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Let (H, ρ) be a normed space. Let $\mathcal{C} \subseteq H$ be a compact subspace of (H, ρ).

Definition 1 (ε-covering and covering number). *Let $\varepsilon > 0$. An ε-covering of $\mathcal{C}$ is a collection $X \subseteq \mathcal{C}$ such that $\forall x \in \mathcal{C}$, there exists $x' \in X$ satisfying $\rho(x - x') \leq \varepsilon$. The ε-covering number of $\mathcal{C}$ is the cardinality of the ε-covering of $\mathcal{C}$ with lowest cardinality, and is denoted by $N(\varepsilon, \mathcal{C}, \rho)$.*

Definition 2 (ε-packing and packing number). *Let $\varepsilon > 0$. An ε-packing of $\mathcal{C}$ is a collection $X \subseteq \mathcal{C}$ such that $\rho(x - x') > \epsilon$, for all $x, x' \in X$, $x \neq x'$. The ε-packing number of $\mathcal{C}$ is the cardinality of the ε-packing of $\mathcal{C}$ with greatest cardinality, and is denoted by $M(\varepsilon, \mathcal{C}, \rho)$.*

Theorem 1.

$$M(2\varepsilon, \mathcal{C}, \rho) \leq N(\varepsilon, \mathcal{C}, \rho) \leq M(\varepsilon, \mathcal{C}, \rho), \tag{1}$$

for all $\varepsilon > 0$.

Theorem 2. *$\varepsilon \mapsto N(\varepsilon, \mathcal{C}, \rho)$ is a non-increasing function.*

Definition 3 (Length of a closed interval). *Let I be a closed interval of $\mathbb{R}$, i.e., there exist $a, b \in \mathbb{R}$, with $a \leq b$, such that $I = [a, b]$. The length of I is defined as $|I| = b - a$.*

Definition 4 (Distance between two intervals). *Let I, J be intervals of $\mathbb{R}$. The distance between I and J is defined as*

$$d(I, J) = \inf_{x \in I, y \in J} |x - y|. \tag{2}$$

Definition 5 (Restricted orthogonality and isometry constants). *Let $m, N \in \mathbb{N}$ and let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{C}^N$. Given a matrix $A \in \mathbb{C}^{m \times N}$, we define its (s, t)-*restricted orthogonality constant *$\theta_{s,t}$ as the smallest $\theta \geq 0$ such that*

$$|\langle Au, Av \rangle| \leq \theta \|u\|_2 \|v\|_2,$$

*for all disjointly supported s-sparse and t-sparse vectors $u \in \mathbb{C}^N$ and $v \in \mathbb{C}^N$, respectively. We further define the s-*restricted isometry constant *δ_s of A as the smallest $\delta \geq 0$ such that*

$$\left| \langle Ax, Ax \rangle - \|x\|_2^2 \right| \leq \delta \|x\|_2^2,$$

for all s-sparse vectors $x \in \mathbb{C}^N$.

Definition 6 (Convex hull)**.** *Let $n \geq 1$ be an integer and $T = \{x_1, \ldots, x_d\}$ a set of d points in $\mathbb{R}^n$. The* convex hull *of T is defined as the set of all convex combinations of points in T, i.e.,*

$$\mathrm{conv}(T) := \left\{ x \in \mathbb{R}^n \mid \exists\, \lambda_1, \ldots, \lambda_d \in [0,1] \text{ s.t. } \sum_{i=1}^{d} \lambda_i = 1 \text{ and } x = \sum_{i=1}^{d} \lambda_i x_i \right\}.$$

Theorem 3 (Radon's Theorem)**.** *Let $n \geq 1$ be an integer. Every set S of $k \geq n+2$ points in $\mathbb{R}^n$ can be partitioned according to $S = T_1 \cup T_2$ such that $T_1 \cap T_2 = \emptyset$ and $\mathrm{conv}(T_1) \cap \mathrm{conv}(T_2) \neq \emptyset$.*

Theorem 4 (Fubini)**.** *Let $\mathcal{N}, \mathcal{K}$ be countable index sets and $a_{n,k} \in \mathbb{R}$ with $a_{n,k} \geq 0$, for all $n \in \mathcal{N}$ and $k \in \mathcal{K}$. Then,*

$$\sum_{n \in \mathcal{N}} \sum_{k \in \mathcal{K}} a_{n,k} = \sum_{k \in \mathcal{K}} \sum_{n \in \mathcal{N}} a_{n,k}\,.$$