Time-Frequency Distributions Based on Conjugate Operators*

Franz Hlawatsch and Helmut Bölcskei

INTHFT, Vienna University of Technology, Gusshausstrasse 25/389, A-1040 Vienna, Austria
Tel.: +43 1 58801 3515; Fax: +43 1 587 05 83; Email: fhlawats@email.tuwien.ac.at

Abstract—New classes of quadratic time-frequency representations (QTFRs), such as the affine, hyperbolic, and power classes, are interesting alternatives to the conventional shift-covariant class (Cohen's class). This paper studies new QTFR classes that retain the inner structure of Cohen's class. These classes are based on a pair of "conjugate" unitary operators and satisfy covariance and marginal properties. For each class, we define a "central member" generalizing the Wigner distribution, and we specify a transformation by which the class can be derived from Cohen's class.

1 Introduction

Cohen's class with signal-independent kernels (briefly called Cohen's class hereafter) is the classical framework for quadratic time-frequency representations (QTFRs) [1]-[4]. Several recently proposed QTFR classes—such as the affine class [5, 6, 2, 3], the hyperbolic class [7, 8], and the power classes [9, 10]—provide interesting alternatives to the constant-bandwidth time-frequency (TF) analysis implemented by Cohen's class. These new QTFRs satisfy important *covariance properties* (e.g., scale covariance), they have specific *TF resolution characteristics* (e.g., constant-Q resolution), they are related to unitary *signal transforms* other than the Fourier transform (e.g., the Mellin transform), and they favor specific *TF geometries* (e.g., the hyperbolic TF geometry of Doppler-invariant signals and self-similar random processes).

This paper presents a general theory of QTFR classes that retain the inner structure of Cohen's class. These QTFR classes are based on pairs of "conjugate" unitary operators related to each other in a specific manner [11]-[14]. Section 2 introduces the concept of conjugate operators. Section 3 discusses the "covariance method" for constructing covariant QTFRs [11, 15]. Section 4 reviews the "characteristic function method" for constructing QTFRs satisfying the marginal properties [16, 17, 13]. Section 5 shows that the two methods coincide in the case of conjugate operators [11, 12, 18]. For any QTFR class based on conjugate operators, a "central QTFR" (generalizing the Wigner distribution) is defined in Section 6 [12]. Section 7 shows that any class based on conjugate operators can be derived from Cohen's class by a unitary transformation [12, 13], and Section 8 considers an example.

Cohen's Class. We first review Cohen's class [1]-[4], which will be generalized subsequently. Cohen's class consists of all QTFRs $C_x(t,f)$ that are *covariant to TF shifts*,

$$C_{\mathbf{S}_{\tau,\nu}x}(t,f) = C_x(t-\tau, f-\nu)\,. \qquad (1)$$

Here, $x(t) \in \mathcal{L}_2(\mathbb{R})$ is a signal with Fourier transform $X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$, and $\mathbf{S}_{\tau,\nu}$ is the TF shift operator, i.e., $\mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$ with the time-shift operator $\mathbf{T}_\tau$ and the frequency-shift operator $\mathbf{F}_\nu$ defined as $(\mathbf{T}_\tau\, x)(t) = x(t-\tau)$ and $(\mathbf{F}_\nu\, x)(t) = x(t)\, e^{j2\pi\nu t}$, respectively. The properties of the operators $\mathbf{T}_\tau$ and $\mathbf{F}_\nu$ entail a characteristic structure of Cohen's class. In particular, any

* This work was supported in part by FWF grants P10012-ÖPH and P10531-ÖPH.

QTFR of Cohen's class can be written as

$$C_x(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)\, x^*(t_2)\, h^*(t_1 - t, t_2 - t)\, e^{-j2\pi f(t_1-t_2)} dt_1 dt_2\,, \qquad (2)$$

where $h(t_1, t_2)$ is a 2-D kernel function independent of $x(t)$. An equivalent expression is

$$C_x(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi(\tau,\nu)\, A_x(\tau,\nu)\, e^{j2\pi(t\nu - f\tau)}\, d\tau\, d\nu\,, \qquad (3)$$

where the kernel $\Psi(\tau,\nu)$ is related to $h(t_1,t_2)$ as $h(t_1,t_2) = \int_{-\infty}^{\infty} \Psi^*(t_1 - t_2, \nu)\, e^{j\pi(t_1+t_2)\nu}\, d\nu$, and

$$A_x(\tau,\nu) = \int_{-\infty}^{\infty} x\left(t+\frac{\tau}{2}\right) x^*\left(t-\frac{\tau}{2}\right) e^{-j2\pi\nu t}\, dt \qquad (4)$$

is the *symmetric ambiguity function* of $x(t)$. The QTFR $C_x(t,f)$ satisfies the *marginal properties*

$$\int_{-\infty}^{\infty} C_x(t,f)\, dt = |X(f)|^2\,, \qquad \int_{-\infty}^{\infty} C_x(t,f)\, df = |x(t)|^2 \qquad (5)$$

if $\Psi(\tau,0) = \Psi(0,\nu) = 1$. A central QTFR of Cohen's class is the *Wigner distribution* [19]

$$W_x(t,f) = \int_{-\infty}^{\infty} x\left(t+\frac{\tau}{2}\right) x^*\left(t-\frac{\tau}{2}\right) e^{-j2\pi f\tau}\, d\tau = \int_{-\infty}^{\infty} X\left(f+\frac{\nu}{2}\right) X^*\left(f-\frac{\nu}{2}\right) e^{j2\pi t\nu}\, d\nu \qquad (6)$$

for which $\Psi(\tau,\nu) \equiv 1$. Any Cohen's class QTFR can be derived from the Wigner distribution as

$$C_x(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(t-t', f-f')\, W_x(t',f')\, dt' df'\,, \qquad (7)$$

with the kernel $\psi(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi(\tau,\nu)\, e^{j2\pi(t\nu - f\tau)}\, d\tau\, d\nu$.

