

# Solutions to the Examination on Mathematics of Information August 23, 2019

## Problem 1

(a) We have

$$\begin{aligned} \int_{-\infty}^{\infty} |f_1(x)|^2 dx &= \int_{-\infty}^{\infty} \frac{1}{(1+|x|)^2} dx \\ &= 2 \int_0^{\infty} \frac{1}{(1+x)^2} dx \\ &= 2 \cdot \left. \frac{-1}{1+x} \right|_0^{\infty} = 2 < \infty, \end{aligned}$$

so  $f_1 \in L^2(\mathbb{R})$  with  $\|f_1\|_{L^2(\mathbb{R})} = \sqrt{2}$ . On the other hand,

$$\begin{aligned} \int_{-\infty}^{\infty} |f_1(x)| dx &= \int_{-\infty}^{\infty} \frac{1}{1+|x|} dx \\ &\geq \int_1^{\infty} \frac{1}{1+x} dx \\ &= \log(1+x) \Big|_1^{\infty} = \infty, \end{aligned}$$

so  $f_1 \notin L^1(\mathbb{R})$ .

(b) Let  $f_2$  be the function given by  $f_2(x) = \mathbb{1}_{(0,1]}(x) \frac{1}{\sqrt{x}}$ ,  $x \in \mathbb{R}$ . We then have

$$\int_{-\infty}^{\infty} |f_2(x)| dx = \int_0^1 \frac{1}{\sqrt{x}} dx = \left. \frac{1}{2} \sqrt{x} \right|_0^1 = \frac{1}{2} < \infty,$$

so  $f_2 \in L^1(\mathbb{R})$ . Moreover,

$$\int_{-\infty}^{\infty} |f_2(x)|^2 dx = \int_0^1 \frac{1}{x} dx = \log x \Big|_0^1 = \infty,$$

so  $f_2 \notin L^2(\mathbb{R})$ .

(c) (i) As  $f$  is an element of  $L^1(\mathbb{R})$ , its Fourier transform  $\hat{f} : \mathbb{R} \rightarrow \mathbb{C}$  is, indeed, defined. Furthermore, as  $f$  is also an element of  $L^2(\mathbb{R})$ , so is  $\hat{f}$ , by Plancherel's theorem. Therefore, applying the triangle inequality in the space  $L^2(\mathbb{R})$ , we have

$$\|G_f\|_{L^2(\mathbb{R})} = \|\hat{f} + H_f\|_{L^2(\mathbb{R})} \leq \|\hat{f}\|_{L^2(\mathbb{R})} + \|H_f\|_{L^2(\mathbb{R})} < \infty,$$

as  $\hat{f} \in L^2(\mathbb{R})$  by the above and  $H_f \in L^2(\mathbb{R})$  by assumption.

(ii) We estimate

$$\begin{aligned}\|\hat{f}\|_{L^1(\mathbb{R})} &= \int_{-\infty}^{\infty} |\hat{f}(\omega)| \, d\omega \\ &= \int_{-\infty}^{\infty} \frac{1}{1+|\omega|} \cdot G_f(\omega) \, d\omega \\ &\stackrel{C-S}{\leq} \left( \int_{-\infty}^{\infty} \frac{1}{(1+|\omega|)^2} \, d\omega \right)^{\frac{1}{2}} \left( \int_{-\infty}^{\infty} |G_f(\omega)|^2 \, d\omega \right)^{\frac{1}{2}} \\ &= \sqrt{2} \|G_f\|_{L^2(\mathbb{R})}.\end{aligned}$$

As  $G_f \in L^2(\mathbb{R})$  by (i) above, we have  $\|\hat{f}\|_{L^1(\mathbb{R})} \leq \sqrt{2} \|G_f\|_{L^2(\mathbb{R})} < \infty$ , and thus  $\hat{f} \in L^1(\mathbb{R})$ .

(iii) Denote the Fourier transform of  $f$  by  $g$ , i.e.,  $g = \hat{f}$ , and let  $f^-$  be the time-reversal of  $f$ , i.e.,  $f^-(x) = f(-x)$ ,  $x \in \mathbb{R}$ . We have shown in (ii) that  $g \in L^1(\mathbb{R})$ , so its Fourier transform  $\hat{g}$  is defined, and additionally we know that  $\hat{g}$  is continuous. Hence  $\hat{g} = \hat{\hat{f}} = f^-$  is continuous. Thus, as the time-reversal of  $f$  is continuous, so is  $f$  itself.

