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Solutions to the **Examination on Mathematics of Information** **February 15, 2022**

Problem 1

(a) Towards a contradiction, we assume that there are a set $S \subset [n]$, with $|S| = s$, and a vector $v \in \ker(D)\backslash 0$, such that $\|v_{S^c}\|_1 = 0$. This implies that v is supported on S exclusively and hence $Dv_S = 0$ with $v_S \neq 0$. The contradiction is now established by noting that for $s < \mathrm{spark}(D)$ (which is by assumption), $Dv_S = 0$ with $v_S \neq 0$ is not possible as $\mathrm{spark}(D)$ is the smallest number of linearly dependent columns of D.

(b) The proof is by the following chain of relations

$$\begin{aligned}\|(x-z)_{S^c}\|_1 &\leq \|x_{S^c}\|_1 + \|z_{S^c}\|_1 \\ &= \|x_{S^c}\|_1 + \|x\|_1 - \|x\|_1 + \|z_{S^c}\|_1 \\ &= 2\|x_{S^c}\|_1 + \|x_S\|_1 - \|x\|_1 + \|z_{S^c}\|_1 \\ &= 2\|x_{S^c}\|_1 - \|x\|_1 + \|(x-z+z)_S\|_1 + \|z_{S^c}\|_1 \\ &\leq 2\|x_{S^c}\|_1 - \|x\|_1 + \|(x-z)_S\|_1 + \|z\|_1,\end{aligned} \tag{1}$$

where we used the triangle inequality twice.

(c) Let $S \subset [n]$ with $|S| = s$ be the indices of the s largest absolute values of x. First, we note that

$$\sigma_s(x) = \|x_{S^c}\|_1. \tag{2}$$

Second, $\widehat{x}$, by virtue of being a solution of (P1) with $y = Dx$, fulfills $\|\widehat{x}\|_1 \leq \|x\|_1$ and

$$D\widehat{x} = y = Dx \quad \Leftrightarrow \quad D(x - \widehat{x}) = 0.$$

Hence, the vector $v := x - \widehat{x}$ lies in the kernel of D. We have $s < \mathrm{spark}(D)$ by assumption and thus, by subproblem (a), $\|v_{S^c}\|_1 \neq 0$. Further, we have

$$\frac{\|v_S\|_1}{\|v_{S^c}\|_1} \leq \Delta_s(D). \tag{3}$$

Next, we bound

$$\begin{aligned}\|v_{S^c}\|_1 &\le 2\|x_{S^c}\|_1 - \|x\|_1 + \|v_S\|_1 + \|\widehat{x}\|_1 \\ &\le 2\|x_{S^c}\|_1 + \|v_S\|_1 \\ &\le 2\|x_{S^c}\|_1 + \Delta_s(D)\|v_{S^c}\|_1,\end{aligned} \tag{4}$$

where the first inequality follows from (1) by setting $z = \widehat{x}$, the second is by $\|\widehat{x}\|_1 \le \|x\|_1$, and the third is due to (3). Using (2) and $1 - \Delta_s(D) > 0$, which is by assumption, we rewrite (4) as

$$\|v_{S^c}\|_1 \le 2\,\frac{1}{1 - \Delta_s(D)}\,\sigma_s(x).$$

The proof is now concluded upon noting that

$$\begin{aligned}\|x - \widehat{x}\|_1 &= \|v\|_1 \\ &= \|v_S\|_1 + \|v_{S^c}\|_1 \\ &\le \Delta_s(D)\|v_{S^c}\|_1 + \|v_{S^c}\|_1 \\ &= (1 + \Delta_s(D))\|v_{S^c}\|_1 \\ &\le 2\,\frac{1 + \Delta_s(D)}{1 - \Delta_s(D)}\,\sigma_s(x).\end{aligned}$$

(d) Let $\gamma := \Delta_s(D) \in (0,1)$. As $\Delta_s(D)$ is well-defined, there exists a set $S \subset [n]$ with $|S| = s$ and a $v \in \ker(D)\backslash 0$ with $\gamma\|v_{S^c}\|_1 = \|v_S\|_1$. We next note, that for every $b \in \mathbb{R}^n$,

$$0 = D(v_S + v_{S^c} + b - b) \qquad \Leftrightarrow \qquad D(v_S + b) = D(b - v_{S^c}).$$