2 Conjugate Operators

Cohen's class is based on the time-shift operator $\mathbf{T}_\tau$ and the frequency-shift operator $\mathbf{F}_\nu$. The characteristic relation existing between these two operators will now be worked out in a generalized setting. We consider two linear operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ indexed by parameters $\alpha \in \mathcal{G}$ and $\beta \in \mathcal{G}$ with $\mathcal{G} \subseteq \mathbb{R}$. These operators are assumed to be *unitary* on a linear signal space $\mathcal{X} \subseteq \mathcal{L}_2(\mathbb{R})$, and to satisfy identical *composition properties*

$$\mathbf{A}_{\alpha_2}\mathbf{A}_{\alpha_1} = \mathbf{A}_{\alpha_1 \bullet \alpha_2} \qquad \text{and} \qquad \mathbf{B}_{\beta_2}\mathbf{B}_{\beta_1} = \mathbf{B}_{\beta_1 \bullet \beta_2}\,,$$

where $(\mathcal{G}, \bullet)$ is a commutative group [16, 11, 20]. The *eigenvalues* $\lambda^A_{\alpha,\tilde{\alpha}}$ and *eigenfunctions* $u^A_{\tilde{\alpha}}(t)$ of $\mathbf{A}_\alpha$ are defined by $(\mathbf{A}_\alpha\, u^A_{\tilde{\alpha}})(t) = \lambda^A_{\alpha,\tilde{\alpha}}\, u^A_{\tilde{\alpha}}(t)$; they are indexed by a "dual parameter" $\tilde{\alpha}$. The **A**-*Fourier transform* (**A**-FT) [16, 17, 14] is defined as[1]

$$X_A(\tilde{\alpha}) \;\triangleq\; \langle x, u^A_{\tilde{\alpha}}\rangle = \int_t x(t)\, u^{A*}_{\tilde{\alpha}}(t)\, dt\,.$$

Analogous definitions apply to $\lambda^B_{\beta,\tilde{\beta}}$, $u^B_{\tilde{\beta}}(t)$, and the **B**-FT $X_B(\tilde{\beta})$.

Conjugate Operators. We now assume that application of one operator to an eigenfunction of the other operator merely produces a shift of the eigenfunction parameter [11, 12]:

> *Definition 1.* Two operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ as described above will be called *conjugate* if $\tilde{\alpha} \in \mathcal{G}$, $\tilde{\beta} \in \mathcal{G}$ and
>
> $$(\mathbf{B}_\beta\, u^A_{\tilde{\alpha}})(t) = u^A_{\tilde{\alpha}\bullet\beta}(t)\,, \qquad (\mathbf{A}_\alpha\, u^B_{\tilde{\beta}})(t) = u^B_{\tilde{\beta}\bullet\alpha}(t)\,.$$

Two conjugate operators $\mathbf{A}_\alpha, \mathbf{B}_\beta$ can be shown to satisfy several remarkable properties [11, 12].

[1] All integrals extend over the entire support of the function integrated.

Specifically, their eigenvalues can be written as

$$\lambda^A_{\alpha,\tilde{\alpha}} = e^{\pm j2\pi\,\mu(\alpha)\,\mu(\tilde{\alpha})} \qquad \text{and} \qquad \lambda^B_{\beta,\tilde{\beta}} = e^{\mp j2\pi\,\mu(\beta)\,\mu(\tilde{\beta})} = (\lambda^A_{\beta,\tilde{\beta}})^* . \tag{8}$$

Here, $\mu(g) \in \mathbb{R}$ maps $(\mathcal{G}, \bullet)$ onto $(\mathbb{R}, +)$ in the sense that $\mu(g_1 \bullet g_2) = \mu(g_1) + \mu(g_2)$, $\mu(g_0) = 0$, and $\mu(g^{-1}) = -\mu(g)$ where g_0 is the identity element in $\mathcal{G}$ and g^{-1} denotes the group-inverse of g. Due to (8), we shall simply write $\lambda^A_{\alpha,\beta} = \lambda_{\alpha,\beta}$ and $\lambda^B_{\alpha,\beta} = \lambda^*_{\alpha,\beta}$ in the following. Furthermore, two conjugate operators can be shown to commute up to a phase factor,

$$\mathbf{A}_\alpha \mathbf{B}_\beta = \lambda_{\alpha,\beta}\, \mathbf{B}_\beta \mathbf{A}_\alpha .$$

Their eigenfunctions are related as $\langle u^B_{\tilde{\beta}}, u^A_{\tilde{\alpha}} \rangle = \lambda_{\tilde{\alpha},\tilde{\beta}}$, $\int_{\mathcal{G}} u^B_{\tilde{\beta}}(t)\, \lambda^*_{\tilde{\alpha},\tilde{\beta}}\, d\mu(\tilde{\beta}) = u^A_{\tilde{\alpha}}(t)$, and $\int_{\mathcal{G}} u^A_{\tilde{\alpha}}(t)\, \lambda_{\tilde{\beta},\tilde{\alpha}}\, d\mu(\tilde{\alpha}) = u^B_{\tilde{\beta}}(t)$, where $d\mu(g) \stackrel{\triangle}{=} |\mu'(g)|\, dg$. The **A**-FT and **B**-FT are related as $X_B(\tilde{\beta}) = \int_{\mathcal{G}} X_A(\tilde{\alpha})\, \lambda^*_{\tilde{\beta},\tilde{\alpha}}\, d\mu(\tilde{\alpha})$ and $X_A(\tilde{\alpha}) = \int_{\mathcal{G}} X_B(\tilde{\beta})\, \lambda_{\tilde{\alpha},\tilde{\beta}}\, d\mu(\tilde{\beta})$ (cf. the equivalent concept of "dual operators" independently introduced in [13, 14]).

