Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Dept. ITET
Prof. Dr. Helmut Bölcskei

Problem solutions
"Mathematics of Information"
August 29, 2018

1. Problem 1

(a) A unitary operator on a general Hilbert space $\mathcal{X}$ is a bounded linear operator $U : \mathcal{X} \to \mathcal{X}$ that is invertible and satisfies $U^{-1} = U^*$.

(b)(i) Let F be a function in the space $L^2(\mathbb{R}^2)$. Then we have

$$\begin{aligned}
\|\mathcal{T}_a F\|^2_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} |F(y, y-x)|^2 \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(y, y-x)|^2 \,\mathrm{d}x \right) \mathrm{d}y \\
&\overset{x \mapsto y-v}{=} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(y, y-(y-v))|^2 \,\mathrm{d}v \right) \mathrm{d}y \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F(y, v)|^2 \,\mathrm{d}y \right) \mathrm{d}v = \|F\|^2_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

Thus $\mathcal{T}_a$ is a well-defined map on $L^2(\mathbb{R}^2)$. Now let F and G be two $L^2(\mathbb{R}^2)$ functions. Then

$$\begin{aligned}
\langle \mathcal{T}_a F, G \rangle &= \int_{\mathbb{R}^2} F(y, y-x)\overline{G(x,y)} \,\mathrm{d}x\,\mathrm{d}y \\
&\overset{x \mapsto y-v}{=} \int_{\mathbb{R}^2} F(y, v)\overline{G(y-v,y)} \,\mathrm{d}v\,\mathrm{d}y \\
&= \langle F, \mathcal{T}_a^* G \rangle,
\end{aligned}$$

where $\mathcal{T}_a^* G(u, v) = G(u - v, u)$. Now for any $F \in L^2(\mathbb{R}^2)$ and $(x, y) \in \mathbb{R}^2$ we have

$$\begin{aligned}
\mathcal{T}_a{}^* \mathcal{T}_a F(x, y) &= (\mathcal{T}_a F)(x-y, x) = F(x, x-(x-y)) = F(x, y), \\
\mathcal{T}_a \mathcal{T}_a{}^* F(x, y) &= (\mathcal{T}_a^* F)(y, y-x) = F(y-(y-x), y) = F(x, y).
\end{aligned}$$

Therefore $\mathcal{T}_a{}^* \mathcal{T}_a = \mathcal{T}_a \mathcal{T}_a{}^* = \mathrm{Id}$ and hence $\mathcal{T}_a$ is invertible with inverse $\mathcal{T}_a{}^*$.

(ii) Take any $f, g \in L^2(\mathbb{R})$. First note that $f \otimes \overline{g} \in L^2(\mathbb{R}^2)$, and thus also $\mathcal{T}_a(f \otimes \overline{g}) \in L^2(\mathbb{R}^2)$. Therefore $\mathcal{T}_a(f \otimes \overline{g})$ lies in the domain of $\mathcal{F}_2$ and so $\mathcal{F}_2 \mathcal{T}_a(f \otimes \overline{g})$ is well defined. Now note that $(\mathcal{T}_a(f \otimes \overline{g}))(x, t) = f(t)\overline{g(t-x)}$. Applying the Cauchy-Schwarz inequality for an arbitrary, but fixed $x \in \mathbb{R}$, yields

$$\begin{aligned}
\int_{\mathbb{R}} |f(t)\overline{g(t-x)}| \mathrm{d}t &= \langle |f|, |g(\cdot - x)| \rangle \\
&\le \|f\|_{L^2(\mathbb{R})} \|g(\cdot - x)\|_{L^2(\mathbb{R})} \\
&= \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} < \infty,
\end{aligned}$$

where in the last step we used $f, g \in L^2(\mathbb{R})$. Therefore $(\mathcal{T}_a(f \otimes \overline{g}))(x, \cdot) \in L^1(\mathbb{R})$, for all $x \in \mathbb{R}$, and thus the partial Fourier transform formula applies to $\mathcal{T}_a(f \otimes \overline{g})$. Finally, for any $(x, \omega) \in \mathbb{R}^2$ we have

$$\begin{aligned} \mathcal{F}_2\mathcal{T}_a(f \otimes \overline{g})(x, \omega) &= \int_{\mathbb{R}} (\mathcal{T}_a(f \otimes \overline{g}))(x, t) e^{-2\pi i \omega t} \mathrm{d}t \\ &= \int_{\mathbb{R}} f(t) \overline{g(t-x)} e^{-2\pi i \omega t} \mathrm{d}t = (V_g f)(x, \omega), \end{aligned}$$

as desired.