Next, let $x = v_S + b$ and $z = b - v_{S^c}$ and choose b such that $\|x\|_1 = \|z\|_1$. This can be effected by setting $b = \alpha v_{S^c}$, calculating

$$\begin{aligned}\|x\|_1 &= \|v_S\|_1 + \alpha\|v_{S^c}\|_1 = (\alpha + \gamma)\|v_{S^c}\|_1 \\ \|z\|_1 &= (1 - \alpha)\|v_{S^c}\|_1,\end{aligned}$$

and finally choosing $\alpha = \frac{1-\gamma}{2}$. Hence,

$$x = v_S + \frac{1-\gamma}{2}v_{S^c} \qquad \text{and} \qquad z = -\frac{1+\gamma}{2}v_{S^c}.$$

Next, we calculate $\sigma_s(x)$. To this end, we first note that $|v_i| \ge |v_j|,\ \forall i \in S, j \in S^c$ as otherwise $\frac{\|v_S\|_1}{\|v_{S^c}\|_1}$ would not be maximal. Since $\frac{1-\gamma}{2} \in (0, 1/2)$ for $\gamma \in (0,1)$, it follows that the indices of the s largest absolute values of x are given by the set S. Hence, we calculate $\sigma_s(x) = \|x_{S^c}\|_1 = \frac{1-\gamma}{2}\|v_{S^c}\|_1$, or equivalently $\frac{2}{1-\gamma}\sigma_s(x) = \|v_{S^c}\|_1$, because $\gamma < 1$ by assumption. Finally, we note that

$$\|x - z\|_1 = \|v_S + v_{S^c}\|_1 = \|v_S\|_1 + \|v_{S^c}\|_1 = (1+\gamma)\|v_{S^c}\|_1 = 2\,\frac{1+\gamma}{1-\gamma}\,\sigma_s(x).$$

Problem 2

(a) Let $(n, \ell) \in \mathbb{Z}^2$ and $(u, \xi) \in [0,1]^2$. We have

$$
\begin{aligned}
\mathcal{Z}_{g_{n,\ell}}(u,\xi) = \sum_{k=-\infty}^{\infty} e^{i2\pi k\xi}\, g_{n,\ell}(u-k) &\overset{(a)}{=} \sum_{k=-\infty}^{\infty} e^{i2\pi k\xi}\, g(u-k-n)\, e^{i2\pi\ell(u-k)} \\
&\overset{(b)}{=} \sum_{k=-\infty}^{\infty} e^{i2\pi(k-n)\xi}\, g(u-k)\, e^{i2\pi\ell(u-k+n)} \\
&\overset{(c)}{=} \sum_{k=-\infty}^{\infty} e^{i2\pi(k-n)\xi}\, g(u-k)\, e^{i2\pi\ell u} \\
&= e^{-i2\pi n\xi}\, e^{i2\pi\ell u} \sum_{k=-\infty}^{\infty} e^{i2\pi k\xi}\, g(u-k) \\
&= e^{-i2\pi n\xi}\, e^{i2\pi\ell u} \mathcal{Z}_g(u,\xi),
\end{aligned}
$$

where (a) is by the definition of $g_{n,\ell}$, (b) follows from the change of variables $k \to k-n$, and (c) by noting that $e^{i2\pi\ell(-k+n)} = 1$.

(b) Let $(n, \ell) \in \mathbb{Z}^2$ and note that

$$
\begin{aligned}
\langle \mathcal{Z}_f, \mathcal{Z}_{g_{n,\ell}} \rangle_{\mathcal{L}^2([0,1]^2)} &= \iint_{[0,1]^2} \mathcal{Z}_f(u,\xi)\mathcal{Z}^*_{g_{n,\ell}}(u,\xi)\, du\, d\xi \\
&\overset{(a)}{=} \iint_{[0,1]^2} e^{i2\pi n\xi} e^{-i2\pi\ell u} \mathcal{Z}_f(u,\xi)\mathcal{Z}^*_g(u,\xi)\, du\, d\xi \\
&\overset{(b)}{=} c^{\mathcal{Z}_f\mathcal{Z}^*_g}_{-n,\ell},
\end{aligned} \tag{5}
$$

where (a) follows from the result in subproblem (a), and (b) is a consequence of the definition of the 2-dimensional Fourier series according to Theorem 1 in the Handout.

