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Solutions to the
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Problem 1

(a) It follows directly from the definition of $\theta_{s,t}$ that $\theta_{s,t}$ is the smallest number $\tilde{\theta} \geq 0$ such that

$$\left| \frac{|\langle Au, Av\rangle|}{\|u\|_2 \|v\|_2} \right| \leq \tilde{\theta},$$

for all disjointly supported s-sparse and t-sparse vectors $u \in \mathbb{C}^N \backslash \{0\}$ and $v \in \mathbb{C}^N \backslash \{0\}$, respectively. Therefore,

$$\begin{aligned}
\theta_{s,t} &= \max_{\substack{u,v\in\mathbb{C}^N \text{ disjointly } s,t\text{-sparse,} \\ \|u\|_2=\|v\|_2=1}} |\langle Au, Av\rangle| \\
&= \max_{\substack{S,T\subset\{1,\dots,N\}, S\cap T=\emptyset, \\ |S|\leq s, |T|\leq t}} \max_{\substack{u\in\mathbb{C}^{|S|}, v\in\mathbb{C}^{|T|}, \\ \|u\|_2=\|v\|_2=1}} |\langle Au, Av\rangle| \\
&= \max_{\substack{S,T\subset\{1,\dots,N\}, S\cap T=\emptyset, \\ |S|\leq s, |T|\leq t}} \max_{\substack{u\in\mathbb{C}^{|S|}, v\in\mathbb{C}^{|T|}, \\ \|u\|_2=\|v\|_2=1}} |\langle A_S u, A_T v\rangle| \\
&= \max_{\substack{S,T\subset\{1,\dots,N\}, S\cap T=\emptyset, \\ |S|\leq s, |T|\leq t}} \max_{\substack{u\in\mathbb{C}^{|S|}, v\in\mathbb{C}^{|T|}, \\ \|u\|_2=\|v\|_2=1}} |\langle (A_T^H A_S)u, v\rangle|.
\end{aligned} \tag{1}$$

Note that for every $S, T \subset \{1, \dots, N\}$ with $S \cap T = \emptyset, |S| \leq s, |T| \leq t$ and $u \in \mathbb{C}^{|S|}, v \in \mathbb{C}^{|T|}$ with $\|u\|_2 = \|v\|_2 = 1$, the Cauchy–Schwarz inequality yields

$$|\langle (A_T^H A_S)u, v\rangle| \leq \|(A_T^H A_S)u\|_2 \|v\|_2 = \|(A_T^H A_S)u\|_2.$$

On the other hand, $A_T^H A_S \in \mathbb{C}^{|T|\times|S|}$ and hence $(A_T^H A_S)u \in \mathbb{C}^{|T|}$. Therefore, if $(A_T^H A_S)u \neq 0$, $v = (A_T^H A_S)u/\|(A_T^H A_S)u\|_2$ satisfies $v \in \mathbb{C}^{|T|}$ with $\|v\|_2 = 1$, and we get

$$\begin{aligned}
|\langle (A_T^H A_S)u, v\rangle| &= |\langle (A_T^H A_S)u, (A_T^H A_S)u/\|(A_T^H A_S)u\|_2\rangle| \\
&= \frac{\|(A_T^H A_S)u\|_2^2}{\|(A_T^H A_S)u\|_2} = \|(A_T^H A_S)u\|_2.
\end{aligned}$$

Hence,

$$\max_{\substack{u\in\mathbb{C}^{|S|}, v\in\mathbb{C}^{|T|}, \\ \|u\|_2=\|v\|_2=1}} |\langle (A_T^H A_S)u, v\rangle| = \max_{\substack{u\in\mathbb{C}^{|S|}, \\ \|u\|_2=1}} \|(A_T^H A_S)u\|_2 = \max_{\substack{u\in\mathbb{C}^{|S|}, \\ \|u\|_2\leq 1}} \|(A_T^H A_S)u\|_2.$$

Combining this with (1), we obtain

$$\begin{aligned}\theta_{s,t} &= \max_{\substack{S,T\subset\{1,\dots,N\},S\cap T=\emptyset,\\ |S|\le s,|T|\le t}} \max_{\substack{u\in\mathbb{C}^{|S|},\\ \|u\|_2\le 1}} \|(A_T^H A_S)u\|_2 \\ &= \max_{\substack{S,T\subset\{1,\dots,N\},S\cap T=\emptyset,\\ |S|\le s,|T|\le t}} \|A_T^H A_S\|_2.\end{aligned}$$

