Oversampled Wilson Expansions

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***Abstract*— Recently orthonormal Wilson bases with good time–frequency localization have been constructed by Daubechies, Jaffard, and Journé. We extend this construction to Wilson sets and frames with arbitrary oversampling (or redundancy). We state conditions under which dual Weyl–Heisenberg (WH) sets induce dual Wilson sets, and we formulate duality conditions in the time domain and frequency domain. We show that the dual frame of a Wilson frame has again Wilson structure, and that it is generated by the dual frame of the underlying Weyl–Heisenberg frame. The Wilson frame construction preserves the numerical properties of the underlying Weyl–Heisenberg frame while halving its redundancy.**

I. Introduction

The well-known Gabor expansion [1]–[7] provides a decomposition of a signal into time–frequency-shifted versions of a prototype window function. The Balian–Low theorem [3], [8] states that there are no orthonormal Gabor (Weyl–Heisenberg) bases for $L_2(\mathbb{R})$ with good localization in both time and frequency. This is unfortunate, since good time–frequency localization of the basis functions is important in many applications [3]. Recently, Daubechies, Jaffard, and Journé demonstrated [9] that a variation on the Gabor scheme resulting in the so-called Wilson bases allows to construct orthonormal bases for $L_2(\mathbb{R})$ having time–shift/frequency-modulation structure and good time–frequency localization. Further results on orthonormal Wilson expansions have been reported in [10] and [11], and discrete-time, integer oversampled Wilson expansions have been introduced by the authors in [12].

Whereas previous work on continuous-time Wilson expansions has been confined to the orthogonal (and, thus, critically sampled or nonredundant) case, this letter introduces Wilson sets and frames for $L_2(\mathbb{R})$ with *arbitrary oversampling/redundancy*. Extending the Daubechies–Jaffard–Journé method for the construction of orthonormal Wilson bases, we demonstrate that oversampled Wilson sets and frames can be constructed from Weyl–Heisenberg (WH) sets and frames, respectively. We formulate duality relations for Wilson expansions in the time domain and frequency domain. It is shown that the dual frame of a Wilson frame has again Wilson structure, and that it is generated by the dual WH frame. We also show that the condition number of the Wilson frame (characterizing important numerical properties of the frame [3]) equals that of the underlying WH frame while the redundancy of the WH frame is halved.

We shall first review some basic results on WH sets and frames that provide a background for our subsequent discussion. The Gabor expansion [1]–[7] of a signal $x(t)$ is

$$x(t) = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \langle x, h_{l,m} \rangle \, g_{l,m}(t). \tag{1}$$

Here, the *WH sets* $\{g_{l,m}(t)\}$ and $\{h_{l,m}(t)\}$ are defined as $g_{l,m}(t) = g(t - lT)\, e^{j2\pi mFt}$ and $h_{l,m}(t) = h(t - lT)\, e^{j2\pi mFt}$, with a *synthesis window* $g(t)$, an *analysis window* $h(t)$, and *grid parameters* T and F satisfying $TF \le 1$. The Gabor expansion (1) exists for all $x(t) \in L_2(\mathbb{R})$ if and only if $g(t)$ and $h(t)$ are *dual windows* [5] in the sense that ($\delta[m]$ denotes the discrete-time unit sample)

$$C_m^{(g,h)}(t) = F\delta[m] \tag{2}$$

where

$$C_m^{(g,h)}(t) = \sum_{l=-\infty}^{\infty} g(t - lT)\, h^*\Big(t - lT - \frac{m}{F}\Big).$$

An equivalent frequency domain formulation of the duality condition (2) is

$$\hat{C}_m^{(g,h)}(f) = T\,\delta[m] \tag{3}$$

where

$$\hat{C}_m^{(g,h)}(f) = \sum_{l=-\infty}^{\infty} G(f - lF)\, H^*\Big(f - lF - \frac{m}{T}\Big)$$

with $G(f)$ and $H(f)$ the Fourier transforms of $g(t)$ and $h(t)$, respectively, e.g., $G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft}\, dt$.