(c) Let $f_1, f_2 \in \mathcal{L}^2(\mathbb{R})$ and note that

$$
\begin{aligned}
\langle \mathbb{T}f_1, \mathbb{T}f_2 \rangle_{\ell^2} &\overset{(a)}{=} \sum_{(n,\ell)\in\mathbb{Z}^2} \langle f_1, g_{n,\ell}\rangle_{\mathcal{L}^2(\mathbb{R})} \langle f_2, g_{n,\ell}\rangle^*_{\mathcal{L}^2(\mathbb{R})} \\
&\overset{(b)}{=} \sum_{(n,\ell)\in\mathbb{Z}^2} \langle \mathcal{Z}_{f_1}, \mathcal{Z}_{g_{n,\ell}}\rangle_{\mathcal{L}^2([0,1]^2)} \langle \mathcal{Z}_{f_2}, \mathcal{Z}_{g_{n,\ell}}\rangle^*_{\mathcal{L}^2([0,1]^2)} \\
&\overset{(c)}{=} \sum_{(n,\ell)\in\mathbb{Z}^2} c^{\mathcal{Z}_{f_1}\mathcal{Z}^*_g}_{-n,\ell} \left(c^{\mathcal{Z}_{f_2}\mathcal{Z}^*_g}_{-n,\ell}\right)^* \\
&\overset{(d)}{=} \sum_{(n,\ell)\in\mathbb{Z}^2} c^{\mathcal{Z}_{f_1}\mathcal{Z}^*_g}_{n,\ell} \left(c^{\mathcal{Z}_{f_2}\mathcal{Z}^*_g}_{n,\ell}\right)^*,
\end{aligned}
$$

where (a) follows from the definition of $\mathbb{T}$, (b) is by unitarity of the Zak transform, (c) follows by application of the result from subproblem (b), and (d) is by the change of variables $n \to -n$.

(d) Let $f \in \mathcal{L}^2(\mathbb{R})$ and note that

$$
\begin{aligned}
\sum_{(n,\ell)\in\mathbb{Z}^2} \left|\langle f, g_{n,\ell}\rangle_{\mathcal{L}^2(\mathbb{R})}\right|^2 &\overset{(a)}{=} \langle \mathbb{T}f, \mathbb{T}f\rangle_{\ell^2} \overset{(b)}{=} \sum_{(n,\ell)\in\mathbb{Z}^2} c_{n,\ell}^{\mathcal{Z}_f\mathcal{Z}_g^*} \left(c_{n,\ell}^{\mathcal{Z}_f\mathcal{Z}_g^*}\right)^* = \sum_{(n,\ell)\in\mathbb{Z}^2} \left|c_{n,\ell}^{\mathcal{Z}_f\mathcal{Z}_g^*}\right|^2 \\
&\overset{(c)}{=} \iint_{[0,1]^2} \left(\mathcal{Z}_f(u,\xi)\mathcal{Z}_g^*(u,\xi)\right)\left(\mathcal{Z}_f(u,\xi)\mathcal{Z}_g^*(u,\xi)\right)^* du\, d\xi \\
&= \iint_{[0,1]^2} \left|\mathcal{Z}_f(u,\xi)\right|^2 \left|\mathcal{Z}_g(u,\xi)\right|^2 du\, d\xi,
\end{aligned}
\tag{6}
$$

where (a) follows from the definition of $\mathbb{T}$, (b) is by the solution of subproblem (c), and (c) is obtained by application of Plancherel's formula.

