ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Prof. Dr. Helmut Bölcskei
Chair for Mathematical Information Science

Examination on Mathematics of Information
September 2, 2022

Please note:
- Exam duration: 180 minutes
- Maximum number of points: 100
- You are allowed to use a summary consisting of 10 handwritten or printed A4 pages (or 5 A4 pages on both sides). Other documents as well as electronic devices (laptops, calculators, cellphones, etc...) are not allowed.
- Your solutions should be explained in detail and your handwriting needs to be clean and legible.
- Please do not use red or green pens. You may use pencils.
- Please note that the "ETH Zurich Ordinance on Disciplinary Measures" applies.

Before you start:
1. The problem statements consist of 5 pages including this page. Please verify that you have received all 5 pages.
2. Please fill in your name, student ID card number and sign below.
3. Please place your student ID card at the front of your desk so we can verify your identity.

During the exam:
4. For your solutions, please use only the empty sheets provided by us. Should you need additional sheets, please let us know.
5. Each problem consists of several subproblems. If you do not provide a solution to a subproblem, you may, whenever applicable, nonetheless assume its conclusion in the ensuing subproblems.
6. All results in the Handout can be used without proof.

After the exam:
7. Please write your name on every sheet and prepare all sheets in a pile. All sheets, including those containing problem statements, must be handed in.
8. Please clean up your desk and remain seated and silent until you are allowed to leave the room in a staggered manner row by row.
9. Please avoid crowding and leave the building by the most direct route.

Family name: .... First name: ....

Legi-No.: ....

Number of additional sheets handed in: ....

Signature: ....

Problem 1 (25 points)

Let $n \geq 2$ be an integer and V an n-dimensional vector space, equipped with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding induced norm $\|x\| = \sqrt{\langle x, x \rangle}$, for $x \in V$. Further, let $\{e_k\}_{k=1}^{n}$ be an orthonormal basis for V.

(a) (6 points) Give an example of a tight frame for V with infinitely many elements. Calculate the frame bound for your example.

(b) (7 points) Let $\{g_k\}_{k \in \mathcal{K}}$ ($\mathcal{K}$ is a countably infinite or finite set) be a frame for V and denote its tightest possible lower and upper frame bounds by A and B, respectively. Show that

$$\frac{1}{n} \sum_{k \in \mathcal{K}} \|g_k\|^2 \leq B \leq \sum_{k \in \mathcal{K}} \|g_k\|^2$$

and

$$A \leq \frac{1}{n} \sum_{k \in \mathcal{K}} \|g_k\|^2.$$

(c) (2 points) Does a frame $\{g_k\}_{k \in \mathcal{K}}$ for V exist that has infinitely many elements and at the same time satisfies $\|g_k\| = 1$, for all $k \in \mathcal{K}$? Justify your answer.

(d) (10 points) For every $\alpha > 0$, find a frame $\{g_k\}_{k \in \mathcal{K}}$ for V, with $\|g_k\| = 1$, for $k \in \mathcal{K}$, such that the tightest possible lower frame bound A satisfies

$$A \leq \alpha.$$

Hint: You also need to establish that your example, indeed, constitutes a frame for V.

Problem 2 (25 points)

(a) (8 points) For $n \in \mathbb{N}_0$, we define $\mathcal{C}_n \subseteq [0,1]$ according to

$$\mathcal{C}_0 = [0,1], \quad \text{and} \quad \mathcal{C}_{n+1} = \left[\frac{1}{3}\,\mathcal{C}_n\right] \bigcup \left[\frac{1}{3}\,\mathcal{C}_n + \frac{2}{3}\right].$$

(i) (2 points) Show that

$$\mathcal{C}_1 = \left[0, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, 1\right]$$

and

$$\mathcal{C}_2 = \left[0, \frac{1}{9}\right] \bigcup \left[\frac{2}{9}, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, \frac{7}{9}\right] \bigcup \left[\frac{8}{9}, 1\right].$$

(ii) (6 points) Let $n \in \mathbb{N}_0$. Prove that there exist 2^n disjoint closed intervals I_j^n, $j \in \{1, \ldots, 2^n\}$, such that $\mathcal{C}_n = \bigcup_{j=1}^{2^n} I_j^n$, satisfying:

1. $I_j^n \subseteq [0,1]$, for all $j \in \{1, \ldots, 2^n\}$,
2. $|I_j^n| = \frac{1}{3^n}$, for all $j \in \{1, \ldots, 2^n\}$,
3. for $n \geq 1$, $d(I_j^n, I_{j'}^n) \geq \frac{1}{3^n}$, for all $j, j' \in \{1, \ldots, 2^n\}$ with $j \neq j'$. Here, d is as in Definition 4 in the Handout.

In the following, we consider the normed space $(\mathbb{R}, |\cdot|)$, where $|\cdot|$ stands for absolute value.