The Operator $\mathbf{D}_\theta = \mathbf{B}_\beta \mathbf{A}_\alpha$. We now compose two conjugate operators $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$ as

$$\mathbf{D}_\theta = \mathbf{D}_{\alpha,\beta} \stackrel{\triangle}{=} \mathbf{B}_\beta \mathbf{A}_\alpha ,$$

where $\theta = (\alpha, \beta) \in \mathcal{D}$ with $\mathcal{D} = \mathcal{G} \times \mathcal{G}$. It is readily shown that $\mathbf{D}_\theta$ is unitary on $\mathcal{X}$ and satisfies the *composition property* [11, 15]

$$\mathbf{D}_{\theta_2} \mathbf{D}_{\theta_1} = \lambda_{\alpha_2,\beta_1}\, \mathbf{D}_{\theta_1 \circ \theta_2} ,$$

where $(\mathcal{D}, \circ)$ is the commutative 2-D group with group operation $\theta_1 \circ \theta_2 = (\alpha_1, \beta_1) \circ (\alpha_2, \beta_2) = (\alpha_1 \bullet \alpha_2, \beta_1 \bullet \beta_2)$, identity element $\theta_0 = (g_0, g_0)$, and inverse elements $\theta^{-1} = (\alpha^{-1}, \beta^{-1})$. Furthermore, $\mathbf{D}_\theta^{-1} = \lambda_{\alpha,\beta}\, \mathbf{D}_{\theta^{-1}}$ and $\mathbf{D}_{\theta_0} = \mathbf{I}$ where $\mathbf{I}$ is the identity operator on $\mathcal{X}$.

Examples. The shift operators $\mathbf{T}_\tau$, $\mathbf{F}_\nu$ underlying Cohen's class are conjugate with $(\mathcal{G}, \bullet) = (\mathbb{R}, +)$, $\mu(g) = g$, eigenvalues $\lambda^T_{\tau,f} = e^{-j2\pi\tau f}$, $\lambda^F_{\nu,t} = e^{j2\pi\nu t}$, eigenfunctions $u^T_f(t) = e^{j2\pi f t}$, $u^F_t(t') = \delta(t' - t)$, and dual parameters $\tilde{\tau} = f$, $\tilde{\nu} = t$. The **T**-FT is the conventional Fourier transform, $X_T(f) = X(f)$, and the **F**-FT is the identity transform, $X_F(t) = x(t)$. All relations claimed to hold for conjugate operators are easily verified: in particular, the operators $\mathbf{T}_\tau$, $\mathbf{F}_\nu$ are conjugate since $(\mathbf{F}_\nu\, u^T_f)(t) = u^T_{f+\nu}(t)$ and $(\mathbf{T}_\tau\, u^F_t)(t') = u^F_{t+\tau}(t')$. They commute up to a phase factor, $\mathbf{T}_\tau \mathbf{F}_\nu = e^{-j2\pi\tau\nu}\, \mathbf{F}_\nu \mathbf{T}_\tau$, and the TF shift operator $\mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$ satisfies the composition property $\mathbf{S}_{\tau_2,\nu_2} \mathbf{S}_{\tau_1,\nu_1} = e^{-j2\pi\nu_1\tau_2}\, \mathbf{S}_{\tau_1+\tau_2,\,\nu_1+\nu_2}$.

The operators underlying the *hyperbolic* QTFR class [7, 8] are conjugate as well, but the operators underlying the *affine class* and the *power classes* [5, 6, 9, 10] are *not* conjugate.

In the next two sections, we shall consider two distinct methods for systematically constructing QTFRs associated to two operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$.

3 Covariance Method

To each pair of conjugate operators $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$, there exists a covariance property[2] for QTFRs that generalizes the TF shift covariance property in (1) [11, 15].

Localization Function. Let $\nu^A_{\tilde{\alpha}}(t)$ denote the instantaneous frequency of the eigenfunction $u^A_{\tilde{\alpha}}(t)$, and let $\tau^B_{\tilde{\beta}}(f)$ denote the group delay of the eigenfunction $u^B_{\tilde{\beta}}(t)$. For any $\tilde{\theta} = (\tilde{\alpha}, \tilde{\beta}) \in \mathcal{D}$, the corresponding functions $\nu^A_{\tilde{\alpha}}(t)$ and $\tau^B_{\tilde{\beta}}(f)$ are assumed[3] to intersect in a unique TF point $z = (t, f)$. Hence, $z = l(\tilde{\theta})$ where $l(\tilde{\theta})$ will be called the *localization function* (LF) of the operator

[2] We note that a covariance property exists also in certain cases where $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are not conjugate [11, 15, 18].

[3] In certain cases, this assumption holds if one uses the group delay of $u^A_{\tilde{\alpha}}(t)$ and the instantaneous frequency of $u^B_{\tilde{\beta}}(t)$; here, an analogous theory can be formulated.

$\mathbf{D}_\theta$ [11]. The LF is constructed by solving the system of equations $\nu_{\tilde{\alpha}}^A(t) = f$, $\tau_{\tilde{\beta}}^B(f) = t$ for $(t, f) = z$ [21, 22, 11]. It is assumed to be invertible, i.e. $z = l(\tilde{\theta}) \Leftrightarrow \tilde{\theta} = l^{-1}(z)$.