## Problem 2

(a) For  $\ell, n, \ell', n' \in \{1, \dots, m\}$ , we have

$$\langle E^{(\ell, n)}, E^{(\ell', n')} \rangle = \sum_{j, k=1}^m E_{jk}^{(\ell, n)} \overline{E_{jk}^{(\ell', n')}} = \begin{cases} 1, & \text{if } \ell = \ell' \text{ and } n = n' \\ 0 & \text{else} \end{cases},$$

which proves that  $\mathcal{E}$  is an orthonormal system. To see that this system is complete, and hence an orthonormal basis, note that every  $A \in \mathbb{C}^{m \times m}$  can be expanded as

$$A = \sum_{j, k=1}^m A_{j, k} E^{(j, k)}.$$

As  $\mathbb{C}^{m \times m}$  has a basis of size  $m \cdot m = m^2$ , namely  $\mathcal{E}$ , the dimension of  $\mathbb{C}^{m \times m}$  is  $m^2$ .

(b) Note that  $D$  acts on vectors  $\mathbf{v} \in \mathbb{C}^m$  as the forward cyclic rotation according to  $D \cdot (v_1, v_2, \dots, v_m)^T = (v_m, v_1, \dots, v_{m-1})^T$ , and so, for  $n \in \mathbb{Z}$ ,  $D^n$  acts on vectors as the forward cyclic rotation by  $n$  places if  $n > 0$ , and as the backward cyclic rotation if  $n < 0$ . For  $\ell, n, \ell', n' \in \{0, \dots, m-1\}$ , we have

$$\begin{aligned} \langle G^{(\ell, n)}, G^{(\ell', n')} \rangle &= \text{tr} \left( (G^{(\ell', n')})^H G^{(\ell, n)} \right) \\ &= \frac{1}{m} \text{tr} \left( D^{-n'} M^{-\ell'} M^\ell D^n \right) \\ &= \frac{1}{m} \text{tr} \left( D^{n-n'} M^{\ell-\ell'} \right) \\ &= \frac{1}{m} \delta_{n, n'} \text{tr} \left( M^{\ell-\ell'} \right) \\ &= \delta_{n, n'} \frac{1}{m} \sum_{k=0}^{m-1} e^{-\frac{2\pi i k(\ell-\ell')}{m}} \\ &= \delta_{n, n'} \delta_{\ell, \ell'}, \end{aligned}$$

where  $\delta_{a, b}$  denotes the Kronecker delta, and we used that, for every  $A \in \mathbb{C}^{m \times m}$ , the matrix  $D^n A$  is obtained by cycling each column of  $A$  by  $n$  places. Concretely, if  $n - n' \neq 0$ , the diagonal of  $D^{n-n'} M^{\ell-\ell'}$  is identically zero. This establishes that  $\mathcal{G}$  is an orthonormal system in  $\mathbb{C}^{m \times m}$ . Noting that  $\#\mathcal{G} = m^2 = \dim(\mathbb{C}^{m \times m})$ , i.e.,  $\mathcal{G}$  is a linearly independent subset of  $\mathbb{C}^{m \times m}$  of the same size as the dimension of  $\mathbb{C}^{m \times m}$ , it follows that  $\mathcal{G}$  is an orthonormal basis for  $\mathbb{C}^{m \times m}$ .

