1$ for critical sampling). We shall see that the *Zak transform*, to be reviewed below, here converts the frame operator to a multiplication operator or, mathematically speaking, it "diagonalizes" the frame operator.

3.1 Discrete-Time Zak Transform

For any $L \in \mathbb{N}$ one can define the *discrete-time Zak transform* (DTZT) (see [14])

$$\mathcal{Z}_x(n, \theta) = \sum_{l=-\infty}^{\infty} x[n + lL]\, e^{-j2\pi l\theta}\,.$$

In the sequel we assume that the parameter L is equal to the time-shift parameter L used in the Gabor expansion (1), (2). The DTZT is quasiperiodic in n and periodic in θ, $\mathcal{Z}_x(n + lL, \theta) = e^{j2\pi l\theta}\, \mathcal{Z}_x(n, \theta)$ and $\mathcal{Z}_x(n, \theta + l) = \mathcal{Z}_x(n, \theta)$ with $l \in \mathbb{Z}$. Hence, it suffices to calculate the DTZT on the "fundamental rectangle" $(n, \theta) \in [0, L-1] \times [0, 1)$. The signal $x[n]$ can be recovered from its DTZT as

$$x[n] = \int_0^1 \mathcal{Z}_x(n, \theta)\, d\theta\,. \tag{5}$$

The DTZT is a unitary mapping and therefore preserves inner products and norms, $\langle x, y\rangle = \langle \mathcal{Z}_x, \mathcal{Z}_y\rangle = \sum_{n=0}^{L-1} \int_0^1 \mathcal{Z}_x(n, \theta)\, \mathcal{Z}_y^*(n, \theta)\, d\theta$ and $\|x\|^2 = \|\mathcal{Z}_x\|^2 = \sum_{n=0}^{L-1} \int_0^1 |\mathcal{Z}_x(n, \theta)|^2 d\theta$. In practical implementations, the DTZT is sampled with respect to the normalized frequency variable θ [15]. The sampled version of the DTZT can be expressed as a discrete Fourier transform, which can be implemented efficiently using FFT algorithms [3, 15].

3.2 Eigenfunctions and Eigenvalues of the Gabor Frame Operator

For critical sampling and integer oversampling, the DTZT is intimately related to the eigenfunctions and eigenvalues of the Gabor frame operator $\mathbf{S}$. A set of eigenfunctions of $\mathbf{S}$ is

$$u_{n,\theta}[n'] = u[n' - n]\, e^{j2\pi \frac{\theta}{L}(n' - n)} \qquad \text{with} \quad u[n] = \sum_{l=-\infty}^{\infty} \delta[n - lL]. \tag{6}$$

Note that the eigenfunctions are parameterized by time n and normalized frequency θ. The DTZT of a signal $x[n]$ can now be written as

$$\mathcal{Z}_x(n, \theta) = \langle x, u_{n,\theta}\rangle. \tag{7}$$

Inserting (6) into the eigenequation [16] $(\mathbf{S}u_{n,\theta})[n'] = \lambda_{n,\theta}\, u_{n,\theta}[n']$ yields the eigenvalues $\lambda_{n,\theta} = L \sum_{k=0}^{K-1} \left| \mathcal{Z}_g\left(n, \theta - \frac{k}{K}\right)\right|^2$. We can see that $\mathcal{Z}_g(n,\theta)$, the DTZT of the Gabor synthesis window $g[n]$, determines the eigenvalues of the Gabor frame operator. The dependence of $\lambda_{n,\theta}$ on $g[n]$ will be emphasized in what follows by writing the eigenvalues as $\lambda_g(n,\theta)$, so that

$$\lambda_g(n,\theta) = L \sum_{k=0}^{K-1} \left| \mathcal{Z}_g\left(n, \theta - \frac{k}{K}\right)\right|^2 .$$

