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Examination on Mathematics of Information

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- Do not turn this page before the official start of the exam.
- The problem statements consist of 6 pages including this page.
  Please verify that you have received all 6 pages.
- Throughout the problem statements there are references to definitions and theorems in the Handout, indicated by e.g. Definition H1 and Theorem H2.

Problem 1

In this problem, you will investigate quantized finite frame expansions. Let $K \in \mathbb{N}$. We define $D := \{-K, -K+1, \ldots, -1, 0, 1, \ldots, K-1, K\}$ and the quantization map

$$\begin{aligned} Q: \ [-K-1/2, K+1/2] &\ \to\ D \\ x &\ \mapsto\ \arg\min_{d \in D} |x - d|. \end{aligned} \tag{1}$$

By definition, Q satisfies the following inequality:

$$|Q(t) - t| \le 1/2, \quad \text{for all } \ t \in [-K-1/2, K+1/2]. \tag{2}$$

Let $x \in \left[-\frac{1}{\sqrt{d}}K, \frac{1}{\sqrt{d}}K\right]^d$, $N \in \mathbb{N}$, $N \ge d$, and let $\mathcal{F} = \{e_1, \ldots, e_N\}$ be a normalized finite tight frame for $\mathbb{R}^d$ with frame bound $A > 0$ and satisfying the zero-sum property (see Definition H8). Denote the analysis operator and the frame operator (see Definition H10) associated with $\mathcal{F}$ by $\mathbb{T}$ and $\mathbb{S}$, respectively. We write $x^{\mathcal{F}} := \mathbb{T}x$.

(a) (4 points) Show that $x^{\mathcal{F}} \in [-K, K]^N$.

We next introduce $\Sigma\Delta$-modulation, a concept widely used in practice to reduce quantization errors. To this end we define $\{u_0, u_1, \ldots, u_N\} \in \mathbb{R}^{N+1}$ and $\{q_1, \ldots, q_N\} \in \mathbb{R}^N$ according to

$$u_0 = 0 \tag{3}$$

$$q_n = Q\left(u_{n-1} + x_n^{\mathcal{F}}\right), \quad \text{for } \ n \in \{1, \ldots, N\} \tag{4}$$

$$u_n = u_{n-1} + x_n^{\mathcal{F}} - q_n, \quad \text{for } \ n \in \{1, \ldots, N\}. \tag{5}$$

Further, we define $\tilde{x}^{\mathcal{F}} \in \mathbb{R}^d$ as

$$\tilde{x}^{\mathcal{F}} = \sum_{n=1}^{N} q_n \, \mathbb{S}^{-1} e_n. \tag{6}$$

(b) (6 points) Show that $|u_n| \le \frac{1}{2}$, for all $n \in \{0, \ldots, N\}$.

(c) (7 points) Show that $u_N = 0$.
Hint: Show that $u_N \in \mathbb{Z}$.

(d) (8 points) Let $\sigma(\mathcal{F})$ be as in Definition H9. Show that

$$\|x - \tilde{x}^{\mathcal{F}}\|_2 \le \frac{\sigma(\mathcal{F})}{2A}, \tag{7}$$

where A denotes the frame bound of $\mathcal{F}$ (see Theorem H11).

Problem 2

For $\mathcal{A} \subseteq \{1, \dots, N\}$, $D_{\mathcal{A}}$ denotes the $N \times N$ diagonal matrix with entries $(D_{\mathcal{A}})_{ii} = 1$, for $i \in \mathcal{A}$, and $(D_{\mathcal{A}})_{ii} = 0$, for $i \in \mathcal{A}^c$, where $\mathcal{A}^c := \{1, \dots, N\} \setminus \mathcal{A}$. We further set $P_{\mathcal{A}}(V) := V D_{\mathcal{A}} V^{\mathsf{H}}$ for a unitary matrix $V \in \mathbb{C}^{N \times N}$. The operator 2-norm of a matrix $C \in \mathbb{C}^{N \times N}$ is $|||C|||_2 := \max_{x \in \mathbb{C}^N, \|x\|_2 = 1} \|Cx\|_2$, where $\|\cdot\|_2$ stands for the 2-norm on $\mathbb{C}^N$.