(c) Let  $\ell, n \in \{1, \dots, m\}$  and  $\ell', n' \in \{0, \dots, m-1\}$  be arbitrary. Then  $|\langle E^{(\ell, n)}, G^{(\ell', n')} \rangle| = |G_{\ell, n}^{(\ell', n')}|$ . As all the nonzero entries of  $G^{(\ell', n')}$  have modulus  $\frac{1}{\sqrt{m}}$ , we find that

$$\mu(\mathcal{E}, \mathcal{G}) = \max_{\substack{(\ell, n) \in \{1, \dots, m\}^2 \\ (\ell', n') \in \{0, \dots, m-1\}^2}} |\langle E^{(\ell, n)}, G^{(\ell', n')} \rangle| = \frac{1}{\sqrt{m}}.$$

(d) Let  $\mathcal{B}_1 = \{U_1, \dots, U_{m^2}\}$  and  $\mathcal{B}_2 = \{V_1, \dots, V_{m^2}\}$  be arbitrary orthonormal bases for  $\mathbb{C}^{m \times m}$ . By way of contradiction, suppose that  $\mu(\mathcal{B}_1, \mathcal{B}_2) < \frac{1}{m}$ . Now, as  $\mathcal{B}_2$  is an

orthonormal basis for  $\mathbb{C}^{m \times m}$ ,  $U_1$  has the following expansion:

$$U_1 = \sum_{n=1}^{m^2} \langle U_1, V_n \rangle V_n,$$

and, moreover, we have the energy conservation relation

$$\|U_1\|^2 := \langle U_1, U_1 \rangle = \sum_{n=1}^{m^2} |\langle U_1, V_n \rangle|^2.$$

Now, using the assumption  $\mu(\mathcal{B}_1, \mathcal{B}_2) < \frac{1}{m}$ , we can conclude that

$$\|U_1\|^2 = \sum_{n=1}^{m^2} |\langle U_1, V_n \rangle|^2 \leq \sum_{n=1}^{m^2} (\mu(\mathcal{B}_1, \mathcal{B}_2))^2 < \sum_{n=1}^{m^2} \frac{1}{m^2} = m^2 \cdot \frac{1}{m^2} = 1.$$

But this contradicts the fact that  $U_1$ , as an element of an orthonormal basis, has unit norm. Therefore, our assumption must be wrong, and hence we deduce that  $\mu(\mathcal{B}_1, \mathcal{B}_2) \geq \frac{1}{m}$ .

### Problem 3

We refer to the chapter “Orthonormal Wavelets” of the discussion session notes as “OW”.

(a) Let  $j, k, \ell \in \mathbb{Z}$  be arbitrary. Then,

$$\begin{aligned} \langle \varphi_{j,k}, \varphi_{j,\ell} \rangle &= \int_{-\infty}^{\infty} 2^{\frac{j}{2}} \varphi(2^j x - k) \overline{2^{\frac{j}{2}} \varphi(2^j x - \ell)} dx \\ &\stackrel{u=2^j x}{=} \int_{-\infty}^{\infty} \varphi(u - k) \overline{\varphi(u - \ell)} dx = \langle \varphi_{0,k}, \varphi_{0,\ell} \rangle, \end{aligned}$$

so it suffices to establish that  $\{\varphi_{0,k} = \varphi(\cdot - k) : k \in \mathbb{Z}\}$  is an orthonormal system. Owing to OW Proposition 2.1, it is sufficient to verify that

$$\sum_{n \in \mathbb{Z}} |(\mathcal{F}\varphi)(\omega + n)|^2 = 1, \quad (1)$$

for all  $\omega \in \mathbb{R}$ . Due to the 1-periodicity of  $\sum_{n \in \mathbb{Z}} |(\mathcal{F}\varphi)(\omega + n)|^2$ , it suffices to verify (1) on an interval of length 1, for instance  $[-\frac{1}{3}, \frac{2}{3}]$ . The identity (1) clearly holds when  $\omega \in [-\frac{1}{3}, \frac{1}{3}]$ , as in this case only one summand participates and equals to 1, while all the remaining ones evaluate to 0. When  $\omega \in [\frac{1}{3}, \frac{2}{3}]$ , we perform an explicit calculation using the properties of  $\beta$  as follows:

$$\begin{aligned} \sum_{n \in \mathbb{Z}} |\hat{\varphi}(\omega + n)|^2 &= \cos^2\left(\frac{\pi}{2}\beta(3\omega - 1)\right) + \cos^2\left(\frac{\pi}{2}\beta(3(1 - \omega) - 1)\right) \\ &= \cos^2\left(\frac{\pi}{2}\beta(3\omega - 1)\right) + \cos^2\left(\frac{\pi}{2}\beta(1 - (3\omega - 1))\right) \\ &= \cos^2\left(\frac{\pi}{2}\beta(3\omega - 1)\right) + \cos^2\left(\frac{\pi}{2}(1 - \beta(3\omega - 1))\right) \\ &= \cos^2\left(\frac{\pi}{2}\beta(3\omega - 1)\right) + \sin^2\left(\frac{\pi}{2}\beta(3\omega - 1)\right) = 1, \end{aligned}$$

as desired.