Therefore, we have

$$
\begin{aligned}
&A \le |\mathcal{Z}_g(u,\xi)|^2 \le B,\ \ \forall (u,\xi)\in[0,1]^2,\ 0<A<B \\
&\overset{(a)}{\Rightarrow} A\iint_{[0,1]^2} |\mathcal{Z}_f|^2 \le \iint_{[0,1]^2} |\mathcal{Z}_f|^2 |\mathcal{Z}_g|^2 \le B\iint_{[0,1]^2} |\mathcal{Z}_f|^2,\ \forall f\in\mathcal{L}^2(\mathbb{R}) \\
&\overset{(b)}{\Rightarrow} A\left\|\mathcal{Z}_f\right\|^2_{\mathcal{L}^2([0,1]^2)} \le \sum_{(n,\ell)\in\mathbb{Z}^2} \left|\langle f, g_{n,\ell}\rangle_{\mathcal{L}^2(\mathbb{R})}\right|^2 \le B\left\|\mathcal{Z}_f\right\|^2_{\mathcal{L}^2([0,1]^2)},\ \forall f\in\mathcal{L}^2(\mathbb{R}) \\
&\overset{(c)}{\Rightarrow} A\left\|f\right\|^2_{\mathcal{L}^2(\mathbb{R})} \le \sum_{(n,\ell)\in\mathbb{Z}^2} \left|\langle f, g_{n,\ell}\rangle_{\mathcal{L}^2(\mathbb{R})}\right|^2 \le B\left\|f\right\|^2_{\mathcal{L}^2(\mathbb{R})},\ \forall f\in\mathcal{L}^2(\mathbb{R}) \\
&\overset{(d)}{\Rightarrow} \{g_{n,\ell}\}_{(n,\ell)\in\mathbb{Z}^2} \text{ is a frame, with frame bounds } A \text{ and } B,
\end{aligned}
$$

where (a) is obtained by multiplying by $|\mathcal{Z}_f|^2$ and integrating, (b) follows from (6), and (c) is by the unitarity of the Zak transform.

(e) Let $f \in \mathcal{L}^2(\mathbb{R})$. For all $\psi \in \mathcal{L}^2(\mathbb{R})$, we have

$$
\begin{aligned}
\langle \mathcal{Z}_{\mathbb{S}f}, \mathcal{Z}_\psi\rangle_{\mathcal{L}^2([0,1]^2)} &\overset{(a)}{=} \langle \mathbb{S}f, \psi\rangle_{\mathcal{L}^2(\mathbb{R})} = \langle \mathbb{T}^*\mathbb{T}f, \psi\rangle_{\mathcal{L}^2(\mathbb{R})} = \langle \mathbb{T}f, \mathbb{T}\psi\rangle_{\ell^2} \\
&\overset{(b)}{=} \sum_{(n,\ell)\in\mathbb{Z}^2} c_{n,\ell}^{\mathcal{Z}_f\mathcal{Z}_g^*} \left(c_{n,\ell}^{\mathcal{Z}_\psi\mathcal{Z}_g^*}\right)^* \\
&\overset{(c)}{=} \iint_{[0,1]^2} \left(\mathcal{Z}_f(u,\xi)\mathcal{Z}_g^*(u,\xi)\right)\left(\mathcal{Z}_\psi(u,\xi)\mathcal{Z}_g^*(u,\xi)\right)^* du\, d\xi \\
&= \iint_{[0,1]^2} \left(\mathcal{Z}_f(u,\xi)\left|\mathcal{Z}_g(u,\xi)\right|^2\right)\left(\mathcal{Z}_\psi(u,\xi)\right)^* du\, d\xi \\
&= \langle \mathcal{Z}_f \left|\mathcal{Z}_g\right|^2, \mathcal{Z}_\psi\rangle_{\mathcal{L}^2([0,1]^2)},
\end{aligned}
$$

where (a) is by the unitarity of the Zak transform, (b) follows from the result in subproblem (c), and (c) is by Plancherel's formula.

Problem 3

(a) Fix $i \in \{1, \dots, m\}$. Then,

$$\begin{aligned}\|f(x_i) - y\|_2 &= \left\| \frac{1}{m}\sum_{j=1}^{m} f(x_i) - \frac{1}{m}\sum_{j=1}^{m} f(x_j) \right\|_2 = \left\| \frac{1}{m}\sum_{j=1}^{m} (f(x_i) - f(x_j)) \right\|_2 \\ &\leq \frac{1}{m}\sum_{j=1}^{m} \|f(x_i) - f(x_j)\|_2 \leq \frac{1}{m}\sum_{j=1}^{m} \sqrt{1+\delta}\,\|x_i - x_j\|_2 \\ &< \frac{1}{m} m\sqrt{2} \cdot \sqrt{2} = 2,\end{aligned}$$

where the first inequality is by the triangle inequality, the second inequality follows from (5) in the problem statement, and the last inequality is by the assumption $\delta < 1/2$ and the relation $\|x_i - x_j\|_2 = \sqrt{2}$, which follows from $\{x_j\}_{j=1}^m$ being an orthonormal basis. We have therefore established that $\|f(x_i) - y\|_2 < 2$, or equivalently $f(x_i) \in \mathcal{B}(y, 2)$, and this holds for all i such that $1 \leq i \leq m$.