We emphasize that the eigenvalues $\lambda_g(n,\theta)$ (suitably discretized with respect to θ) can be computed efficiently using FFT techniques. In the case of critical sampling where $K = 1$, the eigenvalues reduce to

$$\lambda_g(n,\theta) = L \, |\mathcal{Z}_g(n,\theta)|^2 .$$

We note that a second set of eigenfunctions of $\mathbf{S}$ is $\tilde{u}_{n,\theta}[n'] = \tilde{u}[n'-n]\, e^{j2\pi \frac{\theta}{L}(n'-n)}$ with $\tilde{u}[n] = \sum_{m=-\infty}^{\infty} \delta[n - mM]$. These eigenfunctions lead to the "dual DTZT"

$$\tilde{\mathcal{Z}}_x(n,\theta) \;\stackrel{\triangle}{=}\; \langle x, \tilde{u}_{n,\theta}\rangle = \sum_{l=-\infty}^{\infty} x[n + lM]\, e^{-j2\pi l K \theta} ,$$

which also diagonalizes the Gabor frame operator $\mathbf{S}$. The corresponding eigenvalues are as before, $\tilde{\lambda}_g(n,\theta) = \lambda_g(n,\theta)$; they can be expressed in terms of the dual DTZT of $g[n]$ as $\lambda_g(n,\theta) = M \sum_{k=0}^{K-1} |\tilde{\mathcal{Z}}_g(n + kL, \theta)|^2$. In the case of critical sampling, $\tilde{u}[n] = u[n]$ and $\tilde{\mathcal{Z}}_x(n,\theta) = \mathcal{Z}_x(n,\theta)$.

3.3 Diagonalization of the Gabor Frame Operator

We are now ready to show that the Gabor frame operator can be diagonalized using the DTZT. We start from the eigenequation

$$(\mathbf{S}u_{n,\theta})[n'] = \lambda_g(n,\theta)\, u_{n,\theta}[n'] . \tag{8}$$

Taking the inner product of some signal $x[n] \in l^2(\mathbb{Z})$ with both sides of (8), we obtain $\langle x, \mathbf{S}u_{n,\theta}\rangle = \lambda_g^*(n,\theta)\langle x, u_{n,\theta}\rangle$ and further $\langle \mathbf{S}x, u_{n,\theta}\rangle = \lambda_g(n,\theta)\langle x, u_{n,\theta}\rangle$, where we have used the self-adjointness of the frame operator $\mathbf{S}$ (i.e., $\mathbf{S}^* = \mathbf{S}$ and $\lambda_g^*(n,\theta) = \lambda_g(n,\theta)$ where $\mathbf{S}^*$ denotes the adjoint of $\mathbf{S}$ [16]). Using (7), this can be rewritten as

$$\mathcal{Z}_{\mathbf{S}x}(n,\theta) = \lambda_g(n,\theta)\, \mathcal{Z}_x(n,\theta). \tag{9}$$

This shows that the DTZT allows to express $\mathbf{S}$ as a pointwise multiplication in the DTZT domain. In a similar manner, the inverse frame operator $\mathbf{S}^{-1}$ can be shown to become a pointwise division in the DTZT domain,

$$\mathcal{Z}_{\mathbf{S}^{-1}x}(n,\theta) = \frac{\mathcal{Z}_x(n,\theta)}{\lambda_g(n,\theta)} . \tag{10}$$

We shall now summarize two important consequences of (9) and (10).

- The frame condition (3) can be reformulated, using the DTZT of the synthesis window $g[n]$, as

  $$A \,\le\, \lambda_g(n,\theta) \,\le\, B . \tag{11}$$

  In particular, $\{g_{l,m}[n]\}$ is a tight WHF with frame bounds $A = B$ if and only if

  $$\lambda_g(n,\theta) \equiv A .$$

- With (4) and (10), the analysis window $\gamma[n]$ can be computed via the DTZT using

$$\mathcal{Z}_\gamma(n,\theta) = \frac{\mathcal{Z}_g(n,\theta)}{\lambda_g(n,\theta)} \tag{12}$$

and deriving $\gamma[n]$ according to (5), i.e., $\gamma[n] = \int_0^1 \mathcal{Z}_\gamma(n,\theta)\, d\theta$.