(b) Denote  $\mathcal{B}_0 = \{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ . As  $\mathcal{B}_0$  is an orthonormal system by (a), and  $\mathcal{V}_0 = \overline{\text{span}(\mathcal{B}_0)}$ , the set  $\mathcal{B}_0$  is an orthonormal basis for  $\mathcal{V}_0$ . Therefore, we have  $\varphi_{-1,0} \in \mathcal{V}_0$  if and only if there exists a sequence  $\{h[k]\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$  such that

$$\varphi_{-1,0} = \sum_{k \in \mathbb{Z}} h[k] \varphi(\cdot - k). \quad (2)$$

Taking the  $L^2$ -Fourier transform of both sides of (2), we see that (2) is equivalent to

$$\sqrt{2}(\mathcal{F}\varphi)(2\omega) = H(\omega)(\mathcal{F}\varphi)(\omega), \quad \text{for all } \omega \in \mathbb{R}, \quad (3)$$

where the 1-periodic  $L^2[-\frac{1}{2}, \frac{1}{2})$  function  $H$  is given by  $H(\omega) := \text{DTFT}\{h\}(2\pi\omega) = \sum_{k \in \mathbb{Z}} h[k] e^{-2\pi i k \omega}$ . Equality (3) implies that on the interval  $[-\frac{1}{2}, \frac{1}{2})$  the function  $H$  must be given by

$$\begin{aligned} H(\omega) &= \frac{\sqrt{2}(\mathcal{F}\varphi)(2\omega)}{(\mathcal{F}\varphi)(\omega)}, \quad \omega \in [-\frac{1}{2}, \frac{1}{2}) \\ &= \begin{cases} \sqrt{2}(\mathcal{F}\varphi)(2\omega), & \omega \in [-\frac{1}{3}, \frac{1}{3}) \\ 0, & \omega \in [-\frac{1}{2}, -\frac{1}{3}) \cup [\frac{1}{3}, \frac{1}{2}) \end{cases}. \end{aligned} \quad (4)$$

A straightforward check shows that (3) indeed holds for  $H$  as given by (4), and moreover,

$$\begin{aligned}\|H\|_{L^2[-\frac{1}{2}, \frac{1}{2})}^2 &= \int_{-\frac{1}{3}}^{\frac{1}{3}} |\sqrt{2}(\mathcal{F}\varphi)(2\omega)|^2 d\omega \\ &\stackrel{\omega=\theta/2}{=} \int_{-\frac{2}{3}}^{\frac{2}{3}} |(\mathcal{F}\varphi)(\theta)|^2 d\theta = \|\mathcal{F}\varphi\|_{L^2(\mathbb{R})}^2 = \|\varphi\|_{L^2(\mathbb{R})}^2 < \infty.\end{aligned}$$

Thus, by the Parseval relation,  $\{h[k]\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ . This establishes that  $\varphi_{-1,0} \in \mathcal{V}_0$ .

To show that  $\mathcal{V}_j \subset \mathcal{V}_{j+1}$ , for all  $j \in \mathbb{Z}$ , it suffices to establish that  $\mathcal{V}_{-1} \subset \mathcal{V}_0$ , as the general statement then follows by (F2) according to: Suppose  $f \in \mathcal{V}_j$ . Then  $f(2^{-(j+1)}\cdot) \in \mathcal{V}_{-1} \subset \mathcal{V}_0$ , and so  $f = f(2^{-(j+1)}(2^{j+1}\cdot)) \in \mathcal{V}_{j+1}$ , as desired. We thus proceed by showing that  $\mathcal{V}_{-1} \subset \mathcal{V}_0$ . To this end, note that (2) (with  $\{h[k]\}_{k \in \mathbb{Z}}$  as specified in the preceding paragraph) implies