The WH set $\{g_{l,m}(t)\}$ is a *WH frame* for $L_2(\mathbb{R})$ if for all $x(t) \in L_2(\mathbb{R})$

$$A\|x\|^2 \le \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |\langle x, g_{l,m} \rangle|^2 \le B\|x\|^2$$

with the *frame bounds* $A > 0$ and $B < \infty$ [3], [7]. The frame property is desirable since it guarantees completeness of the WH set $\{g_{l,m}(t)\}$ and potentially good numerical properties of the Gabor expansion (characterized by the condition number B/A). For a given WH frame $\{g_{l,m}(t)\}$, the *dual frame* [3], [7] is $\{\tilde{g}_{l,m}(t)\}$ where $\tilde{g}(t) = (\mathbf{S}^{-1}g)(t)$ is the Wexler–Raz

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dual of $g(t)$ [2], [5], [6]. Here, $\mathbf{S}^{-1}$ is the inverse of the frame operator $\mathbf{S}$ defined as

$$(\mathbf{S}x)(t) = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \langle x, g_{l,m}\rangle g_{l,m}(t). \tag{4}$$

The frame $\{g_{l,m}(t)\}$ is *tight* if $A = B$, in which case $\mathbf{S} = A\mathbf{I}$ and, further, $\tilde{g}(t) = (1/A)\, g(t)$. Tight frames provide an easy reconstruction since calculation of the dual frame $\{\tilde{g}_{l,m}(t)\}$ is trivial.

In the *oversampled* case $TF < 1$, one usually observes better numerical stability of the Gabor expansion and better time–frequency concentration of the windows $g(t)$ and $\tilde{g}(t)$ than in the case of *critical sampling*, $TF = 1$ [3]. This, however, comes at the cost of redundant Gabor coefficients.

II. Wilson Expansions

We are now ready to introduce Wilson sets and frames with arbitrary oversampling. Let $\{g_{l,m}(t)\}$ and $\{h_{l,m}(t)\}$ be WH sets with $TF \leq \frac{1}{2}$, i.e., oversampling by a (possibly irrational) factor of at least 2. We define a *Wilson synthesis set* $\{\psi_{l,m}(t)\}$ associated to the synthesis window $g(t)$ by

$$\psi_{l,m}(t) = \begin{cases} g_{2l,0}(t) & l \in \mathbb{Z}, m = 0 \\ \frac{1}{\sqrt{2}}[g_{l,m}(t) + g_{l,-m}(t)] = \sqrt{2}\, g_{l,0}(t) & \\ \quad \cdot \cos(2\pi mFt) \qquad m + l \text{ even}, & \\ \quad l \in \mathbb{Z}, m = 1, 2, \cdots & \\ \frac{-j}{\sqrt{2}}[g_{l,m}(t) + g_{l,-m}(t)] = \sqrt{2}\, g_{l,0}(t) & \\ \quad \cdot \sin(2\pi mFt) \qquad m + l \text{ odd}, & \\ \quad l \in \mathbb{Z}, m = 1, 2, \cdots \end{cases}. \tag{5}$$

We also define a *Wilson analysis set* $\{\phi_{l,m}(t)\}$ associated to the analysis window $h(t)$ according to (5) with $g(t)$ replaced by $2\,h(t)$. The resulting *Wilson expansion* reads

$$x(t) = \sum_{l=-\infty}^{\infty} \sum_{m=0}^{\infty} \langle x, \phi_{l,m}\rangle \psi_{l,m}(t). \tag{6}$$

We note that the redundancy of the Wilson set is half the redundancy of the underlying WH set, which means that a Wilson set is critically sampled if $TF = \frac{1}{2}$.

We shall now provide necessary and sufficient conditions on the windows $g(t)$ and $h(t)$ such that the Wilson expansion (6) exists for all $x(t) \in L_2(\mathbb{R})$.