(b) Let $u, v \in \mathbb{C}^N$ be disjointly supported s-sparse and t-sparse vectors, respectively, let $S := \operatorname{supp}(u) \cup \operatorname{supp}(v)$, and let $u_S, v_S \in \mathbb{C}^{|S|}$ be the restrictions of $u, v \in \mathbb{C}^N$ to S. Since u and v have disjoint supports, we have $\langle u_S, v_S\rangle = 0$ and hence

$$|\langle Au, Av\rangle| = |\langle A_S u_S, A_S v_S\rangle - \langle u_S, v_S\rangle| = |\langle (A_S^H A_S - \mathrm{I}_{|S|})u_S, v_S\rangle|.$$

Applying the Cauchy–Schwarz inequality and the relation

$$\|u_S\|_2 \|A_S^H A_S - \mathrm{I}_{|S|}\|_2 = \|u_S\|_2 \max_{\substack{x\in\mathbb{C}^{|S|},\\ \|x\|_2\le 1}} \|(A_S^H A_S - \mathrm{I}_{|S|})x\|_2 \ge \|(A_S^H A_S - \mathrm{I}_{|S|})u_S\|_2,$$

we get

$$|\langle Au, Av\rangle| \le \|(A_S^H A_S - \mathrm{I}_{|S|})u_S\|_2 \|v_S\|_2 \le \|A_S^H A_S - \mathrm{I}_{|S|}\|_2 \|u_S\|_2 \|v_S\|_2.$$

Based on the lemma in the problem statement and using $\|u_S\|_2 = \|u\|_2$, $\|v_S\|_2 = \|v\|_2$, this allows us to conclude that

$$|\langle Au, Av\rangle| \le \delta_{s+t} \|u\|_2 \|v\|_2,$$

which, in turn, proves

$$\theta_{s,t} \le \delta_{s+t}.$$

(c) (i) Note that each $j \in T$ belongs to exactly s sets S_i, so that

$$u = \frac{1}{s}\sum_{i=1}^{t} u_{S_i} \quad \text{and} \quad \|u\|_2^2 = \frac{1}{s}\sum_{i=1}^{t} \|u_{S_i}\|_2^2.$$

(ii) Using (c)(i) and the triangle inequality, we get

$$\begin{aligned}|\langle Au, Av\rangle| &\le \frac{1}{s}\sum_{i=1}^{t} |\langle Au_{S_i}, Av\rangle| \le \frac{1}{s}\sum_{i=1}^{t} \theta_{s,r}\|u_{S_i}\|_2\|v\|_2 \\ &= \theta_{s,r}\,\frac{1}{s}\left(\sum_{i=1}^{t}\|u_{S_i}\|_2\right)\|v\|_2,\end{aligned} \tag{2}$$

where in the second inequality we used that u_{S_i} and v are disjointly supported s-sparse and r-sparse vectors, respectively. Moreover, note that the Cauchy-

Schwarz inequality yields

$$\left(\sum_{i=1}^{t}\|u_{S_i}\|_2\right)^2 \le \left(\sum_{i=1}^{t}\|u_{S_i}\|_2^2\right)\left(\sum_{i=1}^{t}1\right) = t\left(\sum_{i=1}^{t}\|u_{S_i}\|_2^2\right).$$

This together with (2) and subproblem (c)(i) yields

$$\begin{aligned}|\langle Au, Av\rangle| &\le \theta_{s,r}\,\frac{\sqrt{t}}{s}\left(\sum_{i=1}^{t}\|u_{S_i}\|_2^2\right)^{1/2}\|v\|_2\\ &= \theta_{s,r}\sqrt{\frac{t}{s}}\,\|u\|_2\|v\|_2.\end{aligned}$$

Problem 2

(a) (i) Let us assume that $\|x\|_{2,\infty} = 0$. This implies, that for every $M > 0$,

$$\operatorname{card}\big(\{j \in \{1, \ldots, N\} \colon |x_j| \geq t\}\big) \leq \frac{M^2}{t^2}, \text{ for all } t > 0.$$

Hence, for every $t > 0$,

$$\operatorname{card}\big(\{j \in \{1, \ldots, N\} \colon |x_j| \geq t\}\big) \leq \frac{M^2}{t^2}, \text{ for all } M > 0.$$

Consequently, by choosing $M > 0$ but arbitrarily small, we get

$$\operatorname{card}\big(\{j \in \{1, \ldots, N\} \colon |x_j| \geq t\}\big) = 0, \tag{3}$$

for every $t > 0$. Now, taking t in (3) arbitrarily small, we can conclude that $x = 0$.