$$\varphi_{-1,r} = \varphi_{-1,0}(\cdot - 2r) = \sum_{k \in \mathbb{Z}} h[k] \varphi(\cdot - 2r - k) \stackrel{m=k+2r}{=} \sum_{m \in \mathbb{Z}} h[m - 2r] \varphi(\cdot - m) \in \mathcal{V}_0,$$

for all  $r \in \mathbb{Z}$ . Therefore, as  $\mathcal{V}_0$  is a linear space, we have  $\text{span}\{\varphi_{-1,r} : r \in \mathbb{Z}\} \subset \mathcal{V}_0$ , and as  $\mathcal{V}_0$  is closed, we have  $\overline{\text{span}\{\varphi_{-1,r} : r \in \mathbb{Z}\}} \subset \mathcal{V}_0$ , i.e.,  $\mathcal{V}_{-1} \subset \mathcal{V}_0$ , as desired.

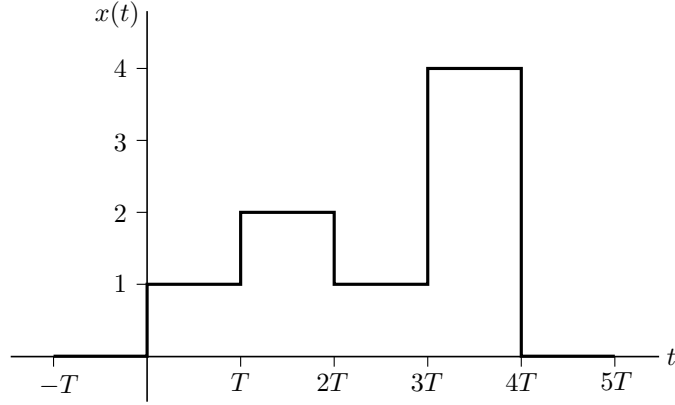
(c) According to OW Definition 1, we need to show that  $\{\mathcal{V}_j\}_{j \in \mathbb{Z}}$  is a sequence of closed subspaces of  $L^2(\mathbb{R})$  satisfying the following five properties:

- (i)  $\mathcal{V}_j \subset \mathcal{V}_{j+1}$ , for all  $j \in \mathbb{Z}$ ,
- (ii)  $\mathcal{V}_{j+1} = \{f(2\cdot) : f \in \mathcal{V}_j\}$ , for all  $j \in \mathbb{Z}$ .
- (iii)  $\bigcap_{j \in \mathbb{Z}} \mathcal{V}_j = \{0\}$ ,
- (iv)  $\bigcup_{j \in \mathbb{Z}} \mathcal{V}_j$  is dense in  $L^2(\mathbb{R})$ , and
- (v)  $\{\varphi(\cdot - k)\}_{k \in \mathbb{Z}}$  is an orthonormal basis for  $\mathcal{V}_0$ .

The spaces  $\mathcal{V}_j$ ,  $j \in \mathbb{Z}$ , are closed by definition. Property (i) follows by part (b) of the problem, Property (ii) is simply (F2), and Property (v) follows from part (a) of the problem and the definition of  $\mathcal{V}_0$ . Now, as Properties (i), (ii), and (v) are satisfied, then so is Property (iii), by OW Theorem 1. Finally, as  $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  and  $\hat{\varphi}(0) = 1 \neq 0$ , Property (iv) follows by OW Theorem 2.

## Problem 4

(a)(i)



We can write  $x = \sum_{n \in \mathbb{Z}} c_n \phi(\cdot - nT)$ , where  $\phi = \mathbb{1}_{[0, T)}$ , and

$$c_n = \begin{cases} 1, & n \in \{0, 2\} \\ 2, & n = 1 \\ 4, & n = 3 \\ 0, & \text{else} \end{cases}.$$