Theorem 1: Let $\{g_{l,m}(t)\}$ and $\{h_{l,m}(t)\}$ be *dual* WH sets with $TF \leq \frac{1}{2}$, i.e., the underlying windows $g(t)$ and $h(t)$ satisfy the duality condition (2) or the equivalent condition (3). Then, the Wilson expansion (6) exists for all $x(t) \in L_2(\mathbb{R})$ if and only if the windows $g(t)$ and $h(t)$ satisfy

$$D_m^{(g,h)}(t) \equiv 0 \quad \forall m \in \mathbb{Z} \tag{7}$$

where

$$D_m^{(g,h)}(t) = \sum_{l=-\infty}^{\infty} (-1)^l\, g(t - lT)\, h^* \cdot \left(-t - lT + \frac{m - \frac{1}{2}}{F}\right). \tag{8}$$

Hence, duality of the WH sets alone is not sufficient for the existence of the Wilson expansion. For the case of *rational* oversampling, i.e., $TF = p/q \leq \frac{1}{2}$ with[1] $\gcd(p, q) = 1$, the additional condition (7) can be reformulated in the frequency domain as follows:

$$\hat{D}_{m,r}^{(g,h)}(f) \equiv 0$$
$$\text{for } m \in \mathbb{Z},\, r = 0, 1, \cdots, q-1 \text{ with } q \text{ odd}$$
$$\hat{D}_{m,r}^{(g,h)}(f) + (-1)^{q/2}\, \hat{D}_{m-p,\, r+q/2}^{(g,h)}(f) \equiv 0$$
$$\text{for } m \in \mathbb{Z},\, r = 0, 1, \cdots, \frac{q}{2} - 1 \text{ with } q \text{ even}$$

where

$$\hat{D}_{m,r}^{(g,h)}(f) = \sum_{l=-\infty}^{\infty} (-1)^{lq}\, G\left(f + \frac{lp + \frac{rp}{q}}{T}\right) \cdot H^*\left(f + \frac{m + \frac{rp}{q} - lp + \frac{1}{2}}{T}\right). \tag{9}$$

The Wilson synthesis set $\{\psi_{l,m}(t)\}$ is complete in $L_2(\mathbb{R})$ if the conditions of Theorem 1 are satisfied; however, it need not possess the (desirable) frame property. We now discuss the derivation of Wilson *frames* from WH frames. We first note the following fundamental result.

Lemma 1: The Wilson frame operator $\mathbf{R}$, defined by $(\mathbf{R}x)(t) = \sum_{l=-\infty}^{\infty} \sum_{m=0}^{\infty} \langle x, \psi_{l,m}\rangle \psi_{l,m}(t)$, can be decomposed as

$$\mathbf{R} = \tfrac{1}{2}(\mathbf{S} + \mathbf{T})$$

where $\mathbf{S}$ is the WH frame operator [see (4)] and the operator $\mathbf{T}$ is defined by

$$(\mathbf{T}x)(t) = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-1)^{m+l} \langle x, g_{l,m}\rangle g_{l,-m}(t).$$

This decomposition of $\mathbf{R}$, which was first found by Auscher for the special case of orthonormal Wilson bases [10], is instrumental in proving the following theorem on Wilson frames.

Theorem 2: Let $\{g_{l,m}(t)\}$ be a WH frame for $L_2(\mathbb{R})$ with $TF \leq \frac{1}{2}$ and frame bounds A and B. Furthermore, let $g(t)$ be such that

$$(\mathbf{T}x)(t) \equiv 0 \qquad \text{for all } x(t) \in L_2(\mathbb{R}) \tag{10}$$

which is satisfied if and only if [cf. (7)]

$$D_m^{(g,g)}(t) \equiv 0 \quad \forall m \in \mathbb{Z}. \tag{11}$$

1) The Wilson synthesis set $\{\psi_{l,m}(t)\}$ associated to $g(t)$ by (5) is a frame for $L_2(\mathbb{R})$ with oversampling factor $2TF$ and frame bounds $A_w = A/2$ and $B_w = B/2$, i.e., for all $x(t) \in L_2(\mathbb{R})$ we have

$$\frac{A}{2}\|x\|^2 \leq \sum_{l=-\infty}^{\infty} \sum_{m=0}^{\infty} \|\langle x, \psi_{l,m}\rangle\|^2 \leq \frac{B}{2}\|x\|^2.$$

