

Examination on Mathematics of Information

August 3, 2026

- Do not turn this page before the official start of the exam.
- The problem statements consist of 8 pages including this page.
Please verify that you have received all 8 pages.
- Throughout the problem statements there are references to definitions and theorems in the Handout, indicated by e.g. Definition H1 and Theorem H2.

Problem 1 (20 points)

In this problem, we study perturbation theory for frames, which involves answering the following informal question. If $\{f_k\}_{k \in \mathcal{K}}$ ($\mathcal{K}$ is a countable set) is a frame for a Hilbert space $\mathcal{H}$ and $\{g_k\}_{k \in \mathcal{K}}$ is a sequence in $\mathcal{H}$ which is close to $\{f_k\}_{k \in \mathcal{K}}$ in a certain sense, must $\{g_k\}_{k \in \mathcal{K}}$ be a frame for $\mathcal{H}$?

Specifically, we answer the general question by proving that the following two statements are equivalent:

- (i) $\{g_k\}_{k \in \mathcal{K}}$ is a frame for $\mathcal{H}$.
- (ii) There is a constant $M > 0$ such that

$$\sum_{k \in \mathcal{K}} |\langle f, f_k - g_k \rangle|^2 \leq M \min \left\{ \sum_{k \in \mathcal{K}} |\langle f, f_k \rangle|^2, \sum_{k \in \mathcal{K}} |\langle f, g_k \rangle|^2 \right\}, \quad \text{for all } f \in \mathcal{H}.$$

In the following, we assume that $\{f_k\}_{k \in \mathcal{K}}$ is a frame for $\mathcal{H}$ with frame bounds A_f, B_f satisfying $0 < A_f \leq B_f < \infty$, i.e.,

$$A_f \|f\|^2 \leq \sum_{k \in \mathcal{K}} |\langle f, f_k \rangle|^2 \leq B_f \|f\|^2, \quad \text{for all } f \in \mathcal{H}.$$

- *Part I:* We prove (i) $\Rightarrow$ (ii), where we assume that $\{g_k\}_{k \in \mathcal{K}}$ has frame bounds $0 < A_g \leq B_g < \infty$, i.e.,

$$A_g \|f\|^2 \leq \sum_{k \in \mathcal{K}} |\langle f, g_k \rangle|^2 \leq B_g \|f\|^2, \quad \text{for all } f \in \mathcal{H}.$$

- (a) (2 points) Prove that for all $f \in \mathcal{H}$, it holds that

$$\sum_{k \in \mathcal{K}} |\langle f, f_k - g_k \rangle|^2 \leq 2 \left(\sum_{k \in \mathcal{K}} |\langle f, f_k \rangle|^2 + \sum_{k \in \mathcal{K}} |\langle f, g_k \rangle|^2 \right).$$

- (b) (3 points) Prove that for all $f \in \mathcal{H}$, it holds that

$$\sum_{k \in \mathcal{K}} |\langle f, g_k \rangle|^2 \leq \frac{B_g}{A_f} \sum_{k \in \mathcal{K}} |\langle f, f_k \rangle|^2.$$

- (c) (5 points) Conclude that (ii) holds. *Hint: Use the results from subproblems (a) and (b).*

- *Part II:* We prove (ii) $\Rightarrow$ (i).

(d) (4 points) Prove that for all $f \in \mathcal{H}$, it holds that

$$\sum_{k \in \mathcal{K}} |\langle f, g_k \rangle|^2 \leq 2(M+1)B_f \|f\|^2,$$

where M is the constant from (ii).

(e) (4 points) Prove that for all $f \in \mathcal{H}$, it holds that

$$A_f \|f\|^2 \leq 2(M+1) \sum_{k \in \mathcal{K}} |\langle f, g_k \rangle|^2,$$

where M is the constant from (ii).