(ii) The Fourier transform of  $\phi = \mathbb{1}_{[0, T)}$  is given by

$$\begin{aligned} \hat{\phi}(\omega) &= \int_0^T e^{-2\pi i \omega t} dt = \left. \frac{e^{-2\pi i \omega t}}{-2\pi i \omega} \right|_{t=0}^T = \frac{1 - e^{-2\pi i T \omega}}{2\pi i \omega} \\ &= T e^{-\pi i T \omega} \frac{e^{\pi i T \omega} - e^{-\pi i T \omega}}{2\pi i T \omega} = T e^{-\pi i T \omega} \operatorname{sinc}(T\omega), \quad \omega \in \mathbb{R}, \end{aligned} \quad (5)$$

where  $\operatorname{sinc}(\theta) := \frac{\sin(\pi\theta)}{\pi\theta}$ ,  $\theta \in \mathbb{R}$ . Therefore, as  $x$  is a linear combination of time-shifted versions of  $\phi$ , we have

$$\hat{x}(\omega) = \underbrace{\left( \sum_{n=0}^3 c_n e^{-2\pi i n T \omega} \right)}_{p(\omega)} \cdot \hat{\phi}(\omega), \quad \omega \in \mathbb{R}.$$

Note that  $\hat{x}(\omega) = 0$  if and only if at least one of  $p(\omega)$  and  $\hat{\phi}(\omega)$  is zero. We have  $\{\omega \in \mathbb{R} : \hat{\phi}(\omega) = 0\} = \{\frac{n}{T}\}_{n \in \mathbb{Z} \setminus \{0\}}$  from the explicit expression (5). Moreover, as  $p(\omega)$  is a non-zero trigonometric polynomial, the set  $\{\omega \in \mathbb{R} : p(\omega) = 0\}$  is discrete. Therefore

$$\{\omega \in \mathbb{R} : \hat{x}(\omega) = 0\} = \{\omega \in \mathbb{R} : \hat{\phi}(\omega) = 0\} \cup \{\omega \in \mathbb{R} : p(\omega) = 0\}$$

is discrete, and hence  $x$  is not bandlimited.

- (iii) The fact that  $x$  can be reconstructed by sampling it at integer multiples of  $T$  even though it is not bandlimited does not contradict the sampling theorem as the sampling theorem only states that bandlimitedness is sufficient for reconstruction, but does not claim necessity.

(b)(i)

Note that

$$x(kT) = \sum_{n \in \mathbb{Z}} c_n \phi(kT - nT), \quad \text{for all } k \in \mathbb{Z},$$

so we simply set  $\phi^n = \{\phi((k - n)T)\}_{k \in \mathbb{Z}}$  to obtain

$$\mathbf{x} = \sum_{n \in \mathbb{Z}} c_n \phi^n. \quad (6)$$

- (ii) Note that, as DTFT :  $\ell^2(\mathbb{Z}) \rightarrow L^2[0, 2\pi)$  is continuous and (6) converges unconditionally, we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}} x(kT) e^{-ik\theta} &= \text{DTFT}\{\mathbf{x}\}(\theta) = \text{DTFT}\left\{\sum_{n \in \mathbb{Z}} c_n \phi^n\right\}(\theta) \\ &= \sum_{n \in \mathbb{Z}} c_n \text{DTFT}\{\phi^n\}(\theta) \\ &= \sum_{n \in \mathbb{Z}} c_n e^{-in\theta} \text{DTFT}\{\phi^0\}(\theta) \\ &= \text{DTFT}\{\{c_n\}_{n \in \mathbb{Z}}\}(\theta) \cdot \sum_{k \in \mathbb{Z}} \phi(kT) e^{-ik\theta}, \quad \theta \in [0, 2\pi). \end{aligned} \quad (7)$$

Now, let  $\alpha > 0$  be such that  $\left| \sum_{k \in \mathbb{Z}} \phi(kT) e^{-ik\theta} \right| \geq \alpha > 0$ , for all  $\theta \in [0, 2\pi)$ , as per the problem assumptions. We can then divide both sides of (7) to obtain

$$\text{DTFT}\{\{c_n\}_{n \in \mathbb{Z}}\}(\theta) = \frac{\sum_{k \in \mathbb{Z}} x(kT) e^{-ik\theta}}{\sum_{k \in \mathbb{Z}} \phi(kT) e^{-ik\theta}}, \quad \theta \in [0, 2\pi).$$