2) For $h(t) = \tilde{g}(t) = (\mathbf{S}^{-1}g)(t)$, the Wilson analysis set $\{\phi_{l,m}(t)\}$ associated to $h(t)$ is the *dual frame* [3], [7] associated to $\{\psi_{l,m}(t)\}$.

[1] Here $\gcd(p, q)$ denotes the greatest common divisor of p and q.

The following important conclusions can be drawn from Theorem 2:

- A WH frame induces a Wilson frame (with half the redundancy of the WH frame) if the additional property $\mathbf{T} = \mathbf{0}$ or, equivalently, $D_m^{(g,g)}(t) \equiv 0$ is satisfied.
- The frame bounds $A_w = A/2$, $B_w = B/2$ of the Wilson synthesis frame $\{\psi_{l,m}(t)\}$ are trivially related to the frame bounds A, B of the underlying WH synthesis frame $\{g_{l,m}(t)\}$. Since $B_w/A_w = B/A$, the Wilson frame inherits the numerical properties of the WH frame even though its redundancy is only half the redundancy of the WH frame.
- In particular, a tight WH frame ($A = B$) with $\mathbf{T} = 0$ induces a tight Wilson frame ($A_w = B_w$).
- The dual frame associated to a Wilson frame has again Wilson structure, and is induced by the dual WH frame.
- Known techniques for calculating the dual WH frame (i.e., the Wexler–Raz dual window [2], [5], [6]) can be applied for constructing the dual Wilson frame.

For the case of rational oversampling, i.e., $TF = p/q \leq \frac{1}{2}$ with $\gcd(p, q) = 1$, condition (11) can be rephrased in the frequency domain as [cf. (9)]

$$\hat{D}_{m,r}^{(g,g)}(f) \equiv 0 \quad \text{for } m \in \mathbb{Z},\ r = 0, 1, \cdots, q-1 \text{ with } q \text{ odd}$$

$$\hat{D}_{m,r}^{(g,g)}(f) + (-1)^{q/2} \hat{D}_{m-p,\, r+q/2}^{(g,g)}(f) \equiv 0 \quad \text{for } m \in \mathbb{Z},\ r = 0, 1, \cdots, \frac{q}{2} - 1 \text{ with } q \text{ even.}$$

For $TF = \frac{1}{2N}$ with $N \in \mathbb{N}$ (critically sampled or integer oversampled Wilson frame), a sufficient condition for (10) or equivalently (11) is the symmetry property

$$g^*(KT - t) = g(t)$$

where $K = 2r$ for N odd and $K = 2r + 1$ for N even, with some $r \in \mathbb{Z}$.

In [9] and [10] it has been shown that an orthonormal Wilson basis can be constructed from a tight WH frame with $TF = \frac{1}{2}$ and $g^*(-t) = g(t)$. This corresponds to the special case $N = 1$ and $K = 0$.

III. Conclusion

We introduced Wilson sets and frames with arbitrary oversampling, and we discussed their derivation from WH sets and frames, respectively. Compared to WH frames, Wilson frames are advantageous since they have half the redundancy of WH frames (but equal condition number) and since the frame functions can be chosen to be real valued. We note that our construction procedure can be rephrased in a discrete-time setting, with the resulting discrete-time Wilson expansions corresponding to a new class of cosine modulated filterbanks [12], [13].

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