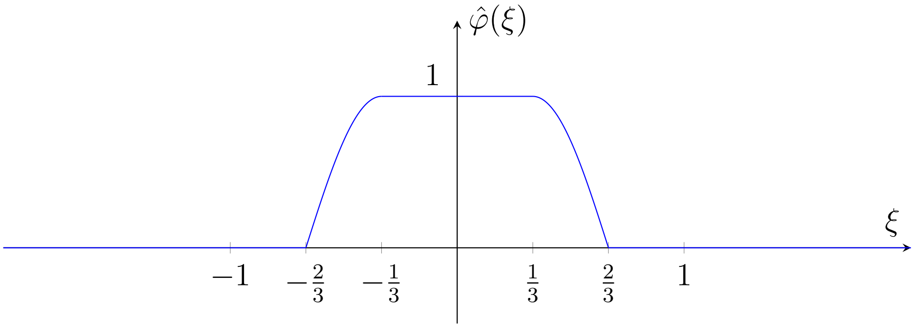
(c) Now, $V_g f \in L^2(\mathbb{R}^2)$ for $f, g \in L^2(\mathbb{R})$ follows since $f \otimes \overline{g} \in L^2(\mathbb{R}^2)$ for such f and g, and $\mathcal{T}_a$ and $\mathcal{F}_2$ are well-defined operators mapping $L^2(\mathbb{R}^2)$ functions to $L^2(\mathbb{R}^2)$ functions. Since $\mathcal{T}_a$ and $\mathcal{F}_2$ are unitary, we have

$$\begin{aligned} \langle V_{g_1} f_1, V_{g_2} f_2 \rangle &= \langle \mathcal{F}_2\mathcal{T}_a(f_1 \otimes \overline{g_1}), \mathcal{F}_2\mathcal{T}_a(f_2 \otimes \overline{g_2}) \rangle \\ &= \langle \mathcal{T}_a(f_1 \otimes \overline{g_1}), \mathcal{T}_a(f_2 \otimes \overline{g_2}) \rangle \\ &= \langle f_1 \otimes \overline{g_1}, f_2 \otimes \overline{g_2} \rangle \\ &= \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}, \end{aligned}$$

for all $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R})$.

Problem 2

(a)(i)



(ii) It is sufficient to verify that $\sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi+n)|^2 = 1$, for all $\xi \in \mathbb{R}$. Due to the 1-periodicity of $\sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi+n)|^2$, it suffices to verify this on an interval of length 1, for instance $[-\frac{1}{3}, \frac{2}{3}]$. Equality clearly holds when $\xi \in [-\frac{1}{3}, \frac{1}{3}]$, as in this case one summand evaluates to 1, and all the remaining ones evaluate to 0. When $\xi \in [\frac{1}{3}, \frac{2}{3}]$, we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi+n)|^2 &= \cos^2\Big(\frac{\pi}{2}(3\xi-1)\Big) + \cos^2\Big(\frac{\pi}{2}(3(1-\xi)-1)\Big)^2 \\ &= \sin^2\left(\frac{3\pi}{2}\xi\right) + \cos^2\left(\frac{3\pi}{2}\xi\right) = 1, \end{aligned} \tag{1}$$

as desired.

(b)(i) First calculate

$$\begin{aligned}\widehat{\varphi_{j,k}}(\xi) &= \int_{\mathbb{R}} 2^{\frac{j}{2}} \varphi(2^j x - k) e^{-2\pi i \xi x}\, \mathrm{d}x \\ &\overset{x\mapsto 2^{-j}(y+k)}{=} \int_{\mathbb{R}} 2^{\frac{j}{2}} \varphi(y) e^{-2\pi i \xi 2^{-j} y} e^{-2\pi i \xi 2^{-j} k} 2^{-j}\, \mathrm{d}y \\ &= 2^{-\frac{j}{2}} e^{\frac{-2\pi i \xi k}{2^j}} \int_{\mathbb{R}} \varphi(y) e^{-2\pi i (\frac{\xi}{2^j}) y}\, \mathrm{d}y \\ &= 2^{-\frac{j}{2}} \hat{\varphi}\left(\frac{\xi}{2^j}\right) e^{\frac{-2\pi i \xi k}{2^j}}.\end{aligned}$$