Finally, inverting the discrete-time Fourier transform, we find

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{\sum_{k \in \mathbb{Z}} x(kT) e^{-ik\theta}}{\sum_{k \in \mathbb{Z}} \phi(kT) e^{-ik\theta}} e^{in\theta} d\theta, \quad n \in \mathbb{Z}.$$

(c)(i)

We first note that

$$\sigma_{\frac{T}{2}}\left(\frac{kT}{2}\right) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}, \quad k \in \mathbb{Z}.$$

Therefore

$$\phi\left(\frac{kT}{2}\right) = \begin{cases} \frac{1}{2}, & k = 0 \\ \frac{k}{4}, & |k| = 1 \\ -\frac{1}{2}, & |k| = 2 \\ 0, & \text{otherwise} \end{cases}, \quad k \in \mathbb{Z}.$$



Now compute

$$\begin{aligned}\sum_{n \in \mathbb{Z}} \phi(nT) e^{-in\theta} &= \sum_{n \in \{-1,0,1\}} \phi(nT) e^{-in\theta} \\ &= -\frac{1}{2}e^{i\theta} + \frac{1}{2} - \frac{1}{2}e^{-i\theta} = \frac{1}{2} - \cos(\theta).\end{aligned}$$

We note that this expression evaluates to 0 at  $\theta \in \{\frac{\pi}{3}, \frac{5\pi}{3}\}$ , so there does not exist an  $\alpha > 0$  such that  $\left| \sum_{n \in \mathbb{Z}} \phi(nT) e^{-in\theta} \right| \geq \alpha$ , for all  $\theta \in [0, 2\pi)$ , i.e., Condition (\*) is not satisfied.

(ii) Recalling (7), we have

$$\begin{aligned}\sum_{k \in \mathbb{Z}} x(kT) e^{-ik\theta} &= \sum_{n \in \mathbb{Z}} c_n e^{-in\theta} \cdot \sum_{k \in \mathbb{Z}} \phi(kT) e^{-ik\theta} \\ &= \text{DTFT}\{\{c_n\}_{n \in \mathbb{Z}}\}(\theta) \cdot \left( \frac{1}{2} - \cos(\theta) \right), \quad \theta \in [0, 2\pi).\end{aligned}\tag{8}$$

A derivation analogous to (7) yields

$$\sum_{k \in \mathbb{Z}} x(kT + \frac{T}{2}) e^{-ik\theta} = \sum_{n \in \mathbb{Z}} c_n e^{-in\theta} \cdot \sum_{k \in \mathbb{Z}} \phi(kT + \frac{T}{2}) e^{-ik\theta}, \quad \theta \in [0, 2\pi).$$

We again compute

$$\begin{aligned}\sum_{k \in \mathbb{Z}} \phi(kT + \frac{T}{2}) e^{-ik\theta} &= \sum_{k \in \{-1,0\}} \phi(kT + \frac{T}{2}) e^{-ik\theta} \\ &= -\frac{1}{4}e^{i\theta} + \frac{1}{4} = -\frac{ie^{i\theta/2}}{2} \sin(\theta/2),\end{aligned}$$

and so

$$\sum_{k \in \mathbb{Z}} x(kT + \frac{T}{2}) e^{-ik\theta} = \text{DTFT}\{\{c_n\}_{n \in \mathbb{Z}}\}(\theta) \cdot \frac{-ie^{i\theta/2}}{2} \sin(\theta/2), \quad \theta \in [0, 2\pi).\tag{9}$$

Now note that

$$\Psi(\theta) := \left| \frac{1}{2} - \cos(\theta) \right|^2 + 4 \left| -\frac{ie^{i\theta/2}}{2} \sin(\theta/2) \right|^2 > 0, \quad \theta \in [0, 2\pi],$$

because there is no  $\theta \in [0, 2\pi]$  for which both  $\cos(\theta) = 1/2$  and  $\sin(\theta/2) = 0$ . Since  $\Psi(\theta) > 0$  on the compact interval  $[0, 2\pi]$ , there must exist  $\alpha > 0$  such that  $\Psi(\theta) \geq \alpha$  on this interval. Now, (8) and (9) can be multiplied by  $\frac{1}{2} - \cos(\theta)$  and  $2ie^{-i\theta/2} \sin(\theta/2)$ , respectively, and added together to yield