(ii) The statement is obvious for $\lambda = 0$. Hence, we can assume that $\lambda \neq 0$. Now, observe that

$$\{j \in \{1, \ldots, N\} \colon |\lambda x_j| \geq t\} = \{j \in \{1, \ldots, N\} \colon |x_j| \geq t/|\lambda|\}.$$

Therefore,

$$\begin{aligned}
\|\lambda x\|_{2,\infty} &= \inf\left\{ M \geq 0 \colon \operatorname{card}\big(\{j \in \{1, \ldots, N\} \colon |\lambda x_j| \geq t\}\big) \leq \frac{M^2}{t^2}, \text{ for all } t > 0 \right\} \\
&= \inf\left\{ M \geq 0 \colon \operatorname{card}\big(\{j \in \{1, \ldots, N\} \colon |x_j| \geq t/|\lambda|\}\big) \leq \frac{M^2}{t^2}, \text{ for all } t > 0 \right\} \\
&= \inf\left\{ M \geq 0 \colon \operatorname{card}\big(\{j \in \{1, \ldots, N\} \colon |x_j| \geq t\}\big) \leq \frac{M^2}{|\lambda|^2 t^2}, \text{ for all } t > 0 \right\} \\
&= |\lambda| \inf\left\{ M \geq 0 \colon \operatorname{card}\big(\{j \in \{1, \ldots, N\} \colon |x_j| \geq t\}\big) \leq \frac{M^2}{t^2}, \text{ for all } t > 0 \right\} \\
&= |\lambda| \|x\|_{2,\infty}.
\end{aligned}$$

(b) (i) Note that

$$\operatorname{card}\big(\{j \in \{1, 2\} \colon |x_j| \geq t\}\big) = \begin{cases} 2, & \text{if} \quad t \leq 2^{-1/2}, \\ 1, & \text{if} \quad 2^{-1/2} < t \leq 1, \\ 0, & \text{if} \quad t > 1. \end{cases}$$

Therefore, for all $t > 0$,

$$\operatorname{card}\big(\{j \in \{1, 2\} \colon |x_j| \geq t\}\big) \leq \frac{1}{t^2}.$$

On the other hand,

$$\operatorname{card}\big(\{j \in \{1, 2\} \colon |x_j| \geq 1\}\big) = 1,$$

and hence, we get

$$\|x\|_{2,\infty} = \inf\left\{ M \geq 0\colon\ \operatorname{card}\bigl(\{j \in \{1,2\}\colon |x_j| \geq t\}\bigr) \leq \frac{M^2}{t^2},\ \text{ for all } t > 0 \right\} = 1.$$

Next, using

$$\operatorname{card}\bigl(\{j \in \{1,2\}\colon |x_j| \geq t\}\bigr) = \operatorname{card}\bigl(\{j \in \{1,2\}\colon |y_j| \geq t\}\bigr),\ \text{ for all } t > 0,$$

yields

$$\|y\|_{2,\infty} = \|x\|_{2,\infty} = 1.$$

(ii) Thanks to (a)(ii), we have

$$\|x+y\|_{2,\infty} = \|(1+2^{-1/2}, 1+2^{-1/2})\|_{2,\infty} = (1+2^{-1/2})\|(1,1)\|_{2,\infty}.$$

We are therefore left with having to calculate $\|(1,1)\|_{2,\infty}$. Let us fix $z := (1,1)$. Then

$$\operatorname{card}\bigl(\{j \in \{1,2\}\colon |z_j| \geq t\}\bigr) = \begin{cases} 2, & \text{if } \quad t \leq 1, \\ 0, & \text{if } \quad t > 1, \end{cases}$$

and hence

$$\|z\|_{2,\infty} = \inf\left\{ M \geq 0\colon\ \operatorname{card}\bigl(\{j \in \{1,2\}\colon |z_j| \geq t\}\bigr) \leq \frac{M^2}{t^2},\ \text{ for all } t > 0 \right\} = 2^{1/2}.$$