Now we have

$$\begin{aligned}
&\|P_{V_j}f\|^2_{L^2(\mathbb{R})} + \|P_{V_j}(f(\cdot - 2^{-(j+1)}))\|^2_{L^2(\mathbb{R})} = \\
=&\sum_{k\in\mathbb{Z}}\left|\int_{\mathbb{R}} f(x)\cdot\overline{\varphi_{j,k}(x)}\mathrm{d}x\right|^2 + \sum_{k\in\mathbb{Z}}\left|\int_{\mathbb{R}} f(x-2^{-(j+1)})\cdot\overline{\varphi_{j,k}(x)}\mathrm{d}x\right|^2 \\
=&\sum_{k\in\mathbb{Z}}\left|\int_{\mathbb{R}} \hat{f}(\xi)\cdot 2^{-\frac{j}{2}}\hat{\varphi}\left(\frac{\xi}{2^j}\right)e^{\frac{2\pi i\xi k}{2^j}}\mathrm{d}\xi\right|^2 + \sum_{k\in\mathbb{Z}}\left|\int_{\mathbb{R}} \hat{f}(\xi)e^{-\frac{2\pi i\xi}{2^{j+1}}}\cdot 2^{-\frac{j}{2}}\hat{\varphi}\left(\frac{\xi}{2^j}\right)e^{\frac{2\pi i\xi k}{2^j}}\mathrm{d}\xi\right|^2 \qquad (2)\\
\overset{k=m/2}{=}&\sum_{\substack{m\in\mathbb{Z}\\ m \text{ even}}}\left\{\left|\int_{\mathbb{R}} \hat{f}(\xi)\cdot 2^{-\frac{j}{2}}\hat{\varphi}\left(\frac{\xi}{2^j}\right)e^{\frac{2\pi i\xi m}{2^{j+1}}}\mathrm{d}\xi\right|^2 + \left|\int_{\mathbb{R}} \hat{f}(\xi)2^{-\frac{j}{2}}\hat{\varphi}\left(\frac{\xi}{2^j}\right)e^{\frac{2\pi i\xi (m-1)}{2^{j+1}}}\mathrm{d}\xi\right|^2\right\} \\
=&\sum_{m\in\mathbb{Z}}\left|\int_{\mathbb{R}} \hat{f}(\xi)\hat{\varphi}\left(\frac{\xi}{2^j}\right)2^{-\frac{j}{2}}e^{\frac{2\pi i\xi m}{2^{j+1}}}\mathrm{d}\xi\right|^2 \qquad (3)\\
=&\sum_{m\in\mathbb{Z}}\left|\int_{-2^j}^{2^j} \hat{f}(\xi)\hat{\varphi}\left(\frac{\xi}{2^j}\right)2^{\frac{1}{2}}2^{-\frac{j+1}{2}}e^{\frac{2\pi i\xi m}{2^{j+1}}}\mathrm{d}\xi\right|^2 \qquad (4)\\
=&2\left\|\hat{\varphi}(2^{-j}\cdot\,)\hat{f}\right\|^2_{L^2([-2^j,2^j])} \qquad (5)\\
=&2\left\|\hat{\varphi}(2^{-j}\cdot\,)\hat{f}\right\|^2_{L^2(\mathbb{R})}, \qquad (6)
\end{aligned}$$

where we used the Plancherel identity in (2), (4) and (6) hold since $\operatorname{supp}\hat{\varphi}(2^{-j}\cdot\,) \subset [-2^j, 2^j]$, and (5) follows since $\{e_m(\xi) = 2^{-\frac{j+1}{2}}e^{-\frac{2\pi i\xi m}{2^{j+1}}} : m \in \mathbb{Z}\}$ is an orthonormal basis for $L^2([-2^j, 2^j])$.