Covariance Property. The LF describes the *TF displacements* caused by $\mathbf{D}_\theta$. If a signal $x(t)$ is localized about a TF point $z = (t, f)$, then $(\mathbf{D}_\theta\, x)(t)$ will be localized about a new TF point $z' = (t', f')$. Since z is the intersection[4] of $u_{\tilde{\alpha}}^A(t)$ and $u_{\tilde{\beta}}^B(t)$ with $(\tilde{\alpha}, \tilde{\beta}) = \tilde{\theta} = l^{-1}(z)$, z' will be the intersection of $(\mathbf{D}_\theta\, u_{\tilde{\alpha}}^A)(t)$ and $(\mathbf{D}_\theta\, u_{\tilde{\beta}}^B)(t)$. Due to the conjugateness of $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$,

$$(\mathbf{D}_\theta\, u_{\tilde{\alpha}}^A)(t) = \lambda_{\alpha,\tilde{\alpha}}\, u_{\tilde{\alpha}\bullet\beta}^A(t) \qquad \text{and} \qquad (\mathbf{D}_\theta\, u_{\tilde{\beta}}^B)(t) = \lambda_{\beta,\tilde{\beta}\bullet\alpha}^*\, u_{\tilde{\beta}\bullet\alpha}^B(t)\,.$$

Hence,

$$z' = l(\tilde{\alpha} \bullet \beta, \tilde{\beta} \bullet \alpha) = l(\tilde{\theta} \circ \theta^T) = l\left(l^{-1}(z) \circ \theta^T\right) \qquad \text{with } \theta^T = (\beta, \alpha)\,.$$

This motivates the following definition [11]:

Definition 2. A QTFR $T_x(z) = T_x(t, f)$ will be called *covariant to* $\mathbf{D}_\theta$ if

$$T_{\mathbf{D}_\theta x}(z) = T_x\Big(l\left(l^{-1}(z) \circ \theta^{-T}\right)\Big) \qquad \text{with } \theta^{-T} = (\theta^{-1})^T = (\beta^{-1}, \alpha^{-1})\,. \tag{9}$$

The Class of All Covariant QTFRs. The class of all QTFRs covariant to $\mathbf{D}_\theta$ is characterized as follows (cf. [11, 15]):

Theorem 1. A QTFR $T_x(z) = T_x(t, f)$ is covariant to an operator $\mathbf{D}_\theta$ if and only if

$$T_x(z) = \left\langle x, \mathbf{H}_z^D x \right\rangle = \int_{t_1}\int_{t_2} x(t_1)\, x^*(t_2)\, h_z^{D*}(t_1, t_2)\, dt_1 dt_2 \tag{10}$$

with $\mathbf{H}_z^D = \mathbf{D}_{[l^{-1}(z)]^T} \mathbf{H}\, \mathbf{D}_{[l^{-1}(z)]^T}^{-1}$. Here, $\mathbf{H}$ is an arbitrary linear operator with kernel $h(t_1, t_2)$, assumed independent of $x(t)$, and the kernel of $\mathbf{H}_z^D$ is given by

$$h_z^D(t_1, t_2) = \int_{t_1'}\int_{t_2'} D_{[l^{-1}(z)]^T}(t_1, t_1')\, h(t_1', t_2')\, D_{[l^{-1}(z)]^T}^{-1}(t_2', t_2)\, dt_1' dt_2'\,, \tag{11}$$

where $D_\theta(t_1, t_2)$ and $D_\theta^{-1}(t_1, t_2)$ are the kernels of $\mathbf{D}_\theta$ and $\mathbf{D}_\theta^{-1}$, respectively.

For given operator $\mathbf{D}_\theta$, (10) and (11) define a class of QTFRs parameterized by the 2-D kernel $h(t_1, t_2)$. This class consists of *all* QTFRs satisfying the covariance (9).

Example. For $\mathbf{D}_\theta = \mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$, (9) becomes the TF shift covariance $T_{\mathbf{S}_{\tau,\nu}x}(t, f) = T_x(t-\tau, f-\nu)$, and (10) becomes Cohen's class as expressed in (2) (note that here $h_z^D(t_1, t_2) = h_z^S(t_1, t_2) = h(t_1 - t, t_2 - t)\, e^{j2\pi f(t_1 - t_2)}$).

4 Characteristic Function Method

Besides the covariance (9), other important properties are the *marginal properties* [16, 17, 11]

$$\int_{\mathcal{G}} T_x(l(\tilde{\theta}))\, d\mu(\tilde{\beta}) = |X_A(\tilde{\alpha})|^2\,, \qquad \int_{\mathcal{G}} T_x(l(\tilde{\theta}))\, d\mu(\tilde{\alpha}) = |X_B(\tilde{\beta})|^2\,. \tag{12}$$

It can be shown that a class of QTFRs satisfying these marginal properties is given by [16, 17, 11]

$$\bar{T}_x(z) = \iint_{\mathcal{D}} \Psi(\theta)\, A_x^D(\theta)\, \Lambda(l^{-1}(z), \theta)\, d\mu^2(\theta) \qquad \text{with } \Lambda(\tilde{\theta}, \theta) = \lambda_{\alpha,\tilde{\alpha}}\, \lambda_{\beta,\tilde{\beta}}^* \tag{13}$$

[4] z is the intersection of $u_{\tilde{\alpha}}^A(t)$ and $u_{\tilde{\beta}}^B(t)$ in the sense that $u_{\tilde{\alpha}}^A(t)$ and $u_{\tilde{\beta}}^B(t)$ are concentrated, in the TF plane, along $\nu_{\tilde{\alpha}}^A(t)$ and $\tau_{\tilde{\beta}}^B(f)$, respectively, and z is the intersection of $\nu_{\tilde{\alpha}}^A(t)$ and $\tau_{\tilde{\beta}}^B(f)$.

where $d\mu^2(\theta) \stackrel{\triangle}{=} d\mu(\alpha)\, d\mu(\beta)$, $\Psi(\theta) = \Psi(\alpha,\beta)$ is a 2-D kernel (independent of $x(t)$) satisfying $\Psi(\alpha, g_0) = \Psi(g_0, \beta) = 1$, and $A_x^D(\theta)$ is the "characteristic function" defined as[5]

$$A_x^D(\theta) \stackrel{\triangle}{=} \langle \mathbf{D}_{\theta^{-1/2}}\, x, \mathbf{D}_{\theta^{1/2}}\, x\rangle = \int_t (\mathbf{D}_{\theta^{-1/2}}\, x)(t)\; (\mathbf{D}_{\theta^{1/2}}\, x)^*(t)\; dt = \lambda_{\alpha,\beta}^{-1/2}\, \langle x, \mathbf{D}_\theta\, x\rangle\,.$$

Example. In the case of $\mathbf{T}_\tau$ and $\mathbf{F}_\nu$, the marginal properties (12) reduce to the conventional marginal properties in (5), $A_x^D(\theta)$ becomes the symmetric ambiguity function $A_x(\tau,\nu)$ in (4), and the QTFR class (13) becomes Cohen's class as expressed in (3).