(b) (9 points) Let $n \in \mathbb{N}_0$. Show that:

(i) (3 points) For $\varepsilon_n = \frac{1}{2 \cdot 3^n}$, there exists an ε_n-covering of $\mathcal{C}_n$ with 2^n elements.
Hint: Use subproblem a)(ii).

(ii) (3 points) For $\varepsilon_n = \frac{1}{3^n}$, there exists an ε_n-packing of $\mathcal{C}_n$ with 2^n elements.
Hint: Use subproblem a)(ii).

(iii) (3 points) Show that $N(\frac{1}{2 \cdot 3^n}, |\cdot|, \mathcal{C}_n) = 2^n$.

(c) (8 points) We now define the Cantor set $\mathcal{C}_\infty = \bigcap_{n \geq 0} \mathcal{C}_n$.

(i) (4 points) Show that for $\varepsilon \in \left[\frac{1}{2 \cdot 3^{n+1}}, \frac{1}{2 \cdot 3^n}\right]$, one has $N(\varepsilon, |\cdot|, \mathcal{C}_\infty) \leq 2^{n+1}$, for all $n \in \mathbb{N}_0$.

(ii) (4 points) Show that $\log_2 N(\varepsilon, |\cdot|, \mathcal{C}_\infty) \leq 1 + \log_3(\varepsilon^{-1})$, for all $\varepsilon \in (0, 1/2]$.

Problem 3 (25 points)

Let $m, N \in \mathbb{N}$, let $s, t \in \{1, \dots, N\}$, and let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{C}^N$. Given a matrix $A \in \mathbb{C}^{m \times N}$, the goal of this problem is to prove the relation

$$\delta_{s+t} \leq \frac{1}{s+t}\left(s\delta_s + t\delta_t + 2\sqrt{st}\,\theta_{s,t}\right), \tag{1}$$

where $\theta_{s,t}$, δ_{s+t}, δ_s, and δ_t are, respectively, the (s,t)-restricted orthogonality constant, the $(s+t)$-restricted isometry constant, the s-restricted isometry constant, and the t-restricted isometry constant of the matrix A (see the Handout for the corresponding definitions).

(a) (5 points) We define the function $f \colon [0,1] \to \mathbb{R}$ as

$$f \colon y \mapsto \delta_s y + \delta_t(1-y) + 2\theta_{s,t}\sqrt{y(1-y)}. \tag{2}$$

Prove that f is concave on $[0,1]$, which implies that there exists $y^* \in [0,1]$ such that f is nondecreasing on $[0, y^*]$ and nonincreasing on $[y^*, 1]$.

In what follows, we will assume that $y^* \leq s/(s+t) \leq 1$. (The case $0 \leq s/(s+t) \leq y^* \leq 1$ can be treated similary, but will not be considered here).

(b) (7 points) Fix $x \in \mathbb{C}^N$ to be an $(s+t)$-sparse vector of unit norm, i.e., $\|x\|_2 = 1$. Show that there exist disjointly supported s-sparse and t-sparse vectors $u \in \mathbb{C}^N$ and $v \in \mathbb{C}^N$, respectively, such that $x = u + v$ and $\|u\|_2^2 \in [s/(s+t), 1]$.

(c) (3 points) Prove that, for x, u, and v according to subproblem (b), we have

$$\|Ax\|_2^2 = \|Au\|_2^2 + \|Av\|_2^2 + 2\,\mathrm{Re}\langle Au, Av\rangle.$$

(d) (3 points) Use the result from subproblem (c) to establish that

$$\left|\|Ax\|_2^2 - \|x\|_2^2\right| \leq \delta_s\|u\|_2^2 + \delta_t\|v\|_2^2 + 2\theta_{s,t}\|u\|_2\|v\|_2.$$

(e) (4 points) Prove that

$$\left|\|Ax\|_2^2 - \|x\|_2^2\right| \leq f\left(\frac{s}{s+t}\right),$$

with f as defined in (2).

(f) (3 points) Prove the desired result (1).

Problem 4 (25 points)

Let $n \geq 1$ be an integer. We define the class of closed balls in $\mathbb{R}^n$ as

$$\mathcal{B} := \{B(x_0, r) \mid x_0 \in \mathbb{R}^n, r > 0\},$$

where

$$B(x_0, r) := \{x \in \mathbb{R}^n \mid \|x - x_0\|_2 \leq r\}$$

is the closed ball centered at x_0 and of radius r.

Compute the VC dimension of the class $\mathcal{B}$.

Hint: This question is difficult. First, treat the cases $n = 1$ and $n = 2$ and then generalize the results to arbitrary n. Radon's theorem (see the Handout) can be used to establish the general result. You will get credit for partial results if the ideas are exposed in a clear manner.