(ii) Since $\|\hat{\varphi}(2^{-j}\cdot)\hat{f} - \hat{f}\|_{L^2(\mathbb{R}^2)} \to 0$ as $j \to \infty$, we have $\lim_{j\to\infty}\|\hat{\varphi}(2^{-j}\cdot\,)\hat{f}\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$. Now we have

$$\begin{aligned}
\liminf_{j\to\infty}\|P_{V_j}f\|^2_{L^2(\mathbb{R})} &= \liminf_{j\to\infty}\left[2\|\hat{\varphi}(2^{-j}\cdot\,)\hat{f}\|^2_{L^2(\mathbb{R})} - \|P_{V_j}(f(\cdot - 2^{-(j+1)}))\|^2_{L^2(\mathbb{R})}\right] \\
&= 2\|f\|^2_{L^2(\mathbb{R})} - \limsup_{j\to\infty}\|P_{V_j}f(\cdot - 2^{-(j+1)}))\|^2_{L^2(\mathbb{R})} \\
&\geq 2\|f\|^2_{L^2(\mathbb{R})} - \limsup_{j\to\infty}\|f(\cdot - 2^{-(j+1)}))\|^2_{L^2(\mathbb{R})} \qquad (7)\\
&= 2\|f\|^2_{L^2(\mathbb{R})} - \limsup_{j\to\infty}\|f\|^2_{L^2(\mathbb{R})} \\
&= \|f\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

where in (7) we used the fact that projections are norm-bounded by 1. On the other hand $\|P_{V_j}f\|_{L^2(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})}$ for all $j \in \mathbb{Z}$, so $\|P_{V_j}f\|_{L^2(\mathbb{R})} \to \|f\|_{L^2(\mathbb{R})}$ as $j \to \infty$. We can now finally deduce

$$\|f - P_{V_j}f\|^2_{L^2(\mathbb{R})} = \|f\|^2_{L^2(\mathbb{R})} - \underbrace{\langle f - P_{V_j}f, P_{V_j}f\rangle}_{=0} - \|P_{V_j}f\|^2_{L^2(\mathbb{R})} \to 0, \text{ as } j \to \infty, \qquad (8)$$

where $\langle f - P_{V_j}f, P_{V_j}f\rangle = 0$ follows by the properties ${P_{V_j}}^2 = P_{V_j}$ and $P^*_{V_j} = P_{V_j}$.

(iii) We have

$$\begin{aligned}\|P_{V_j}f\|^2_{L^2(\mathbb{R})} &\le \|P_{V_j}f\|^2_{L^2(\mathbb{R})} + \|P_{V_j}(f(\cdot - 2^{-(j+1)}))\|^2_{L^2(\mathbb{R})} \\ &= 2\left\|\hat{\varphi}(2^{-j}\,\cdot\,)\hat{f}\right\|^2_{L^2(\mathbb{R})} \to 0 \quad \text{as } j \to -\infty, \qquad (9)\end{aligned}$$

where the last step follows from the assumption in the problem statement.

(c) We verify the four conditions in the definition of the multiresolution approximation:

(I) Let $f \in \bigcap_{j\in\mathbb{Z}} V_j$, i.e., $f \in V_j$ for all $j \in \mathbb{Z}$. Then $f = P_{V_j}f$ for all $j \in \mathbb{Z}$, so by (b)(iii) we have $\|f\|_{L^2(\mathbb{R})} = \|P_{V_j}f\|_{L^2(\mathbb{R})} \to 0$ as $j \to -\infty$, and hence $\|f\|_{L^2(\mathbb{R})} = 0$. Therefore $\bigcap_{j\in\mathbb{Z}} V_j = 0$, which implies $f = 0$.
Now, let $f \in L^2(\mathbb{R})$ be arbitrary. Then by (b)(ii) we have that for any $\epsilon > 0$ we can find a $j_\epsilon \in \mathbb{Z}$ such that $\|f - P_{V_{j_\epsilon}}f\|_{L^2(\mathbb{R})} < \epsilon$. Since $P_{V_{j_\epsilon}}f \in V_{j_\epsilon} \subset \bigcup_{j\in\mathbb{Z}} V_j$, and ϵ was arbitrary, we deduce that $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$.