5 Equivalence of Methods

So far, we have discussed two distinct approaches to the systematic construction of QTFR classes corresponding to two operators $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$: the *covariance method* results in the QTFR class $\mathcal{T} = \{T_x(z)\}$ in (10) that consists of all QTFRs satisfying the covariance property (9), while the *characteristic function method* results in the QTFR class $\bar{\mathcal{T}} = \{\bar{T}_x(z)\}$ in (13) that is related to the marginal properties (12). Although we have considered only the case of conjugate operators, these two methods are in fact more generally valid [11, 13, 15, 16, 18]. However, the conjugate case is an important special case since *here the two methods are equivalent* [11, 12]:

Theorem 2. For conjugate operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$, there is

$$\mathcal{T} = \bar{\mathcal{T}} \qquad \text{or equivalently} \qquad T_x(z) \;\equiv\; \bar{T}_x(z)$$

where the kernel $h(t_1, t_2)$ of $T_x(z)$ and the kernel $\Psi(\theta)$ of $\bar{T}_x(z)$ are related as

$$h(t_1,t_2) = \iint_{\mathcal{D}} \Psi^*(\theta)\, D_\theta(t_1,t_2)\, \lambda_{\alpha,\beta}^{1/2}\, d\mu^2(\theta)\,. \tag{14}$$

Examples. In the case of the conjugate operators $\mathbf{T}_\tau$ and $\mathbf{F}_\nu$, both the covariance method and the characteristic function method result in Cohen's class (see (2) and (3), respectively). In the case of $\mathbf{T}_\tau$ and the TF scaling operator $\mathbf{C}_a$ defined as $(\mathbf{C}_a x)(t) = \sqrt{e^a}\, x(e^a t)$, which are not conjugate, the covariance method results in the affine class [5, 6, 2, 3] whereas the characteristic function method results in a different class [17].

6 The Central Member

In what follows, we consider the QTFR class $\mathcal{T} = \bar{\mathcal{T}}$ corresponding to conjugate operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$. We define the "central member" of this QTFR class, denoted $W_x^D(z)$, via its kernel $\Psi(\theta) \equiv 1$ [12]. Inserting in (13), the central member is obtained as

$$W_x^D(z) = \iint_{\mathcal{D}} A_x^D(\theta)\; \Lambda(l^{-1}(z),\theta)\; d\mu^2(\theta)\,. \tag{15}$$

This can be expressed in terms of the $\mathbf{A}$-FT $X_A(\tilde{\alpha})$ and the $\mathbf{B}$-FT $X_B(\tilde{\beta})$ as

$$W_x^D(z) = \int_{\mathcal{G}} X_A\Big(\tilde{\alpha}\bullet\beta^{1/2}\Big)\, X_A^*\Big(\tilde{\alpha}\bullet\beta^{-1/2}\Big)\, \lambda_{\beta,\tilde{\beta}}^*\, d\mu(\beta) = \int_{\mathcal{G}} X_B\Big(\tilde{\beta}\bullet\alpha^{1/2}\Big)\, X_B^*\Big(\tilde{\beta}\bullet\alpha^{-1/2}\Big)\, \lambda_{\alpha,\tilde{\alpha}}\, d\mu(\alpha)$$

where $(\tilde{\alpha},\tilde{\beta}) = l^{-1}(z)$. Furthermore, any QTFR $T_x(z)$ of $\mathcal{T} = \bar{\mathcal{T}}$ can be derived from $W_x^D(z)$ as

$$T_x(z) = \iint_{\mathcal{D}} \psi\Big(l^{-1}(z)\circ\tilde{\theta}^{-1}\Big)\; W_x^D\Big(l(\tilde{\theta})\Big)\; d\mu^2(\tilde{\theta}) \tag{16}$$

where $\psi(\tilde{\theta}) = \iint_{\mathcal{D}} \Psi(\theta)\, \Lambda(\tilde{\theta},\theta)\, d\mu^2(\theta)$.

Example. In the case of the conjugate operators $\mathbf{T}_\tau$ and $\mathbf{F}_\nu$, the central member becomes the Wigner distribution in (6), and relation (16) reduces to the convolution relation (7).

[5] We note that $\theta^{1/2}$ is defined by $\theta^{1/2}\circ\theta^{1/2} = \theta$, and that $\lambda_{\alpha,\beta}^{-1/2} = (e^{\pm j2\pi\,\mu(\alpha)\,\mu(\beta)})^{-1/2} = e^{\mp j\pi\,\mu(\alpha)\,\mu(\beta)}$.