We furthermore note that the frame operator corresponding to the dual frame $\{\gamma_{l,m}[n]\}$ is $\mathbf{S}^{-1}$, and its eigenvalues are $\lambda_\gamma(n,\theta) = L\sum_{k=0}^{K-1}\left|\mathcal{Z}_\gamma\left(n,\theta-\frac{k}{K}\right)\right|^2 = 1/\lambda_g(n,\theta)$. Hence, the DTZT of the synthesis window $g[n]$ can be obtained from the DTZT of the analysis window $\gamma[n]$ as

$$\mathcal{Z}_g(n,\theta) = \frac{\mathcal{Z}_\gamma(n,\theta)}{\lambda_\gamma(n,\theta)}\,.$$

Using a different approach, equations analogous to (9)-(12) have been found for the continuous-time case in [3]. In the case of critical sampling ($K=1$), the above relations simplify to

$$A \,\le\, L\,|\mathcal{Z}_g(n,\theta)|^2 \,\le\, B$$

and

$$\mathcal{Z}_\gamma(n,\theta) = \frac{1}{L\mathcal{Z}_g^*(n,\theta)}\,, \qquad \mathcal{Z}_g(n,\theta) = \frac{1}{L\mathcal{Z}_\gamma^*(n,\theta)}\,.$$

3.4 Frame Bounds

The frame bounds in (3) are important since they characterize the numerical properties of the Gabor expansion [6]. From (11) it follows that the frame bounds A and B are given by the infimum and supremum, respectively, of the eigenvalues $\lambda_g(n,\theta) = L\sum_{k=0}^{K-1}\left|\mathcal{Z}_g\left(n,\theta-\frac{k}{K}\right)\right|^2$,

$$A = \inf\{\lambda_g(n,\theta)\}\,, \qquad B = \sup\{\lambda_g(n,\theta)\}\,.$$

3.5 Construction of Tight Weyl-Heisenberg Frames

Next we describe the construction of a tight WHF in the case of critical sampling or integer oversampling. The procedure outlined below extends the procedure in [6] to the case of integer oversampling.

If $\{g_{l,m}[n]\}$ is a WHF for $l^2(\mathbb{Z})$, then it can be shown that $\{h_{l,m}[n]\}$ with $h[n] = (\mathbf{S}^{-1/2}g)[n]$ is a *tight* WHF for $l^2(\mathbb{Z})$ with frame bound $A=1$. It can furthermore be shown that $\mathbf{S}^{-1/2}$ corresponds to a pointwise division by $\sqrt{\lambda_g(n,\theta)}$ in the DTZT domain, i.e.

$$\mathcal{Z}_{\mathbf{S}^{-1/2}x}(n,\theta) = \frac{\mathcal{Z}_x(n,\theta)}{\sqrt{\lambda_g(n,\theta)}}\,.$$

Hence, if a synthesis window $g[n]$ gives rise to a WHF, then a synthesis window $h[n]$ corresponding to a *tight* WHF with frame bound $A=1$ can be constructed by calculating

$$\mathcal{Z}_h(n,\theta) = \frac{\mathcal{Z}_g(n,\theta)}{\sqrt{\lambda_g(n,\theta)}} \tag{13}$$

and $h[n] = \int_0^1 \mathcal{Z}_h(n,\theta)\, d\theta$. Since this procedure can be implemented using the FFT, it is much more efficient than taking the inverse square root of a general matrix.

In the case of an *exact* WHF $\{g_{l,m}[n]\}$, the set $\{h_{l,m}[n]\}$ with $h[n] = (\mathbf{S}^{-1/2}g)[n]$ is an orthonormal basis for $l^2(\mathbb{Z})$ [8]. Since a WHF with critical sampling is exact, the above procedure applied to a WHF with critical sampling will result in an orthonormal basis for $l^2(\mathbb{Z})$. In the case of critical sampling, (13) simplifies to

$$\mathcal{Z}_h(n,\theta) = \frac{\mathcal{Z}_g(n,\theta)}{\sqrt{L}\,|\mathcal{Z}_g(n,\theta)|} = \frac{1}{\sqrt{L}}\,e^{j\arg\{\mathcal{Z}_g(n,\theta)\}}\,.$$

4 Diagonalization of the Gabor Frame Operator by Imposing Support Restrictions on the Synthesis Window

In the general case (specifically, the case of rational oversampling where $\frac{M}{L} = \frac{q}{p} > 1$ is a rational number larger than 1), the DTZT does not diagonalize the Gabor frame operator for general Gabor windows. However, the Gabor frame operator itself will reduce to a simple multiplication operator (without any intermediate transformation) if certain support restrictions on the Gabor synthesis window $g[n]$ or its Fourier transform are imposed [5], [9]-[11].