(b) We have established in the previous subproblem that $\{f(x_j)\}_{j=1}^m \subset \mathcal{B}(y, 2)$, so that we are left with having to show that $\|f(x_i) - f(x_j)\|_2 > 1$, for $1 \leq i \neq j \leq m$. From (5) in the problem statement, we have

$$\|f(x_i) - f(x_j)\|_2^2 \geq (1-\delta)\|x_i - x_j\|_2^2 = 2(1-\delta) > 1,$$

where the last inequality is thanks to the assumption $\delta < 1/2$. Taking the square root yields the desired result.

(c) Using the volume ratio estimate provided in the Handout (Lemma 1), one gets

$$M(1; \mathcal{B}(y,2), \|\cdot\|_2) \leq \frac{\mathit{vol}\,(2\mathcal{B}(y,2) + \mathcal{B}(y/2,1))}{\mathit{vol}(\mathcal{B}(y/2,1))},$$

where we chose $\mathcal{B} = \mathcal{B}(y,2)$ and $\mathcal{B}' = \mathcal{B}(y/2,1)$. Simplifying with the equalities $\mathcal{B}(y,2) = 2\mathcal{B}(y/2,1)$ and $\mathit{vol}(5\mathcal{B}(y/2,1)) = 5^k\mathit{vol}(\mathcal{B}(y/2,1))$, we obtain

$$M(1; \mathcal{B}(y,2), \|\cdot\|_2) \leq 5^k.$$

Taking the logarithm on both sides, and setting $C := (\log(5))^{-1}$, yields the desired result according to

$$C\log\left(M(1; \mathcal{B}(y,2), \|\cdot\|_2)\right) \leq k.$$

(d) Assume for the sake of contradiction that a function f satisfying (5) in the problem statement exists. From the result of subproblem (b), there would hence exist a 1-packing of $(\mathcal{B}(y,2), \|\cdot\|_2)$ with m elements, namely $\{f(x_j)\}_{j=1}^m$ with $y = (1/m)\sum_{j=1}^m f(x_j)$, which, in turn, would imply

$$C\log(m) \leq C\log\left(M(1; \mathcal{B}(y,2), \|\cdot\|_2)\right) \overset{(*)}{\leq} k.$$

Here, $(*)$ has been established in subproblem (c). Therefore, if $C\log(m) > k$, a function $f\colon \mathbb{R}^m \to \mathbb{R}^k$ satisfying (5) in the problem statement cannot exist.

Problem 4

(a) We verify that d satisfies the defining properties of a metric (cf. Handout, Definition 2) on $\mathbb{H}^n$:

- $d(x, y) \geq 0$ for $x, y \in \mathbb{H}^n$ is satisfied by definition;
- $d(x, y) = 0$ implies that $x_i = y_i$ for all $i \in [n]$ which, in turn, implies $x = y$;
- $d(x, y) = \#\{i \in [n] \,|\, x_i \neq y_i\} = \#\{i \in [n] \,|\, y_i \neq x_i\} = d(y, x)$;
- Fix $x, y, z \in \mathbb{H}^n$ and note that, for all $i \in [n]$ such that $x_i \neq z_i$, one has either $x_i \neq y_i$ or $y_i \neq z_i$. Thus, we get

$$\begin{aligned} d(x, z) &= \#\{i \in [n] \,|\, x_i \neq z_i\} \\ &\leq \# \{\{i \in [n] \,|\, x_i \neq y_i\} \cup \{i \in [n] \,|\, y_i \neq z_i\}\} \\ &\leq \#\{i \in [n] \,|\, x_i \neq y_i\} + \#\{i \in [n] \,|\, y_i \neq z_i\} \\ &= d(x, y) + d(y, z). \end{aligned}$$

(b) $\mathcal{B}(x, m)$ is the set of points at distance d less than or equal to m from x and as such can be expressed as

$$\mathcal{B}(x, m) = \bigcup_{k=0}^{m} \{y \in \mathbb{H}^n \,|\, d(x, y) = k\}.$$

