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Handout
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Definition H1 (ε-covering and covering number). *Let (H, ρ) be a normed space, $\mathcal{C} \subseteq H$ a compact subset of (H, ρ), and $\varepsilon > 0$. An ε-covering of $\mathcal{C}$ is a collection $X \subseteq \mathcal{C}$ such that for every $x \in \mathcal{C}$, there exists an $x' \in X$ satisfying $\rho(x - x') \leq \varepsilon$. The ε-covering number of $\mathcal{C}$ is the cardinality of an ε-covering of $\mathcal{C}$ with smallest cardinality, and is denoted by $N(\varepsilon, \mathcal{C}, \rho)$.*

Theorem H2 (Weak law of large numbers). *Let $\{X_i\}_{i=1}^n$ be i.i.d. samples taken according to a distribution $\mathcal{P}$ over domain $\mathcal{X}$ and let $g \in L_1(\mathcal{P})$. Then, $\frac{1}{n}\sum_{i=1}^n g(X_i) \xrightarrow{p} \mathbb{E}[g(X)]$. That is, $\forall \delta > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\left(\frac{1}{n}\sum_{i=1}^n g(X_i)\right) - \mathbb{E}[g(X)]\right| > \delta\right) = 0.$$

Definition H3. *Let $d \in \mathbb{N}$. We define the inner product on $\mathbb{C}^d$ as*

$$\langle x, y\rangle := \sum_{j=1}^d x_j y_j^*, \ \text{ for } \ x, y \in \mathbb{C}^d. \tag{1}$$

We define the Euclidean norm on $\mathbb{C}^d$ as

$$\|x\|_2 := \sqrt{\langle x, x\rangle}, \ \text{ for } \ x \in \mathbb{C}^d, \tag{2}$$

as well as the 1-norm according to

$$\|x\|_1 := \sum_{j=1}^d |x_j|.$$

Definition H4 (s-th restricted isometry constant). *Let m, N, and $s \in \{1, \dots, N\}$ be natural numbers and $\Phi \in \mathbb{C}^{m\times N}$. For each integer $s = 1, \dots, N$, we define the s-th restricted isometry constant of Φ as the smallest number δ_s such that*

$$\left|\|\Phi x\|_2^2 - \|x\|_2^2\right| \leq \delta_s \|x\|_2^2$$

holds for all s-sparse vectors x. A vector is said to be s-sparse if it has at most s nonzero entries.

Theorem H5. *Let $d \in \mathbb{N}$ and $x, y \in \mathbb{C}^d$. Then,*

$$|\langle x, y\rangle| \leq \|x\|_2 \|y\|_2. \tag{3}$$

Definition H6. *Let $d \in \mathbb{N}$. We define the operator norm $|||\cdot|||_2$ for $M \in \mathbb{C}^{d\times d}$ according to*

$$|||M|||_2 := \sup_{\substack{x\in\mathbb{C}^d \\ \|x\|_2=1}} \|Mx\|_2. \tag{4}$$

Theorem H7. *Let $d \in \mathbb{N}$, $M \in \mathbb{C}^{d\times d}$, and $x \in \mathbb{C}^d$. Then,*

$$\|Mx\|_2 \le |||M|||_2 \|x\|_2. \tag{5}$$

Definition H8. *Let $d \in \mathbb{N}$, $N \in \mathbb{N}$, and let $\mathcal{F} := \{e_1, \dots, e_N\}$ be a collection of N vectors in $\mathbb{R}^d$, with $N \ge d$. If there exists $A > 0$ such that*

$$\sum_{n=1}^{N} |\langle x, e_n\rangle|^2 = A\|x\|_2^2, \text{ for all } x \in \mathbb{R}^d, \tag{6}$$

then $\mathcal{F}$ is said to be a finite tight frame in $\mathbb{R}^d$. The quantity A is called the frame bound of $\mathcal{F}$. If additionally,

$$\|e_n\| = 1, \text{ for all } n \in \{1, \dots, N\}, \tag{7}$$

then $\mathcal{F}$ is said to be normalized. If moreover

$$\sum_{n=1}^{N} e_n = 0, \tag{8}$$

then $\mathcal{F}$ is said to have the zero-sum property.

Definition H9. *Let $d \in \mathbb{N}$, $N \in \mathbb{N}$, and let $\mathcal{F} = \{e_1, \dots, e_N\}$ be a finite tight frame in $\mathbb{R}^d$. We define the variation of $\mathcal{F}$ as*

$$\sigma(\mathcal{F}) := \sum_{n=1}^{N-1} \|e_n - e_{n+1}\|_2. \tag{9}$$

Definition H10. *Let $d \in \mathbb{N}$, $N \in \mathbb{N}$, and let $\mathcal{F} = \{e_1, \dots, e_N\}$ be a finite tight frame in $\mathbb{R}^d$. The analysis operator associated with $\mathcal{F}$ is*

$$\mathbb{T} = \begin{pmatrix} e_1^T \\ \vdots \\ e_N^T \end{pmatrix} \in \mathbb{R}^{N\times d}. \tag{10}$$

We define the frame operator of $\mathcal{F}$ according to

$$\mathbb{S} := \mathbb{T}^T\mathbb{T} \in \mathbb{R}^d. \tag{11}$$

Theorem H11. *Let $\mathcal{F}$ be a finite tight frame with frame bound $A > 0$. Then,*

$$\left\| \left\| \mathbb{S}^{-1} \right\| \right\|_2 = \frac{1}{A}. \tag{12}$$

Theorem H12. *Let $d \in \mathbb{N}$, $N \in \mathbb{N}$, and let $\mathcal{F} = \{e_1, \ldots, e_N\}$ be a finite frame in $\mathbb{R}^d$. Then,*

$$x = \sum_{n=1}^{N} \langle x, e_n \rangle \mathbb{S}^{-1} e_n, \ \textit{for all} \ x \in \mathbb{R}^d. \tag{13}$$

Definition H13. *For $A = (a_1, \ldots, a_n) \in \mathbb{C}^{m \times n}$ with columns $\| \cdot \|_2$-normalized to 1, the coherence is defined as*

$$\mu(A) = \max_{\substack{i,j \\ i \neq j}} |a_i^{\mathsf{H}} a_j|.$$