(II) Let $f \in L^2(\mathbb{R})$. By definition of V_j, we have $f \in V_j$ if and only if $f(2^{-j}\cdot) \in V_0$. But $f(2^{-j}\cdot) = f(2^{-(j+1)}(2\cdot))$, so $f \in V_j$ is equivalent to $f(2^{-(j+1)}(2\,\cdot)) \in V_0$. Now, by definition of V_{j+1}, we see that $f(2^{-(j+1)}(2\,\cdot)) \in V_0$ is equivalent to $f(2\,\cdot) \in V_{j+1}$. Therefore, we have shown that $f \in V_j$ if and only if $f(2\,\cdot) \in V_{j+1}$.

(III) Since $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0, we can expand an arbitrary $f \in V_0$ according to

$$f = \sum_{l\in\mathbb{Z}} a_l\,\varphi(\cdot - l),$$

for some $\{a_l\}_{l\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. Then, we have

$$f(\cdot - k) = \sum_{l\in\mathbb{Z}} a_l\,\varphi(\cdot - k - l) \overset{l\mapsto l+k}{=} \sum_{l\in\mathbb{Z}} a_{l+k}\,\varphi(\cdot - l),$$

and so $f(\cdot - k) \in V_0$, for all $k \in \mathbb{Z}$ and $f \in V_0$.

(IV) Since $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0, it is, in particular, a Riesz basis for V_0.

Problem 3

(a) Let $f, g \in L^2(\mathbb{R})$ and $x, y \in \mathbb{R}$ be arbitrary. We have the following reconstruction formula:

$$\begin{aligned}
f_{-x,-y} &= \sum_{m,n\in\mathbb{Z}} \langle f_{-x,-y}, \tilde{g}_{mT,nF}\rangle\, g_{mT,nF} \\
&= \sum_{m,n\in\mathbb{Z}} \langle f_{-x,0}, \tilde{g}_{mT,nF+y}\rangle\, g_{mT,nF} \\
&= \sum_{m,n\in\mathbb{Z}} \langle f, e^{-2\pi i(nF+y)x}\, \tilde{g}_{mT+x,nF+y}\rangle\, g_{mT,nF}.
\end{aligned}$$

Now we take the inner product of both sides of this equation with $h_{-x,-y}$ to obtain

$$\begin{aligned}
\langle f_{-x,-y}, h_{-x,-y}\rangle &= \sum_{m,n\in\mathbb{Z}} \langle f, e^{-2\pi i(nF+y)x}\, \tilde{g}_{mT+x,nF+y}\rangle \langle g_{mT,nF}, h_{-x,-y}\rangle \\
&= \sum_{m,n\in\mathbb{Z}} \langle f, e^{-2\pi i(nF+y)x}\, \tilde{g}_{mT+x,nF+y}\rangle \langle g_{mT,nF+y}, h_{-x,0}\rangle \\
&= \sum_{m,n\in\mathbb{Z}} \langle f, e^{-2\pi i(nF+y)x}\, \tilde{g}_{mT+x,nF+y}\rangle \langle e^{-2\pi i(nF+y)x}\, g_{mT+x,nF+y}, h\rangle \\
&= \sum_{m,n\in\mathbb{Z}} \langle f, \tilde{g}_{mT+x,nF+y}\rangle \langle g_{mT+x,nF+y}, h\rangle.
\end{aligned}$$

Finally, note that $\langle f_{-x,-y}, h_{-x,-y}\rangle = \langle f, h\rangle$. Together with the previous equation, this yields the desired identity.

(b) Integrating as suggested, we get

$$\begin{aligned}
TF\langle f, h\rangle &= \int_0^F \int_0^T \sum_{m,n\in\mathbb{Z}} \langle f, \tilde{g}_{mT+x,nF+y}\rangle \langle g_{mT+x,nF+y}, h\rangle \,\mathrm{d}x\,\mathrm{d}y \\
&= \sum_{m,n\in\mathbb{Z}} \int_0^F \int_0^T \langle f, \tilde{g}_{mT+x,nF+y}\rangle \langle g_{mT+x,nF+y}, h\rangle \,\mathrm{d}x\,\mathrm{d}y \qquad (10) \\
&= \sum_{m,n\in\mathbb{Z}} \int_{nF}^{(n+1)F} \int_{mT}^{(m+1)T} \langle f, \tilde{g}_{x,y}\rangle \langle g_{x,y}, h\rangle \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^2} \langle f, \tilde{g}_{x,y}\rangle \langle g_{x,y}, h\rangle \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^2} \langle f, \tilde{g}_{x,y}\rangle \overline{\langle h, g_{x,y}\rangle} \,\mathrm{d}x\,\mathrm{d}y \\
&= \langle f, h\rangle \overline{\langle \tilde{g}, g\rangle},
\end{aligned}$$