Consequently, we get

$$\|x+y\|_{2,\infty} = 2^{1/2}(1+2^{-1/2}) = 2^{1/2}+1 > 2 = \|x\|_{2,\infty} + \|y\|_{2,\infty}.$$

(c) (i) If $|x_j^1 + \cdots + x_j^k| \geq t$ for some $j \in \{1,\ldots,N\}$, then we have that $|x_j^i| \geq t/k$ for this j and some $i \in \{1,\ldots,k\}$. This allows us to conclude that

$$\bigl\{j \in \{1,\ldots,N\}\colon |x_j^1 + \cdots + x_j^k| \geq t\bigr\} \subset \bigcup_{i \in \{1,\ldots,k\}} \bigl\{j \in \{1,\ldots,N\}\colon |x_j^i| \geq t/k\bigr\}.$$

(ii) From (c)(i) we get that

$$\begin{aligned} &\operatorname{card}\Bigl(\bigl\{j \in \{1,\ldots,N\}\colon |x_j^1 + \cdots + x_j^k| \geq t\bigr\}\Bigr) \\ &\leq \sum_{i \in \{1,\ldots,k\}} \operatorname{card}\Bigl(\bigl\{j \in \{1,\ldots,N\}\colon |x_j^i| \geq t/k\bigr\}\Bigr) \\ &\leq \sum_{i \in \{1,\ldots,k\}} \frac{\|x^i\|_{2,\infty}^2}{(t/k)^2} \\ &= \frac{k^2\bigl(\|x^1\|_{2,\infty}^2 + \cdots + \|x^k\|_{2,\infty}^2\bigr)}{t^2}. \end{aligned}$$

We therefore obtain

$$\|x^1 + \cdots + x^k\|_{2,\infty} \leq k\big(\|x^1\|_{2,\infty}^2 + \cdots + \|x^k\|_{2,\infty}^2\big)^{1/2}.$$

(iii) We have

$$\begin{aligned}
&\frac{\big(\|x^1\|_{2,\infty}^2 + \cdots + \|x^k\|_{2,\infty}^2\big)}{\big(\|x^1\|_{2,\infty} + \cdots + \|x^k\|_{2,\infty}\big)^2} \\
&= \left(\frac{\|x^1\|_{2,\infty}}{\big(\|x^1\|_{2,\infty} + \cdots + \|x^k\|_{2,\infty}\big)}\right)^2 + \cdots + \left(\frac{\|x^k\|_{2,\infty}}{\big(\|x^1\|_{2,\infty} + \cdots + \|x^k\|_{2,\infty}\big)}\right)^2 \\
&\leq \frac{\|x^1\|_{2,\infty}}{\big(\|x^1\|_{2,\infty} + \cdots + \|x^k\|_{2,\infty}\big)} + \cdots + \frac{\|x^k\|_{2,\infty}}{\big(\|x^1\|_{2,\infty} + \cdots + \|x^k\|_{2,\infty}\big)} = 1,
\end{aligned}$$

and hence

$$\big(\|x^1\|_{2,\infty}^2 + \cdots + \|x^k\|_{2,\infty}^2\big)^{1/2} \leq \|x^1\|_{2,\infty} + \cdots + \|x^k\|_{2,\infty}. \tag{4}$$

Combining this with (c)(ii), we obtain

$$\|x^1 + \cdots + x^k\|_{2,\infty} \leq k\big(\|x^1\|_{2,\infty} + \cdots + \|x^k\|_{2,\infty}\big).$$

(d) For every $k \in \{1, \ldots, N\}$, we can write

$$\|x\|_2^2 = \sum_{j=1}^{N} (x_j^*)^2 \geq \sum_{j=1}^{k} (x_j^*)^2 \geq k(x_k^*)^2.$$

Raising to the power $1/2$ and taking the maximum over k yields the desired result.