The starting point for this type of diagonalization is the following general expression for the Gabor frame operator, derived in [3] for the continuous-time case:

$$(\mathbf{S}x)[n] = \sum_{m=-\infty}^{\infty} x[n-mM] \left[M \sum_{l=-\infty}^{\infty} g[n-lL]\, g^*[n-lL-mM] \right].$$

If the support of the synthesis window $g[n]$ is restricted to an interval of length $\leq M$, this expression simplifies to

$$(\mathbf{S}x)[n] = \lambda_g[n]\, x[n] \qquad \text{with} \quad \lambda_g[n] = M \sum_{l=-\infty}^{\infty} |g[n-lL]|^2.$$

As our notation indicates, the $\lambda_g[n]$ are in fact the eigenvalues of the frame operator $\mathbf{S}$ in the case that $g[n]$ has length $\leq M$; the corresponding eigenfunctions are $u_n[n'] = \delta[n'-n]$ with $0 \leq n \leq M-1$. The following conclusions can now be shown to hold:

- The frame condition can be rewritten as $A \leq \lambda_g[n] \leq B$.
- The set $\{g_{l,m}[n]\}$ is a tight WHF for $l^2(\mathbb{Z})$ with frame bound A if and only if $\lambda_g[n] \equiv A$.
- The dual Gabor window $\gamma[n]$ is obtained as $\gamma[n] = \frac{g[n]}{\lambda_g[n]}$.
- The frame bounds are given by $A = \inf\{\lambda_g[n]\}$ and $B = \sup\{\lambda_g[n]\}$.
- If $\{g_{l,m}[n]\}$ is an arbitrary WHF for $l^2(\mathbb{Z})$, then a tight WHF $\{h_{l,m}[n]\}$ with $A=1$ can be constructed according to $h[n] = \frac{g[n]}{\sqrt{\lambda_g[n]}}$.

A similar diagonalization is valid for band-limited Gabor windows [11]. The key observation is that the frequency-domain counterpart $\hat{\mathbf{S}}$ of the Gabor frame operator $\mathbf{S}$ can be expressed as

$$(\hat{\mathbf{S}}X)(\theta) = \sum_{l=0}^{L-1} X\left(\theta - \frac{l}{L}\right) \left[\frac{1}{L} \sum_{m=0}^{M-1} G\left(\theta - \frac{m}{M}\right) G^*\left(\theta - \frac{m}{M} - \frac{l}{L}\right) \right],$$

where $X(\theta) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j2\pi\theta n}$ and $G(\theta) = \sum_{n=-\infty}^{\infty} g[n]\, e^{-j2\pi\theta n}$ are the Fourier transforms of $x[n]$ and $g[n]$, respectively. If, within the frequency-domain period $|\theta| \leq \frac{1}{2}$, the window $g[n]$ is bandlimited according to $G(\theta) = 0$ for $|\theta| \geq \frac{1}{L}$, we obtain

$$(\hat{\mathbf{S}}X)(\theta) = \lambda_g(\theta)\, X(\theta) \qquad \text{with} \quad \lambda_g(\theta) = \frac{1}{L} \sum_{m=0}^{M-1} \left| G\left(\theta - \frac{m}{M}\right) \right|^2.$$

The $\lambda_g(\theta)$ are the eigenvalues of both $\hat{\mathbf{S}}$ and $\mathbf{S}$; the eigenfunctions of $\hat{\mathbf{S}}$ are $u_\theta(\theta') = \delta(\theta' - \theta)$ and those of $\mathbf{S}$ are $u_\theta[n] = e^{j2\pi\theta n}$. The following conclusions can be shown to hold:

- The frame condition can be rewritten as $A \leq \lambda_g(\theta) \leq B$.

- The set $\{g_{l,m}[n]\}$ is a tight WHF for $l^2(\mathbb{Z})$ with frame bound A if and only if $\lambda_g(\theta) \equiv A$.
- The Fourier transform of the dual Gabor window $\gamma[n]$ is given by $\Gamma(\theta) = \frac{G(\theta)}{\lambda_g(\theta)}$.
- The frame bounds are given by $A = \inf\{\lambda_g(\theta)\}$ and $B = \sup\{\lambda_g(\theta)\}$.
- If $\{g_{l,m}[n]\}$ is an arbitrary WHF for $l^2(\mathbb{Z})$, then a tight WHF $\{h_{l,m}[n]\}$ with $A = 1$ can be constructed by calculating $H(\theta) = \frac{G(\theta)}{\sqrt{\lambda_g(\theta)}}$ and $h[n] = \int_0^1 H(\theta)\, e^{j2\pi n\theta}\, d\theta$.