where the last step follows by the identity (IR) given in the problem statement. Since f and h were arbitrary (and in particular can be chosen so that $\langle f, h\rangle = 1$), we deduce $\langle g, \tilde{g}\rangle = TF$. To justify the change of order of summation and integration in (10), observe that

$$\sum_{m,n\in\mathbb{Z}} \int_0^F \int_0^T |\langle f, \tilde{g}_{mT+x,nF+y}\rangle \langle g_{mT+x,nF+y}, h\rangle| \, \mathrm{d}x \, \mathrm{d}y =$$

$$\begin{aligned} &= \int_{\mathbb{R}^2} |\langle f, \tilde{g}_{x,y}\rangle| \, |\langle g_{x,y}, h\rangle| \, \mathrm{d}x \, \mathrm{d}y \\ &= \langle F_1, F_2\rangle_{L^2(\mathbb{R}^2)} \leq \|F_1\|_{L^2(\mathbb{R}^2)} \|F_2\|_{L^2(\mathbb{R}^2)} < \infty, \end{aligned} \tag{11}$$

where $F_1(x, y) = |\langle f, \tilde{g}_{x,y}\rangle|$ and $F_2(x, y) = |\langle h, g_{x,y}\rangle|$ are $L^2(\mathbb{R}^2)$ functions by the assumption at the beginning of the problem statement.

(c) We have the following two expansions:

$$g = \sum_{m,n\in\mathbb{Z}} \langle g, \tilde{g}_{mT,nF}\rangle \, g_{mT,nF} = 1 \cdot g_{0,0} + \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} 0 \cdot g_{mT,nF}.$$

Now, by the Lemma given in the problem statement, we have

$$\begin{aligned} 1^2 + \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} 0^2 &= \sum_{m,n\in\mathbb{Z}} |\langle g, \tilde{g}_{mT,nF}\rangle|^2 + |\langle g, \tilde{g}_{0,0}\rangle - 1|^2 + \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} |\langle g, \tilde{g}_{mT,nF}\rangle - 0|^2 \\ &\geq |\langle g, \tilde{g}_{0,0}\rangle|^2 + |\langle g, \tilde{g}_{0,0}\rangle - 1|^2 \geq |\langle g, \tilde{g}\rangle|^2 = (TF)^2, \end{aligned} \tag{12}$$

where the first inequality is obtained by discarding all the terms on the right-hand side of the first equality except for the summands with $(m, n) = (0, 0)$. Thus, we have shown that $TF \leq 1$.

Problem 4

(a) Let $e_n = \mathbb{1}_{[n-1/2,\, n+1/2]}$ be the indicator function of the interval $[n-1/2, n+1/2]$. Note that $\|e_n\|_{L^2(\mathbb{R})} = 1$, and so $e_n \in L^2(\mathbb{R})$. Then, by the Cauchy-Schwarz inequality we have

$$\left| \int_{n-1/2}^{n+1/2} x(t)\,\mathrm{d}t \right| = |\langle e_n, x\rangle| \le \|e_n\|_{L^2(\mathbb{R})} \|x\|_{L^2(\mathbb{R})} = \|x\|_{L^2(\mathbb{R})},$$

for any $x \in L^2(\mathbb{R})$ and $n \in \mathbb{Z}$, and thus $(\mathcal{A}x)_n$ is well-defined for all $x \in L^2(\mathbb{R})$ and $n \in \mathbb{Z}$. To show that $\mathcal{A}x \in \ell^2(\mathbb{Z})$ whenever $x \in L^2(\mathbb{R})$, we again use the Cauchy-Schwarz inequality to get