Problem 3

(a) (i) We take two indices $i \neq j$ and assume that there exists at least one element $\theta \in \mathbb{H}^d$ in the intersection $\mathbb{H}_i \cap \mathbb{H}_j$. By the triangle inequality and the definition of $\mathbb{H}_i$ and $\mathbb{H}_j$, we have

$$d_H(\theta^i, \theta^j) \leq d_H(\theta^i, \theta) + d_H(\theta^j, \theta) \leq \varepsilon/2 + \varepsilon/2 \leq \varepsilon.$$

However, by definition of an ε-packing, we must have $d_H(\theta^i, \theta^j) > \varepsilon$, which results in a contradiction and thereby concludes the proof of the sets $\{\mathbb{H}_i\}_{i=1}^M$ being disjoint.

(ii) For a given integer $k \in [0, d]$, the points in $\mathbb{H}^d$ that are at distance k/d of θ^i are exactly the θ obtained by flipping k different coordinates of θ^i. We can therefore observe that there are exactly $\binom{d}{k}$ points at distance k/d from θ^i. Summing over all the integers k such that $k/d \leq \varepsilon/2$ or, equivalently, $k \leq d\varepsilon/2$, one obtains

$$|\mathbb{H}_i| = \sum_{k=0}^{n} \binom{d}{k},$$

with $n = \lfloor d\varepsilon/2 \rfloor$.

(iii) We argue as follows:

$$\begin{aligned}
\log M(\varepsilon; \mathbb{H}^d, d_H) + \log \binom{d}{n} &= \log \left[M(\varepsilon; \mathbb{H}^d, d_H) \binom{d}{n} \right] \\
&\leq \log \left[M(\varepsilon; \mathbb{H}^d, d_H) \sum_{k=0}^{n} \binom{d}{k} \right] \\
&= \log \left[\sum_{i=1}^{M} \sum_{k=0}^{n} \binom{d}{k} \right] \\
&\overset{(a)(ii)}{=} \log \left[\sum_{i=1}^{M} |\mathbb{H}_i| \right] \\
&\overset{(b)(i)}{\leq} \log |\mathbb{H}^d| = \log 2^d = d \log 2,
\end{aligned}$$

where we used the shorthand M for $M(\varepsilon; \mathbb{H}^d, d_H)$. The last inequality comes from the fact that the $\mathbb{H}_i$ are disjoint subsets of $\mathbb{H}^d$, so that the sum of their cardinalities is bounded by the cardinality of $\mathbb{H}^d$. Rearranging terms yields the desired result according to

$$\log M(\varepsilon; \mathbb{H}^d, d_H) \leq d \log 2 - \log \binom{d}{n}. \tag{5}$$

(b) (i) Following the hint, we note that

$$\begin{aligned}\frac{\mathbb{P}[Y=\ell]}{\mathbb{P}[Y=\ell-1]} &= \frac{\binom{d}{\ell}(n/d)^{\ell}(1-n/d)^{d-\ell}}{\binom{d}{\ell-1}(n/d)^{\ell-1}(1-n/d)^{d-\ell+1}} \\ &= \frac{d-\ell+1}{\ell}\frac{n/d}{1-n/d} \\ &= \frac{d-\ell+1}{d-n}\ \frac{n}{\ell}. \end{aligned} \tag{6}$$

Both of the fractions in (6) are larger than 1 for $\ell \le n$ and smaller than 1 for $\ell > n$. This means that $\mathbb{P}[Y=\ell]$ is maximized at $\ell = n$, which is the desired result.

(ii) Following the hint, we have

$$\begin{aligned} 1 &= \sum_{\ell=0}^{d} \mathbb{P}[Y=\ell] \\ &\overset{(b)(i)}{\le} (d+1)\mathbb{P}[Y=n] \\ &= (d+1)\binom{d}{n}(n/d)^n(1-n/d)^{d-n}. \end{aligned}$$

Taking the logarithm, we obtain

$$\begin{aligned} \log\binom{d}{n} &\ge -\log\left\{(d+1)(n/d)^n(1-n/d)^{d-n}\right\} \\ &= -n\log(n/d)-(d-n)\log(1-n/d)-\log(d+1) \\ &= d\left\{-(n/d)\log(n/d)-(1-n/d)\log(1-n/d)\right\}-\log(d+1) \\ &= d\,\phi(n/d)-\log(d+1), \end{aligned}$$

which is the desired result.