7 Transformation of Cohen's Class

The QTFR class $\mathcal{T} = \bar{\mathcal{T}}$ can be constructed using a transformation approach, a fact linking our theory to the "warping" theory in [21, 22]. Let $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ be conjugate operators on a signal space $\mathcal{X}$, with group $(\mathcal{G}, \bullet)$, and consider the operators $\mathbf{C}_\gamma \stackrel{\triangle}{=} \mathbf{V}\,\mathbf{A}_{s(\gamma)}\mathbf{V}^{-1}$ and $\mathbf{D}_\delta \stackrel{\triangle}{=} \mathbf{V}\,\mathbf{B}_{s(\delta)}\mathbf{V}^{-1}$. Here, $\mathbf{V}$ is an isometric isomorphism (i.e., a norm-preserving one-to-one transformation) mapping $\mathcal{X}$ onto some other space $\mathcal{Y}$, and $s(\cdot)$ is a one-to-one function mapping some other commutative group $(\mathcal{H}, *)$ onto $(\mathcal{G}, \bullet)$, in the sense that $s(h_1 * h_2) = s(h_1) \bullet s(h_2)$ for all $h_1, h_2 \in \mathcal{H}$. Assuming suitable choice of the dual parameters $\tilde{\gamma}$ and $\tilde{\delta}$, it can be shown that the eigenvalues and eigenfunctions of $\mathbf{C}_\gamma$ and $\mathbf{D}_\delta$ are $\lambda^C_{\gamma,\tilde{\gamma}} = \lambda^A_{s(\gamma),s(\tilde{\gamma})}$, $u^C_{\tilde{\gamma}}(t) = \left(\mathbf{V}u^A_{s(\tilde{\gamma})}\right)(t)$ and $\lambda^D_{\delta,\tilde{\delta}} = \lambda^B_{s(\delta),s(\tilde{\delta})}$, $u^D_{\tilde{\delta}}(t) = \left(\mathbf{V}u^B_{s(\tilde{\delta})}\right)(t)$, respectively. Furthermore, $\mathbf{C}_\gamma$ and $\mathbf{D}_\delta$ are *conjugate* operators on $\mathcal{Y}$, with group $(\mathcal{H}, *)$. Thus, isometric isomorphisms $\mathbf{V}$ and one-to-one group transformations $s(\cdot)$ preserve the conjugateness property of two operators.

The following theorem [12] states that *any* arbitrary pair of conjugate operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ can be derived from the shift operators underlying Cohen's class, $\mathbf{T}_\tau$ and $\mathbf{F}_\nu$, using such a transformation, and furthermore, that the QTFR class $\mathcal{T} = \bar{\mathcal{T}}$ corresponding to $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$ can be derived from Cohen's class using a transformation. Similar results have been derived independently in [13, 14].

Theorem 3. Let $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$ be conjugate with group $(\mathcal{G}, \bullet)$ corresponding to function $\mu(\cdot)$, so that $\lambda^A_{\alpha,\tilde{\alpha}} = e^{\pm j2\pi\,\mu(\alpha)\,\mu(\tilde{\alpha})}$.

Case 1. If $\lambda^A_{\alpha,\tilde{\alpha}} = e^{-j2\pi\,\mu(\alpha)\,\mu(\tilde{\alpha})}$ ($-$ sign), then

$$\mathbf{A}_\alpha = \mathbf{V}\,\mathbf{T}_{t_r\mu(\alpha)}\mathbf{V}^{-1} \qquad \text{and} \qquad \mathbf{B}_\beta = \mathbf{V}\,\mathbf{F}_{\mu(\beta)/t_r}\mathbf{V}^{-1}$$

where $t_r > 0$ is an arbitrary reference time constant and the kernel of $\mathbf{V}$ is

$$V(t,t') = \frac{1}{\sqrt{t_r}}\,u^B_{\mu^{-1}(t'/t_r)}(t)$$

with $\mu^{-1}(\cdot)$ denoting the function inverse to $\mu(\cdot)$. Furthermore, any QTFR $T_x(z) = T_x(t,f)$ of the QTFR class $\mathcal{T} = \bar{\mathcal{T}}$ associated to $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$ can be derived from a corresponding QTFR $C_x(t,f)$ of Cohen's class as

$$T_x(z) = C_{\mathbf{V}^{-1}x}\left(t_r\mu(\tilde{\beta}), \frac{\mu(\tilde{\alpha})}{t_r}\right)\bigg|_{\tilde{\theta}=l^{-1}(z)}.$$

Case 2. If $\lambda^A_{\alpha,\tilde{\alpha}} = e^{j2\pi\,\mu(\alpha)\,\mu(\tilde{\alpha})}$ ($+$ sign), then the relations valid in Case 1 have to be replaced by $\mathbf{A}_\alpha = \mathbf{V}\,\mathbf{F}_{\mu(\alpha)/t_r}\mathbf{V}^{-1}$ and $\mathbf{B}_\beta = \mathbf{V}\,\mathbf{T}_{t_r\mu(\beta)}\mathbf{V}^{-1}$, $V(t,t') = \frac{1}{\sqrt{t_r}}\,u^A_{\mu^{-1}(t'/t_r)}(t)$, and $T_x(z) = C_{\mathbf{V}^{-1}x}\left(t_r\mu(\tilde{\alpha}), \frac{\mu(\tilde{\beta})}{t_r}\right)\Big|_{\tilde{\theta}=l^{-1}(z)}$.

8 An Example

We shall finally illustrate the application of our theory by considering a specific example. Let the operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ be defined on the space $\mathcal{X} = \mathcal{L}_2(\mathbb{R}_+)$ as

$$(\mathbf{A}_\alpha x)(t) = e^{j2\pi \ln\alpha\,\ln(t/t_r)}\,x(t) \qquad \text{and} \qquad (\mathbf{B}_\beta\,x)(t) = \frac{1}{\sqrt{\beta}}\,x\!\left(\frac{t}{\beta}\right), \qquad t,\alpha,\beta > 0\,,$$

where $t_r > 0$ is a fixed reference time constant. The operators satisfy the identical composition properties $\mathbf{A}_{\alpha_2}\mathbf{A}_{\alpha_1} = \mathbf{A}_{\alpha_1\alpha_2}$ and $\mathbf{B}_{\beta_2}\mathbf{B}_{\beta_1} = \mathbf{B}_{\beta_1\beta_2}$, so that the underlying group is the multiplicative group, $(\mathcal{G}, \bullet) = (\mathbb{R}_+, \cdot)$, with identity element $g_0 = 1$ and inverse elements $g^{-1} = 1/g$. The eigenvalues/functions of $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are $\lambda^A_{\alpha,\tilde{\alpha}} = e^{j2\pi\ln\alpha\ln\tilde{\alpha}}$, $u^A_{\tilde{\alpha}}(t) = \frac{1}{\sqrt{t}}\,\delta(\ln\frac{t}{t_r} - \ln\tilde{\alpha})$ and

