Addendum to
Beurling-Type Density Criteria for System Identification

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Proofs omitted in the paper

Proof of Lemma 8. Item (i) is a direct consequence of the definition of the upper Beurling class density. To show item (ii), let $\epsilon > 0$ be arbitrary, and set $\theta = \mathcal{D}^+(\mathcal{L}) + \epsilon$. Then, by the definition of $\mathcal{D}^+(\mathcal{L})$, we have

$$\frac{n^+(\Lambda, (0,R)^2)}{R^2} \leqslant \mathcal{D}^+(\mathcal{L}) + \epsilon$$

for every $\Lambda \in \mathcal{L}$ and sufficiently large R, and thus

$$D^+(\Lambda) = \limsup_{R\to\infty} \frac{n^+(\Lambda, (0,R)^2)}{R^2} \leqslant \mathcal{D}^+(\mathcal{L}) + \epsilon. \tag{1}$$

Now, as (1) holds for every $\Lambda \in \mathcal{L}$, we deduce that $\sup_{\Lambda\in\mathcal{L}} D^+(\Lambda) \leqslant \mathcal{D}^+(\mathcal{L}) + \epsilon$, and hence, as $\epsilon > 0$ was arbitrary, we obtain $\sup_{\Lambda\in\mathcal{L}} D^+(\Lambda) \leqslant \mathcal{D}^+(\mathcal{L})$. □

Proof of Lemma 9. We identify $\mathbb{C}$ with $\mathbb{R}^2$ for ease of exposition. For a positive integer q, define the set $\mathcal{S}_q$ according to

$$\mathcal{S}_q = \{[qkR, q(k+1)R] \times [q\ell R, q(\ell+1)R] \,:\, k,\ell \in \mathbb{Z}\},$$

and note that $\mathcal{S}_q$ is a collection of squares in $\mathbb{R}^2$ of side length qR tessellating the plane. A simple counting argument now yields, for all $K \in \mathcal{S}_q$,

$$\#(\Omega_\gamma \cap K) \geqslant (qR\gamma^{-1})^2 - 4\left(qR\gamma^{-1} + 1\right),$$

where the subtracted term accounts for the $\lceil qR\gamma^{-1}\rceil$ points adjacent to each of the edges of the square K, but which might not be inside it. On the other hand, as every element of $\mathcal{S}_q$ can be covered by $(q+2)^2$ translates of $(0,R)^2$, using $n^+(\Lambda,(0,R)^2) \leqslant \theta R^2$ we have $\#(\Lambda \cap K) \leqslant (q+2)^2 \cdot \theta R^2 =$

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$((q+2)R\theta^{1/2})^2$ for all $K \in \mathcal{S}_q$. Therefore, as $\gamma^{-1} > \theta^{1/2}$ by assumption, there exists a positive integer $q' = q'(\theta, R, \gamma)$ such that

$$(q'R\gamma^{-1})^2 - 4\left(q'R\gamma^{-1} + 1\right) \geqslant \left((q'+2)R\theta^{1/2}\right)^2.$$

We thus have $\#(\Omega_\gamma \cap K) \geqslant \#(\Lambda \cap K)$, for all $K \in \mathcal{S}_{q'}$, and can therefore enumerate $\Lambda = \{\lambda_{m,n}\}_{(m,n)\in\mathcal{I}}$ so that, for every $(m,n) \in \mathcal{I}$, $\lambda_{m,n}$ and $\omega_{m,n}$ are contained in the same square $K_{m,n} \in \mathcal{S}_{q'}$. Setting $R' = \sqrt{2}q'R$ to be the length of the diagonal of the squares in $\mathcal{S}_{q'}$ now yields $|\lambda_{m,n} - \omega_{m,n}| \leqslant R'$, for all $(m,n) \in \mathcal{I}$, as desired. $\square$

Proof of Lemma 10. We again identify $\mathbb{C}$ with $\mathbb{R}^2$ for ease of exposition. Note that we have $n^+(\Lambda, (0,R')^2) \leqslant n^+(\Omega_\gamma, (0, R'+2R)^2)$, for all $R' > 0$ and $\Lambda \in \mathcal{L}$, by the uniform closeness assumption, and so

$$\mathcal{D}^+(\mathcal{L}) = \limsup_{R'\to\infty} \sup_{\Lambda\in\mathcal{L}} \frac{n^+(\Lambda, (0,R')^2)}{R'^2} \leqslant \limsup_{R'\to\infty} \frac{n^+(\Omega_\gamma, (0,R'+2R)^2)}{(R'+2R)^2} \left(1 + \frac{2R}{R'}\right)^2 = \gamma^{-2}.$$

$\square$

Proof of Lemma 14. (i) Suppose that $\mathcal{A}$ is bounded below. Then the operator $\tilde{\mathcal{A}} : X \to \mathrm{Im}(\mathcal{A})$ given by $\tilde{\mathcal{A}}(x) = \mathcal{A}(x)$, for $x \in X$, is a continuous map between Banach spaces, and has a continuous inverse. In other words, $\tilde{\mathcal{A}}$ is an isomorphism between Banach spaces. Thus $\tilde{\mathcal{A}}^* : (\mathrm{Im}(\mathcal{A}))^* \to X^*$ is also an isomorphism between Banach spaces, and so, by the inverse mapping theorem [**?**, Cor. 2.12], so is $(\tilde{\mathcal{A}}^*)^{-1} : X^* \to (\mathrm{Im}(\mathcal{A}))^*$. Consider now an arbitrary $f \in X^*$, and set $h = (\tilde{\mathcal{A}}^*)^{-1} f$. As h is a continuous linear functional on $\mathrm{Im}(\mathcal{A}) \subset Y$, it follows by the Hahn-Banach theorem [**?**, Thm. 3.6] that h can be extended to a continuous linear functional h_Y defined on Y. Now, since $h_Y |_{\mathrm{Im}(\mathcal{A})} = h$, we have

$$\langle \mathcal{A}^* h_Y, x\rangle = \langle h_Y, \underbrace{\mathcal{A}x}_{\in \mathrm{Im}(\mathcal{A})}\rangle = \langle h, \mathcal{A}x\rangle = \langle h, \tilde{\mathcal{A}}x\rangle =$$
$$= \langle \tilde{\mathcal{A}}^* h, x\rangle = \langle f, x\rangle \quad \text{for } x \in X,$$

and thus, as $x \in X$ was arbitrary, we deduce that $\mathcal{A}^* h_Y = f$. Finally, since f was arbitrary, we have that $\mathcal{A}^*$ is surjective.

(ii) Let f be an arbitrary element of X^* with $\|f\| = 1$, and let $g \in Y^*$ be such that $\mathcal{A}^* g = f$ and $a\|g\| \leqslant 1$. Note that then $g \neq 0$, and so $g/\|g\|$ is a well-defined element of Y^* of unit norm. Therefore,

$$\|\mathcal{A}x\| \geqslant \left|\left\langle \mathcal{A}x, \frac{g}{\|g\|}\right\rangle\right| = \|g\|^{-1} \left|\langle x, \mathcal{A}^* g\rangle\right| \geqslant a \left|\langle x, f\rangle\right|,$$

for all $x \in X$. Taking the supremum of the right-hand side over $f \in X^*$ and using the fact that $\sup_{f\in X^*, \|f\|=1} |\langle x, f\rangle| = \|x\|$ yields $\|\mathcal{A}x\| \geqslant a\|x\|$, as desired. $\square$