(c) We argue as follows:

$$\begin{aligned} \frac{\log M(\varepsilon;\mathbb{H}^d,d_H)}{d} &\overset{(5)}{\le} \log 2 - \frac{\log\binom{d}{n}}{d} \\ &\overset{(b)(ii)}{\le} \log 2 - \phi(n/d) + \frac{\log(d+1)}{d} \\ &= (n/d)\log 2 + (1-n/d)\log 2 + (n/d)\log(n/d) \\ &\quad + (1-n/d)\log(1-n/d) + \frac{\log(d+1)}{d} \\ &= (n/d)\log\left(\frac{n/d}{1/2}\right) + (1-n/d)\log\left(\frac{1-n/d}{1/2}\right) + \frac{\log(d+1)}{d} \\ &= D\big((n/d)\|(1/2)\big) + \frac{\log(d+1)}{d}. \end{aligned}$$

Problem 4

(a) Consider the points $x_1 = 0$ and $x_2 = 1$ in $\mathbb{R}$. The four possible labelings

$$\begin{cases} h_{(-1,-1)}(x_1) = 0 & \text{and} \quad h_{(-1,-1)}(x_2) = 0, \\ h_{(0,0)}(x_1) = 1 & \text{and} \quad h_{(0,0)}(x_2) = 0, \\ h_{(1,1)}(x_1) = 0 & \text{and} \quad h_{(1,1)}(x_2) = 1, \\ h_{(0,1)}(x_1) = 1 & \text{and} \quad h_{(0,1)}(x_2) = 1, \end{cases}$$

are produced by $\mathcal{H}_1$. Therefore, there is a set of 2 points shattered by $\mathcal{H}_1$, which implies $\dim_{VC}(\mathcal{H}_1) \geq 2$.

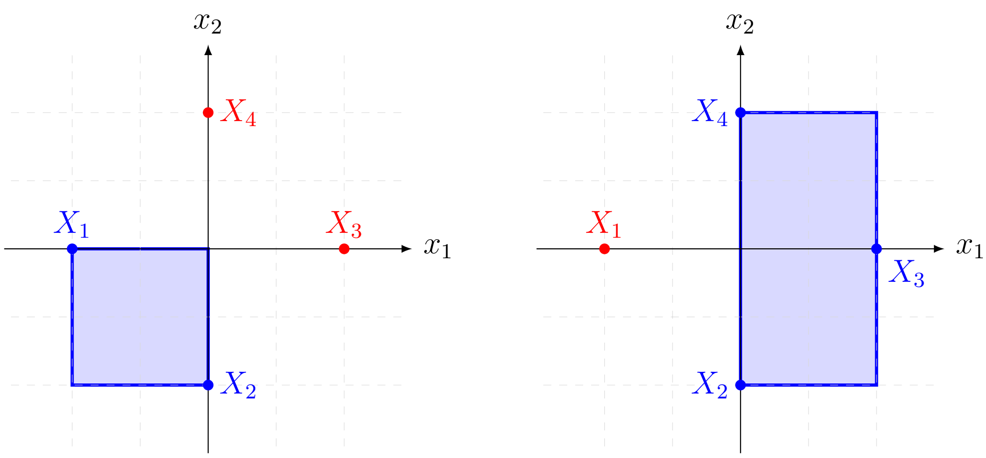
On the other hand, for any set of three distinct points x_1, x_2, and x_3 that we choose without loss of generality such that $x_1 < x_2 < x_3$, there is no closed interval containing x_1 and x_3 but not x_2. Therefore, there is no set of 3 points shattered by $\mathcal{H}_1$, which implies $\dim_{VC}(\mathcal{H}_1) < 3$.

We have therefore proven that $\dim_{VC}(\mathcal{H}_1) = 2$.