We note that the dual window $\gamma[n]$ and the "tight" window $h[n]$ obtained by the above methods will have the same finite time or frequency supports as the original window $g[n]$.

The diagonalization discussed in Section 3 (valid for critical sampling and integer oversampling) is consistent with the diagonalization discussed above (valid for synthesis windows with suitably restricted time or frequency supports). In fact, in the case of critical sampling and integer oversampling it can be shown that a window $g[n]$ with finite time length $\leq M$ yields $\lambda_g(n,\theta) = \lambda_g[n]$, and a window $g[n]$ with $G(\theta) = 0$ for $|\theta| \geq \frac{1}{L}$ satisfies $\lambda_g(n,\theta) = \lambda_g(\theta/L)$.

5 Implementation and Simulation Results

For the implementation of the Gabor expansion using the FFT, we used the cyclic definition of the discrete Gabor expansion proposed in [2]. In this setting the signal $x[n]$ to be expanded and the Gabor windows $g[n]$ and $\gamma[n]$ are periodic signals with the same period N, where N has to be an integer multiple of both L and M. The Gabor frame operator $\mathbf{S}$ here reduces to a matrix of size $N \times N$. Hence, straightforward calculation of the dual window $\gamma[n]$ requires the inversion of an $N \times N$ matrix, which is on the order of N^3. In contrast, the DTZT-based method discussed in Section 3 can be implemented using the FFT, which is on the order of $N \log_2 \frac{N}{L}$, and the methods using finite-support windows (i.e., windows with finite support within the fundamental time or frequency period) are even of order N.

Fig. 1 illustrates the DTZT-based calculation of a "tight" window for the case of integer oversampling by a factor of 2. Fig. 2 illustrates the calculation of the dual window for a finite-length synthesis window in the case of rational oversampling.

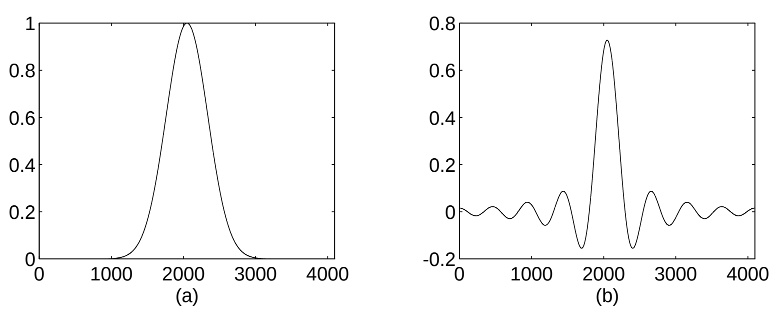

Fig. 1: a) Gaussian $g[n]$ with period $N = 4096$, b) corresponding tight Gabor window $h[n]$ for oversampling by 2 and $L = 256, M = 512$.

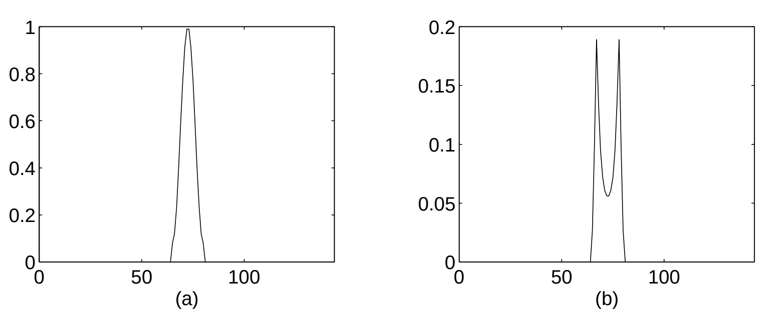

Fig. 2: a) Finite-length Gabor window $g[n]$ with period $N = 144$ and length 16, b) corresponding dual Gabor window $\gamma[n]$ for oversampling by 4/3 and $L = 12, M = 16$.

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