Let $A, B \in \mathbb{C}^{N \times N}$ be unitary matrices whose columns are denoted by $\{a_j\}_{j=1}^N$, $\{b_j\}_{j=1}^N$, respectively, and define $U := AB^{\mathsf{H}}$. Let $\mathcal{S}, \mathcal{T} \subseteq \{1, \dots, N\}$ be such that $|\mathcal{S}||\mathcal{T}| < \frac{1}{\mu([A\ B])^2}$, where $\mu([A\ B])$ is the coherence of $[A\ B]$, see Definition H13. The goal of this problem is to derive the following uncertainty relation:

$$\|x\|_2 \leq \left(1 + \frac{1}{1 - \mu([A\ B])\sqrt{|\mathcal{S}||\mathcal{T}|}}\right)(\|P_{\mathcal{S}^c}(A)x\|_2 + \|P_{\mathcal{T}^c}(B)x\|_2), \quad x \in \mathbb{C}^N. \tag{8}$$

(a) (2 points) Show that for every $x \in \mathbb{C}^N$,

$$\|P_{\mathcal{T}^c}(B)x\|_2 = \|P_{\mathcal{T}^c}(A)Ux\|_2.$$

(b) (5 points) Assume that $x \in \mathbb{C}^N$ is such that $\mathrm{supp}(A^{\mathsf{H}}x) \subseteq \mathcal{S}$. Show that

$$\|P_{\mathcal{T}^c}(B)x\|_2 \geq (1 - |||P_{\mathcal{T}}(A)UP_{\mathcal{S}}(A)|||_2)\|P_{\mathcal{S}}(A)x\|_2.$$

Hint: Use the result from subproblem (a) and the reverse triangle inequality.

(c) (6 points) Show that for every $x \in \mathbb{C}^N$, it holds that

$$\|x\|_2 \leq \frac{\|P_{\mathcal{S}^c}(A)x\|_2 + \|P_{\mathcal{T}^c}(B)x\|_2}{1 - |||P_{\mathcal{T}}(A)UP_{\mathcal{S}}(A)|||_2} + \|P_{\mathcal{S}^c}(A)x\|_2.$$

Hint: Use the result from subproblem (b).

(d) (5 points) Show that for every matrix $C \in \mathbb{C}^{N \times N}$,

$$|||C|||_2 \leq \sqrt{\sum_{i,j=1}^N |\langle a_i, Ca_j\rangle|^2}.$$

Hint: First show that $|||C|||_2 \leq \sqrt{\mathrm{Tr}(CC^{\mathsf{H}})}$.

(e) (5 points) Use the result in subproblem (d) to show that

$$|||P_{\mathcal{T}}(A)UP_{\mathcal{S}}(A)|||_2 \leq \mu([A\ B])\sqrt{|\mathcal{S}||\mathcal{T}|}.$$

(f) (2 points) Combine the results in subproblems (c) and (e) to derive (8).

Problem 3

Let $\mathcal{F}$ be a class of real-valued, measurable functions with common domain $\mathcal{X}$. Further, let $\mathcal{P}$ be a fixed probability distribution over $\mathcal{X}$. We consider the $L_1(\mathcal{P})$-norm on $\mathcal{F}$ that assigns to $f \in \mathcal{F}$ the value

$$\|f\|_{\mathcal{P}} := \mathbb{E}[|f(X)|],$$

where the expectation is taken with respect to the random variable X distributed according to $\mathcal{P}$. Throughout, we assume that $\|f\|_{\mathcal{P}} < \infty$ for every $f \in \mathcal{F}$, i.e., $\mathcal{F} \subseteq L_1(\mathcal{P})$.

In this problem, we establish a sufficient condition for $\mathcal{F}$ to be Glivenko-Cantelli with respect to the fixed distribution $\mathcal{P}$. The condition is formulated in terms of the so-called bracketing number of $\mathcal{F}$, which we define next.

Definition 1. *Let $\epsilon > 0$. An ϵ-bracket $[\ell, u]$ with respect to $\|\cdot\|_{\mathcal{P}}$ is a pair of functions $\ell, u \in L_1(\mathcal{P})$ with $\ell(x) \leq u(x)$, $\forall x \in \mathcal{X}$, and $\|u - \ell\|_{\mathcal{P}} \leq \epsilon$. We say that a collection of ϵ-brackets $\{[\ell_j, u_j]\}_{j=1}^m$ (with cardinality $m \in \mathbb{N}$) is an ϵ-bracket-covering of $\mathcal{F}$ with respect to $\|\cdot\|_{\mathcal{P}}$ if, for every $f \in \mathcal{F}$, there is a $j \in \{1, \dots, m\}$ such that*

$$\ell_j(x) \leq f(x) \leq u_j(x), \quad \textit{for all } x \in \mathcal{X}.$$

The bracketing number *$N_{[\,]}(\epsilon, \mathcal{F}, \|\cdot\|_{\mathcal{P}})$ is the cardinality of an ϵ-bracket-covering of $\mathcal{F}$ with respect to $\|\cdot\|_{\mathcal{P}}$ with smallest cardinality.*

Throughout the problem, $\mathcal{P}$ is fixed and all expectations $\mathbb{E}[\cdot]$ are understood to be with respect to $\mathcal{P}$. Furthermore, $\mathcal{F}$ is such that $N_{[\,]}(\epsilon, \mathcal{F}, \|\cdot\|_{\mathcal{P}}) < \infty$ for all $\epsilon > 0$.