(b) (i) The 4 points $X_1 = (-1, 0)$, $X_2 = (0, -1)$, $X_3 = (1, 0)$, and $X_4 = (0, 1)$ are shattered by $\mathcal{H}_2$. To see this, we fix a labeling $(y_1, y_2, y_3, y_4) \in \{0, 1\}^4$. The rectangle $h_{(-y_1,-y_2,y_3,y_4)} \in \mathcal{H}_2$ labels correctly all four points:

$$\begin{cases} h_{(-y_1,-y_2,y_3,y_4)}(X_1) = y_1, \\ h_{(-y_1,-y_2,y_3,y_4)}(X_2) = y_2, \\ h_{(-y_1,-y_2,y_3,y_4)}(X_3) = y_3, \\ h_{(-y_1,-y_2,y_3,y_4)}(X_4) = y_4. \end{cases}$$

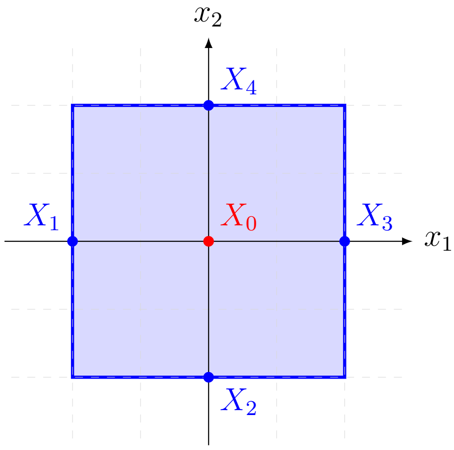
The figures below picture the rectangle $h_{(-1,-1,0,0)}$ (left) and the rectangle $h_{(0,-1,1,1)}$ (right).



(ii) Assume that we can find a_1, b_1, a_2, and b_2 such that

$$\begin{cases} h_{(a_1,a_2,b_1,b_2)}(0,0) = 0, \\ h_{(a_1,a_2,b_1,b_2)}(-1,0) = h_{(a_1,a_2,b_1,b_2)}(1,0) = 1, \\ h_{(a_1,a_2,b_1,b_2)}(0,-1) = h_{(a_1,a_2,b_1,b_2)}(0,1) = 1. \end{cases}$$

Since $h_{(a_1,a_2,b_1,b_2)}(-1,0) = h_{(a_1,a_2,b_1,b_2)}(1,0) = 1$, we must have $a_1 \leq -1$ and $1 \leq b_1$, which implies $a_1 \leq 0 \leq b_1$. Likewise, since $h_{(a_1,a_2,b_1,b_2)}(0,-1) = h_{(a_1,a_2,b_1,b_2)}(0,1) = 1$, we must have $a_2 \leq -1$ and $1 \leq b_2$, which implies $a_2 \leq 0 \leq b_2$. The inequalities $a_1 \leq 0 \leq b_1$ and $a_2 \leq 0 \leq b_2$ together imply $h_{(a_1,a_2,b_1,b_2)}(0,0) = 1$, which yields a contradiction, thereby concluding the proof.



(iii) Take any set of five distinct points in $\mathbb{R}^2$. We call X_1 the "leftmost" point (the point of smallest first coordinate x_1^-), X_2 the "lowest" point (the point of smallest second coordinate x_2^-), X_3 the "rightmost" point (the point of largest first coordinate x_1^+) and X_4 the "highest" point (the point of largest second coordinate x_2^+). Note that these extremal points X_1, X_2, X_3, and X_4 are not necessarily distinct (e.g., there could be a point of both largest first and largest second coordinates, i.e., $X_3 = X_4$). We consider the labeling y that assigns 1 to X_1, X_2, X_3 and X_4 and 0 to a point X_0 distinct from X_1, X_2, X_3, and X_4 (which exists since we consider 5 points in total). If it exists, a rectangle $h_{(a_1,a_2,b_1,b_2)}$ realizing the desired labeling has to be such that $a_1 \leq x_1^-$, $x_1^+ \leq b_1$, $a_2 \leq x_2^-$, and $x_2^+ \leq b_2$. Since by construction of X_1, X_2, X_3 and X_4, we must have $x_1^- \leq x_1^0 \leq x_1^+$ and $x_2^- \leq x_2^0 \leq x_2^+$, we necessarily get $a_1 \leq x_1^0 \leq b_1$ and $a_2 \leq x_2^0 \leq b_2$, which does not produce the correct labeling for the point X_0. This contradiction proves that there is no rectangle $h_{(a_1,a_2,b_1,b_2)}$ yielding the desired labelling y. Therefore, no set of 5 points can be shattered by $\mathcal{H}_2$.

(iv) The class $\mathcal{H}_2$ shatters *a* set of 4 points but does not shatter *any* set of 5 points. By definition of VC dimension, we therefore have $\dim_{VC}(\mathcal{H}_2) = 4$.