$\lambda^B_{\beta,\tilde\beta} = e^{-j2\pi \ln\beta \ln\tilde\beta}$, $u^B_{\tilde\beta}(t) = \frac{1}{\sqrt{t}}\, e^{j2\pi \ln\tilde\beta\, \ln(t/t_r)}$. Note that $\mu(g) = \ln g$ and $d\mu(g) = \frac{dg}{g}$. The **A**-FT and **B**-FT are $X_A(\tilde\alpha) = \sqrt{t_r\tilde\alpha}\; x(t_r\tilde\alpha)$ and $X_B(\tilde\beta) = \int_0^\infty x(t)\, e^{-j2\pi \ln\tilde\beta\, \ln(t/t_r)}\, \frac{dt}{\sqrt{t}}$, respectively. The operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are *conjugate* since $(\mathbf{B}_\beta\, u^A_{\tilde\alpha})(t) = u^A_{\tilde\alpha\beta}(t)$ and $(\mathbf{A}_\alpha\, u^B_{\tilde\beta})(t) = u^B_{\tilde\beta\alpha}(t)$. They commute up to a phase factor, $\mathbf{A}_\alpha\mathbf{B}_\beta = e^{j2\pi\ln\alpha\ln\beta}\,\mathbf{B}_\beta\mathbf{A}_\alpha$. The combined operator $\mathbf{D}_\theta = \mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta\mathbf{A}_\alpha$ satisfies the composition property $\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = e^{j2\pi\ln\alpha_2\ln\beta_1}\,\mathbf{D}_{\alpha_1\alpha_2,\beta_1\beta_2}$.

The localization function and inverse localization function of $\mathbf{D}_\theta = \mathbf{B}_\beta\mathbf{A}_\alpha$ are obtained as

$$(t,f) = l(\tilde\alpha,\tilde\beta) = \left(t_r\tilde\alpha,\, \frac{\ln\tilde\beta}{t_r\tilde\alpha}\right), \qquad (\tilde\alpha,\tilde\beta) = l^{-1}(t,f) = \left(\frac{t}{t_r},\, e^{tf}\right).$$

The covariance property (9) associated to $\mathbf{D}_\theta$ reads

$$T_{\mathbf{D}_\theta x}(t,f) = T_x\Big(\frac{t}{\beta},\, \beta\Big(f - \frac{\ln\alpha}{t}\Big)\Big),$$

and the class of all covariant QTFRs is obtained from (10) as

$$T_x(t,f) = \frac{t_r}{t}\int_0^\infty\!\!\int_0^\infty x(t_1)\, x^*(t_2)\; h^*\Big(t_r\frac{t_1}{t}, t_r\frac{t_2}{t}\Big)\, e^{-j2\pi tf\,\ln(t_1/t_2)}\, dt_1 dt_2\,, \qquad t>0\,.$$

The marginal properties (12) associated to $\mathbf{D}_\theta$ read (after simplification where possible)

$$\int_{-\infty}^\infty T_x(t,f)\, df = |x(t)|^2\,, \qquad \int_0^\infty T_x\Big(t,\frac{b}{t}\Big)\,\frac{dt}{t} = \left|\int_0^\infty x(t)\, e^{-j2\pi b\,\ln(t/t_r)}\,\frac{dt}{\sqrt{t}}\right|^2.$$

The characteristic function method (see (13)), with the simplifying substitution $a = \ln\alpha$, $b = \ln\beta$, yields the QTFRs

$$\bar T_x(t,f) = \int_{-\infty}^\infty\!\!\int_{-\infty}^\infty \tilde\Psi(a,b)\; \tilde A_x(a,b)\; e^{j2\pi[\ln(t/t_r)\,a\, -\, tf\, b]}\, da\, db\,, \qquad t>0$$

with

$$\tilde A_x(a,b) = \int_0^\infty x(t\, e^{b/2})\, x^*(t\, e^{-b/2})\; e^{-j2\pi a\ln(t/t_r)}\, dt$$

(note that $\tilde\Psi(a,b) = \Psi(e^a,e^b)$ and $\tilde A_x(a,b) = A^D_x(e^a,e^b)$ where $\Psi(\alpha,\beta)$ and $A^D_x(\alpha,\beta)$ are the quantities used in (13)). It is readily verified that the QTFRs $T_x(t,f)$ and $\bar T_x(t,f)$ are identical with the kernels related as $h(t_1,t_2) = \frac{1}{\sqrt{t_1t_2}}\int_{-\infty}^\infty \tilde\Psi^*(a,\ln\frac{t_1}{t_2})\, e^{j2\pi\left(\ln\frac{\sqrt{t_1t_2}}{t_r}\right)a}\, da$ (see (14)). The central member (15) is obtained as

$$\begin{aligned} W^D_x(t,f) &= \int_{-\infty}^\infty\!\!\int_{-\infty}^\infty \tilde A_x(a,b)\; e^{j2\pi[\ln(t/t_r)\,a\, -\, tf\, b]}\, da\, db \\ &= t\int_{-\infty}^\infty x(t\, e^{b/2})\, x^*(t\, e^{-b/2})\; e^{-j2\pi tfb}\, db = \int_{-\infty}^\infty X_B(e^{tf+a/2})\; X_B^*(e^{tf-a/2})\; e^{j2\pi\ln(t/t_r)a}\, da \end{aligned}$$

where $X_B(\tilde\beta) = \int_0^\infty x(t)\, e^{-j2\pi\ln\tilde\beta\,\ln(t/t_r)}\,\frac{dt}{\sqrt{t}}$. Any QTFR $T_x(t,f) = \bar T_x(t,f)$ can be derived from $W^D_x(t,f)$ as (see (16))

