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Handout: Results from "Orthonormal Wavelets"
Examination on Mathematics of Information
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Definition 1 (Multiresolution approximation). *A multiresolution approximation of $L^2(\mathbb{R})$ is a sequence $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ of closed linear subspaces of $L^2(\mathbb{R})$ with the following properties:*

(i) $\mathcal{V}_j \subset \mathcal{V}_{j+1}$, *for all* $j \in \mathbb{Z}$,

(ii) *for all* $f \in L^2(\mathbb{R})$ *and all* $j \in \mathbb{Z}$, $f \in \mathcal{V}_j \iff f(2\,\cdot) \in \mathcal{V}_{j+1}$,

(iii) $\bigcap_{j\in\mathbb{Z}} \mathcal{V}_j = \{0\}$,

(iv) $\bigcup_{j\in\mathbb{Z}} \mathcal{V}_j$ *is dense in* $L^2(\mathbb{R})$, *and*

(v) *there exists a function* $\varphi \in \mathcal{V}_0$, *known as the scaling function, such that* $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ *is an orthonormal basis of the space* $\mathcal{V}_0$.

Proposition 2.1. *Let* $g \in L^2(\mathbb{R})$. *Then* $\{g(\cdot - k) : k \in \mathbb{Z}\}$ *is an orthonormal system if and only if*

$$\sum_{n\in\mathbb{Z}} |(\mathcal{F}g)(\omega + n)|^2 = 1, \quad \text{for all } \omega \in \mathbb{R}.$$

Theorem 1. *Let* $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ *be a sequence of closed linear subspaces of* $L^2(\mathbb{R})$ *satisfying conditions (i), (ii), and (v) of Definition 1. Then (iii) is satisfied as well.*

Theorem 2. *Let* $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ *be a sequence of closed linear subspaces of* $L^2(\mathbb{R})$ *satisfying conditions (i), (ii), and (v) of Definition 1. Assume* $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, *and* $\hat{\varphi}$ *is continuous at* 0. *Then*

$$\hat{\varphi}(0) \neq 0 \iff \overline{\bigcup_{j\in\mathbb{Z}} \mathcal{V}_j} = L^2(\mathbb{R}).$$

Moreover, if either of the two equivalent statements holds, then $\hat{\varphi}(0) = 1$.