(a) (7 points) Prove that, for all $\epsilon > 0$,

$$N(\epsilon, \mathcal{F}, \|\cdot\|_{\mathcal{P}}) \leq N_{[\,]}(2\epsilon, \mathcal{F}, \|\cdot\|_{\mathcal{P}}).$$

Hint: Use that $f(x) \leq g(x)$, $\forall x \in \mathcal{X}$, implies $\mathbb{E}[f(X)] \leq \mathbb{E}[g(X)]$.

(b) (16 points) Fix $\epsilon > 0$ and consider a minimal ϵ-bracket-covering $\{[\ell_j, u_j]\}_{j=1}^m$ of $\mathcal{F}$ with respect to $\|\cdot\|_{\mathcal{P}}$, where $m := N_{[\,]}(\epsilon, \mathcal{F}, \|\cdot\|_{\mathcal{P}}) < \infty$. Further, let $\{X_i\}_{i=1}^n$ be i.i.d. samples taken according to distribution $\mathcal{P}$.

i. (5 points) Show that, for every $f \in \mathcal{F}$, there exists a $j \in \{1, \dots, m\}$ such that

$$\left(\frac{1}{n}\sum_{i=1}^n f(X_i)\right) - \mathbb{E}[f(X)] \leq \left(\frac{1}{n}\sum_{i=1}^n u_j(X_i)\right) - \mathbb{E}[u_j(X)] + \epsilon.$$

ii. (7 points) Use the weak law of large numbers (Theorem H2 in the Handout) to prove that

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{f\in\mathcal{F}} \left(\left(\frac{1}{n}\sum_{i=1}^n f(X_i)\right) - \mathbb{E}[f(X)]\right) > 2\epsilon\right) = 0.$$

Similarly, one can show that

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{f\in\mathcal{F}}\left(\mathbb{E}[f(X)] - \left(\frac{1}{n}\sum_{i=1}^{n} f(X_i)\right)\right) > 2\epsilon\right) = 0. \tag{9}$$

You may from now on assume (9) to be true without proof.

iii. (4 points) Prove that

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{f\in\mathcal{F}}\left|\left(\frac{1}{n}\sum_{i=1}^{n} f(X_i)\right) - \mathbb{E}[f(X)]\right| > 2\epsilon\right) = 0.$$

(c) (2 points) Prove that $N_{[\,]}(\epsilon, \mathcal{F}, \|\cdot\|_{\mathcal{P}}) < \infty$ for all $\epsilon > 0$ implies that $\mathcal{F}$ is Glivenko-Cantelli for $\mathcal{P}$, i.e., that $\forall\delta > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{f\in\mathcal{F}}\left|\left(\frac{1}{n}\sum_{i=1}^{n} f(X_i)\right) - \mathbb{E}[f(X)]\right| > \delta\right) = 0.$$

Problem 4

Let m, N, and $s \in \{1, \dots, N\}$ be natural numbers. Let $x \in \mathbb{C}^N$ and $\Phi \in \mathbb{C}^{m \times N}$ with s-th restricted isometry constant δ_s (see Definition H4 in the Handout). Prove that

$$\|\Phi x\|_2 \leq \sqrt{1+\delta_s}\left(\|x\|_2 + \frac{\|x\|_1}{\sqrt{s}}\right),$$

where the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are specified in Definition H3 in the Handout.

Hint: This question is difficult. It might be useful to first prove that, for $u \in \mathbb{C}^s$ and $v \in \mathbb{C}^s$,

$$\text{if } \max_{i=1,\dots,s} |u_i| \leq \min_{i=1,\dots,s} |v_i|, \quad \text{then } \|u\|_2 \leq \frac{\|v\|_1}{\sqrt{s}}, \tag{10}$$

and then write x as a sum of disjoint (in terms of their support) s-sparse vectors, the norms of which can be bounded individually using (10). You will get credit for partial results if the ideas are exposed in a clear manner.