$$T_x(t,f) = \int_{t'=0}^\infty\int_{f'=-\infty}^\infty \psi\Big(\frac{t}{t'},\, e^{tf-t'f'}\Big)\; W^D_x(t',f')\; dt'df'\,, \qquad t>0\,,$$

where $\psi(\tilde\alpha,\tilde\beta) = \int_{-\infty}^\infty\int_{-\infty}^\infty \tilde\Psi(a,b)\; e^{j2\pi[(\ln\tilde\alpha)a - (\ln\tilde\beta)b]}\, da\, db$. Finally, any QTFR $T_x(t,f) = \bar T_x(t,f)$ can be derived from a corresponding Cohen's class QTFR $C_x(t,f)$ as (see Theorem 3, Case 2)

$$T_x(t,f) = C_{\mathbf{V}^{-1}x}\Big(t_r\ln\frac{t}{t_r},\, \frac{tf}{t_r}\Big) \qquad \text{with}\;\; (\mathbf{V}^{-1}x)(t) = \sqrt{e^{t/t_r}}\; x(t_r\, e^{t/t_r})\,.$$

We note that the QTFR class constructed above is the time-domain counterpart of the *hyperbolic class* [7, 8], and $\tilde A_x(a,b)$ and $W^D_x(t,f)$ are the time-domain counterparts of the *hyperbolic ambiguity function* and the *Q-distribution*, respectively [7, 8].

References

[1] L. Cohen, *Time-Frequency Analysis.* Prentice-Hall, 1995.

[2] P. Flandrin, *Temps-fréquence*. Paris: Hermès, 1993.

[3] F. Hlawatsch and G.F. Boudreaux-Bartels, "Linear and quadratic time-frequency signal representations," *IEEE Signal Processing Mag.*, vol. 9, no. 2, pp. 21-67, April 1992.

[4] F. Hlawatsch, "Duality and classification of bilinear time-frequency signal representations," *IEEE Trans. Signal Processing*, vol. 39, no. 7, pp. 1564-1574, July 1991.

[5] J. Bertrand and P. Bertrand, "Affine time-frequency distributions," in *Time-Frequency Signal Analysis — Methods and Applications*, ed. B. Boashash, Longman-Cheshire, Melbourne, 1992, pp. 118-140.

[6] O. Rioul and P. Flandrin, "Time-scale energy distributions: A general class extending wavelet transforms," *IEEE Trans. Signal Processing*, vol. 40, no. 7, pp. 1746-1757, July 1992.

[7] A. Papandreou, F. Hlawatsch, and G.F. Boudreaux-Bartels, "The hyperbolic class of quadratic time-frequency representations, Part I," *IEEE Trans. Signal Processing*, vol. 41, no. 12, pp. 3425-3444, Dec. 1993.

[8] F. Hlawatsch, A. Papandreou, and G.F. Boudreaux-Bartels, "The hyperbolic class of quadratic time-frequency representations, Part II," *IEEE Trans. Signal Processing*, submitted.

[9] F. Hlawatsch, A. Papandreou, and G.F. Boudreaux-Bartels, "The power classes of quadratic time-frequency representations: A generalization of the affine and hyperbolic classes," *Proc. 27th Asilomar Conf.*, Pacific Grove, CA, pp. 1265-1270, Nov. 1993.

[10] A. Papandreou, F. Hlawatsch, and G.F. Boudreaux-Bartels, "A unified framework for the scale covariant affine, hyperbolic, and power class quadratic time-frequency representations using generalized time shifts," *Proc. IEEE ICASSP-95*, Detroit, MI, May 1995, vol. 2, pp. 1017-1020.

[11] F. Hlawatsch and H. Bölcskei, "Displacement-covariant time-frequency energy distributions," *Proc. IEEE ICASSP-95*, Detroit, MI, May 1995, vol. 2, pp. 1025-1028.

[12] F. Hlawatsch and H. Bölcskei, "Covariant time-frequency distributions based on conjugate operators," to appear in *IEEE Signal Processing Letters.*

[13] A.M. Sayeed and D.L. Jones, "On the equivalence of generalized joint signal representations," *Proc. IEEE ICASSP-95*, Detroit, MI, May 1995, vol. 3, pp. 1533-1536.

[14] A.M. Sayeed and D.L. Jones, "Integral transforms covariant to unitary operators and their implications for joint signal representations," submitted to *IEEE Trans. Signal Processing.*

[15] F. Hlawatsch and H. Bölcskei, "Unified theory of displacement-covariant time-frequency analysis," *Proc. IEEE Int. Sympos. Time-Frequency Time-Scale Analysis*, Philadelphia, PA, Oct. 1994, pp. 524-527.

[16] R.G. Baraniuk, "Beyond time-frequency analysis: Energy densities in one and many dimensions," *Proc. IEEE ICASSP-94*, Adelaide, Australia, Apr. 1994, vol. 3, pp. 357-360.

[17] L. Cohen, "The scale representation," *IEEE Trans. Signal Processing*, vol. 41, no. 12, pp. 3275-3292, Dec. 1993.

[18] R.G. Baraniuk, "Marginals vs. covariance in joint distribution theory," *Proc. IEEE ICASSP-95*, Detroit, MI, May 1995, vol. 2, pp. 1021-1024.

[19] T.A.C.M. Claasen and W.F.G. Mecklenbräuker, "The Wigner distribution—A tool for time-frequency signal analysis," Parts I–III, *Philips J. Res.*, vol. 35, pp. 217-250, 276-300, 372-389; 1980.

[20] W. Rudin, *Fourier Analysis on Groups.* Wiley, 1967.

[21] R.G. Baraniuk and D.L. Jones, "Unitary equivalence: A new twist on signal processing," to appear in *IEEE Trans. Signal Processing.*

[22] R.G. Baraniuk, "Warped perspectives in time-frequency analysis," *Proc. IEEE Int. Sympos. Time-Frequency Time-Scale Analysis*, Philadelphia, PA, Oct. 1994, pp. 528-531.