As this union is over disjoint sets, it follows that

$$\#\mathcal{B}(x, m) = \sum_{k=0}^{m} \#\{y \in \mathbb{H}^n \,|\, d(x, y) = k\}.$$

The set $\{y \in \mathbb{H}^n \,|\, d(x, y) = k\}$ is the set of points in $\mathbb{H}^n$ which differ from x in exactly k coordinates. There are $\binom{n}{k}$ possible choices for these coordinates, so that

$$\#\mathcal{B}(x, m) = \sum_{k=0}^{m} \binom{n}{k}. \tag{7}$$

(c) By definition, for every $x \in \mathbb{H}^n$, one can find $x_j \in \{x_1, \ldots, x_{N(m;\mathbb{H}^n,d)}\}$ such that $d(x, x_j) \leq m$, or equivalently, $x \in \mathcal{B}(x_j, m)$. Therefore,

$$\mathbb{H}^n \subseteq \bigcup_{j=1}^{N(m\,;\mathbb{H}^n,d)} \mathcal{B}(x_j, m),$$

which implies

$$\#\mathbb{H}^n \leq \sum_{j=1}^{N(m\,;\mathbb{H}^n,d)} \#\mathcal{B}(x_j, m).$$

Now using $\#\mathbb{H}^n = 2^n$ and, from (7), $\#\mathcal{B}(x_j, m) = \sum_{k=0}^{m} \binom{n}{k}$, we get

$$2^n \leq N(m; \mathbb{H}^n, d) \sum_{k=0}^{m} \binom{n}{k},$$

which, after rearranging terms, yields the desired result

$$\frac{2^n}{\sum_{k=0}^{m} \binom{n}{k}} \leq N(m; \mathbb{H}^n, d). \tag{8}$$

(d) Take i, j such that $1 \leq i \neq j \leq M(m; \mathbb{H}^n, d)$, so that $d(x_i, x_j) > m$, and fix $x \in \mathcal{B}(x_i, \lfloor m/2 \rfloor)$. We need to show that $x \notin \mathcal{B}(x_j, \lfloor m/2 \rfloor)$, or, equivalently, that $d(x, x_j) > \lfloor m/2 \rfloor$. By the triangle inequality, one has

$$d(x, x_j) \geq d(x_i, x_j) - d(x, x_i) > m - \lfloor m/2 \rfloor = \lceil m/2 \rceil \geq \lfloor m/2 \rfloor.$$

Therefore, the balls $\{\mathcal{B}(x_j, \lfloor m/2 \rfloor)\}_{j=1}^{M(m;\mathbb{H}^n,d)}$ are, indeed, disjoint.

(e) Since $\mathcal{B}(x_j, \lfloor m/2 \rfloor) \subseteq \mathbb{H}^n$ by definition, one has $\bigcup_{j=1}^{M(m;\mathbb{H}^n,d)} \mathcal{B}(x_j, \lfloor m/2 \rfloor) \subseteq \mathbb{H}^n$. This implies the following estimate on the cardinalities:

$$\#\left\{ \bigcup_{j=1}^{M(m;\mathbb{H}^n,d)} \mathcal{B}(x_j, \lfloor m/2 \rfloor) \right\} \leq \#\mathbb{H}^n,$$

which, using that the balls are disjoint, yields

$$\sum_{j=1}^{M(m;\mathbb{H}^n,d)} \#\mathcal{B}(x_j, \lfloor m/2 \rfloor) \leq 2^n.$$

With (7), one therefore gets

$$M(m; \mathbb{H}^n, d) \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{n}{k} \leq 2^n,$$

which, after rearranging terms, yields the desired result

$$M(m; \mathbb{H}^n, d) \leq \frac{2^n}{\sum_{k=0}^{\lfloor m/2 \rfloor} \binom{n}{k}}. \tag{9}$$

(f) Following the hint, we have $N(m; \mathbb{H}^n, d) \leq M(m; \mathbb{H}^n, d)$. Combining this result with (8) and (9) yields the desired result

$$\frac{2^n}{\sum_{k=0}^{m} \binom{n}{k}} \leq N(m; \mathbb{H}^n, d) \leq M(m; \mathbb{H}^n, d) \leq \frac{2^n}{\sum_{k=0}^{\lfloor m/2 \rfloor} \binom{n}{k}}.$$