Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Prof. Dr. Helmut Bölcskei
Chair for Mathematical Information Science

Handout
Examination on Mathematics of Information
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Definition 1. *The spark of a matrix A, denoted by* $\operatorname{spark}(A)$, *is defined as the cardinality of the smallest set of linearly dependent columns.*

Definition 2. *For $A = (a_1 \dots a_n) \in \mathbb{C}^{m\times n}$ with columns $\|\cdot\|_2$-normalized to 1, the coherence is defined as $\mu(A) = \max_{i\neq j} |a_i^H a_j|$.*

Definition 3 (Fourier transform). *Given a complex-valued signal $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, its Fourier transform is defined as*

$$\hat{f}(w) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i w t}\, dt.$$

Theorem 1 (Plancherel's theorem). *The Fourier transform on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is an isometry with respect to the L^2-norm, i.e.,*

$$\|f\|_2 = \|\hat{f}\|_2, \quad \text{for all } f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

Theorem 2 (Fubini's theorem). *Given a Lebesgue-measurable function $h \in L^1(\mathbb{R}^2)$, the following equalities hold*

$$\int\int_{\mathbb{R}^2} h(t,\omega)\, d\big(\lambda_{\{t,\omega\}}\big) = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} h(t,\omega)\, dt \right\} d\omega = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} h(t,\omega)\, d\omega \right\} dt,$$

where λ is the Lebesgue measure on $\mathbb{R}^2$.

Theorem 3 (Equivalence of norms in finite dimensions). *Let $\|\cdot\|$ and $\|\cdot\|'$ be norms on $\mathbb{R}^n$, where $n \geq 1$ is an integer. Then, there exists a constant $C \geq 1$ such that*

$$C^{-1}\|x\| \leq \|x\|' \leq C\|x\|, \quad \text{for all } x \in \mathbb{R}^n.$$

Theorem 4 (Relation between metric entropy and volume ratio). *Consider the norms $\|\cdot\|$ and $\|\cdot\|'$ on $\mathbb{R}^n$, where $n \geq 1$ is an integer, and let $\mathcal{B}$ and $\mathcal{B}'$ be their corresponding unit balls, i.e., $\mathcal{B} = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ and $\mathcal{B}' = \{x \in \mathbb{R}^n \mid \|x\|' \leq 1\}$. Then, the ε-covering number of $\mathcal{B}$ in the $\|\cdot\|'$-norm satisfies*

$$\left(\frac{1}{\varepsilon}\right)^n \frac{\mathit{vol}\,(\mathcal{B})}{\mathit{vol}\,(\mathcal{B}')} \leq \mathcal{N}(\varepsilon; \mathcal{B}, \|\cdot\|') \leq \frac{\mathit{vol}\,\left(\frac{2}{\varepsilon}\mathcal{B} + \mathcal{B}'\right)}{\mathit{vol}\,(\mathcal{B}')}.$$