$$\begin{aligned}& \left( \frac{1}{2} - \cos(\theta) \right) \sum_{k \in \mathbb{Z}} x(kT) e^{-ik\theta} + \left( 2ie^{-i\theta/2} \sin(\theta/2) \right) \sum_{k \in \mathbb{Z}} x(kT + \frac{T}{2}) e^{-ik\theta} \\ &= \sum_{n \in \mathbb{Z}} c_n e^{-in\theta} \cdot \left[ \left| \frac{1}{2} - \cos(\theta) \right|^2 + 4 \left| -\frac{ie^{i\theta/2}}{2} \sin(\theta/2) \right|^2 \right] = \text{DTFT}\{\{c_n\}_{n \in \mathbb{Z}}\}(\theta) \cdot \Psi(\theta),\end{aligned}$$

for  $\theta \in [0, 2\pi)$ . The coefficients  $\{c_n\}_{n \in \mathbb{Z}}$  can therefore be extracted by inverting the discrete-time Fourier transform according to

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{1/2 - \cos(\theta)}{\Psi(\theta)} \sum_{k \in \mathbb{Z}} x(kT) e^{-ik\theta} + \frac{2ie^{-i\theta/2} \sin(\theta/2)}{\Psi(\theta)} \sum_{k \in \mathbb{Z}} x(kT + \frac{T}{2}) e^{-ik\theta} \right] e^{in\theta} d\theta, \quad n \in \mathbb{Z}.$$

## Appendix: Results from “Orthonormal Wavelets”

**Definition 1** (Multiresolution approximation). *A multiresolution approximation of  $L^2(\mathbb{R})$  is a sequence  $\{\mathcal{V}_j\}_{j \in \mathbb{Z}}$  of closed linear subspaces of  $L^2(\mathbb{R})$  with the following properties:*

- (i)  $\mathcal{V}_j \subset \mathcal{V}_{j+1}$ , for all  $j \in \mathbb{Z}$ ,
- (ii) for all  $f \in L^2(\mathbb{R})$  and all  $j \in \mathbb{Z}$ ,  $f \in \mathcal{V}_j \iff f(2 \cdot) \in \mathcal{V}_{j+1}$ ,
- (iii)  $\bigcap_{j \in \mathbb{Z}} \mathcal{V}_j = \{0\}$ ,
- (iv)  $\bigcup_{j \in \mathbb{Z}} \mathcal{V}_j$  is dense in  $L^2(\mathbb{R})$ , and
- (v) there exists a function  $\varphi \in \mathcal{V}_0$ , known as the scaling function, such that  $\{\varphi(\cdot - k)\}_{k \in \mathbb{Z}}$  is an orthonormal basis of the space  $\mathcal{V}_0$ .

**Proposition 2.1.** *Let  $g \in L^2(\mathbb{R})$ . Then  $\{g(\cdot - k) : k \in \mathbb{Z}\}$  is an orthonormal system if and only if*

$$\sum_{n \in \mathbb{Z}} |(\mathcal{F}g)(\omega + n)|^2 = 1, \quad \text{for all } \omega \in \mathbb{R}.$$

**Theorem 1.** *Let  $\{\mathcal{V}_j\}_{j \in \mathbb{Z}}$  be a sequence of closed linear subspaces of  $L^2(\mathbb{R})$  satisfying conditions (i), (ii), and (v) of Definition 1. Then (iii) is satisfied as well.*

**Theorem 2.** *Let  $\{\mathcal{V}_j\}_{j \in \mathbb{Z}}$  be a sequence of closed linear subspaces of  $L^2(\mathbb{R})$  satisfying conditions (i), (ii), and (v) of Definition 1. Assume  $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ , and  $\hat{\varphi}$  is continuous at 0. Then*

$$\hat{\varphi}(0) \neq 0 \iff \overline{\bigcup_{j \in \mathbb{Z}} \mathcal{V}_j} = L^2(\mathbb{R}).$$

*Moreover, if either of the two equivalent statements holds, then  $\hat{\varphi}(0) = 1$ .*