$$\begin{aligned} \|\mathcal{A}x\|_{\ell^2(\mathbb{Z})}^2 = \sum_{n\in\mathbb{Z}} |(\mathcal{A}x)_n|^2 &= \sum_{n\in\mathbb{Z}^2} \left| \int_{n-1/2}^{n+1/2} x(t)\,\mathrm{d}t \right|^2 \\ &\le \sum_{n\in\mathbb{Z}^2} \int_{n-1/2}^{n+1/2} |x(t)|^2\,\mathrm{d}t = \int_{\mathbb{R}} |x(t)|^2\,\mathrm{d}t = \|x\|_{L^2(\mathbb{R})}^2, \end{aligned}$$

for all $x \in L^2(\mathbb{R})$. In particular, we have $\|\mathcal{A}x\|_{\ell^2(\mathbb{Z})} < \infty$, and so $\mathcal{A}$ is well-defined. Also, $\mathcal{A}$ is linear, because integration is a linear operation. Finally, since $\|\mathcal{A}x\|_{\ell^2(\mathbb{Z})}^2 \le \|x\|_{L^2(\mathbb{R})}$, for all $x \in L^2(\mathbb{R})$, we have that $\mathcal{A}$ is bounded.

By definition of adjoint operators, $\mathcal{A}^*$ is the unique operator such that

$$\langle \mathcal{A}x, y\rangle_{\ell^2(\mathbb{Z})} = \langle x, \mathcal{A}^* y\rangle_{L^2(\mathbb{R})},$$

for all $x \in L^2(\mathbb{R})$ and $y \in \ell^2(\mathbb{Z})$. For arbitrary, but fixed x and y, we calculate

$$\begin{aligned} \langle \mathcal{A}x, y\rangle_{\ell^2(\mathbb{Z})} &= \sum_{n\in\mathbb{Z}} (\mathcal{A}x)_n \cdot \overline{y_n} = \sum_{n\in\mathbb{Z}} \int_{n-1/2}^{n+1/2} x(t)\,\mathrm{d}t \cdot \overline{y_n} \\ &= \sum_{n\in\mathbb{Z}} \int_{n-1/2}^{n+1/2} x(t)\,\overline{y_n}\,\mathrm{d}t \\ &= \int_{\mathbb{R}} x(t)(\mathcal{A}^* y)(t)\,\mathrm{d}y, \end{aligned} \tag{13}$$

where $\mathcal{A}^* y$ is the piecewise-constant function given by $(\mathcal{A}^* y)(t) = y_n$ for $t \in [n - \frac{1}{2}, n + \frac{1}{2})$.

(b) It follows from the (a) that

$$\|\mathcal{A}^* y\|_{L^2(\mathbb{R})}^2 = \sum_{n\in\mathbb{Z}} \int_{n-1/2}^{n+1/2} |y_n|^2\,\mathrm{d}t = \sum_{n\in\mathbb{Z}} |y_n|^2 = \|y\|_{\ell^2(\mathbb{Z})}^2 \quad \text{for all } y \in \ell^2(\mathbb{Z}),$$

as desired.

(c) Let $\{e_n\}_{n\in\mathbb{Z}}$ be the indicator functions defined at the beginning of (a). To show that $\mathcal{G} = \{e_n : n \in \mathbb{Z}\}$ is a frame for $\operatorname{Im}(\mathcal{A}^*)$, take an arbitrary $x = \mathcal{A}^* y \in \operatorname{Im}(\mathcal{A}^*)$.

Then, we have

$$\sum_{n\in\mathbb{Z}} |\langle x, e_n\rangle|^2 = \sum_{n\in\mathbb{Z}} |\langle \mathcal{A}^* y, e_n\rangle|^2 = \sum_{n\in\mathbb{Z}} \left| \int_{n-1/2}^{n+1/2} y_n \,\mathrm{d}t \right|^2$$
$$= \|y\|^2_{\ell^2(\mathbb{Z})} = \|\mathcal{A}^* y\|^2_{L^2(\mathbb{R})} = \|x\|^2_{L^2(\mathbb{R})},$$

and so $\mathcal{G}$ is a (tight) frame for $\mathrm{Im}(\mathcal{A}^*)$. Thus we have

$$\mathcal{A}x = \{\langle x, e_n\rangle : n \in \mathbb{Z}\},$$

for all $x \in \mathrm{Im}(\mathcal{A}^*) \subset L^2(\mathbb{R})$, and so $\mathcal{A}$ is the analysis operator associated with the frame $\mathcal{G}$ for the Hilbert space $\mathrm{Im}(\mathcal{A}^*)$.