(f) (2 points) Conclude that (i) holds. *Hint: Use the results from subproblems (d) and (e).*

Problem 2 (20 points)

Let $\mathcal{H}$ be a class of functions $h: \mathcal{X} \subset \mathbb{R}^d \rightarrow [-1, 1]$. Throughout this problem, we fix the sample $S = (x_1, \dots, x_n) \in \mathcal{X}^n$. Define the metric

$$d_S(h, h') = \left(\frac{1}{n} \sum_{i=1}^n (h(x_i) - h'(x_i))^2 \right)^{1/2}, \quad h, h' \in \mathcal{H}.$$

Suppose that for every $\delta > 0$, there exists a finite δ -covering of $(\mathcal{H}, d_S)$, which we denote by $\mathcal{H}_\delta = \{h_1, \dots, h_N\}$, for some $N \in \mathbb{N}$. With the empirical Rademacher complexity as in Definition H4, the goal of this problem is to show that

$$\mathcal{R}(\mathcal{H}(\{x_i\}_{i=1}^n)/n) \leq \inf_{\delta > 0} \left(\delta + \sqrt{\frac{2 \log(2|\mathcal{H}_\delta|)}{n}} \right). \quad (1)$$

Here, and throughout this problem, $\log(\cdot)$ denotes the logarithm to base e .

- (a) In this subproblem, we show that for a finite set $A \subset \mathbb{R}^m$ with $\max_{a \in A} \|a\|_2 =: r$, it holds that

$$\mathbb{E}_\varepsilon \left[\sup_{a \in A} \left(\frac{1}{m} \sum_{i=1}^m \varepsilon_i a_i \right) \right] \leq \frac{r \sqrt{2 \log(|A|)}}{m}, \quad (2)$$

where $\{\varepsilon_i\}_{i=1}^m$ are i.i.d. Rademacher random variables.

- i. (9 points) Show that for every $\lambda > 0$,

$$\exp \left(\lambda \mathbb{E}_\varepsilon \left[\sup_{a \in A} \sum_{i=1}^m \varepsilon_i a_i \right] \right) \leq |A| \exp \left(\frac{\lambda^2 r^2}{2} \right).$$

Hint: Apply Jensen's inequality (Lemma H5). Furthermore, you may use without proof that $e^x + e^{-x} \leq 2e^{x^2/2}$, $x \in \mathbb{R}$.

- ii. (5 points) Use the result of subproblem (a)i. to establish (2).

- (b) (6 points) Show that (1) holds.

Hint: Apply the result (2) in subproblem (a). You may use without proof that for $B \subset \mathbb{R}^m$,

$$\mathbb{E}_\varepsilon \left[\sup_{b \in B} \left| \frac{1}{m} \sum_{i=1}^m \varepsilon_i b_i \right| \right] = \mathbb{E}_\varepsilon \left[\sup_{b \in B \cup -B} \left(\frac{1}{m} \sum_{i=1}^m \varepsilon_i b_i \right) \right],$$

where $-B := \{-b: b \in B\}$ and $\{\varepsilon_i\}_{i=1}^m$ are i.i.d. Rademacher random variables.

Problem 3 (30 points)

In the exercise session, we derived metric entropy scaling of Lipschitz functions defined locally on the real line. In this problem, we extend these results to Lipschitz functions defined on the unit cube in d -dimensional space.

To this end, we consider the following class of functions¹

$$\mathcal{F}_L([0, 1]^d) := \{f : [0, 1]^d \rightarrow \mathbb{R} \mid f(0) = 0, |f(x) - f(x')| \leq L \|x - x'\|_\infty, \forall x, x' \in [0, 1]^d\}.$$

Further, let

$$\|f\|_{L^\infty} := \sup_{x \in [0, 1]^d} |f(x)|, \quad f \in \mathcal{F}_L([0, 1]^d)$$

be the sup norm on $\mathcal{F}_L([0, 1]^d)$ and $\rho_\infty(f, g) := \|f - g\|_{L^\infty}$ the corresponding induced metric. Without loss of generality, we focus on $L = 1$ and henceforth write $\mathcal{F}([0, 1]^d)$ instead of $\mathcal{F}_1([0, 1]^d)$. The goal of this problem is to prove that the metric entropy of $\mathcal{F}([0, 1]^d)$ satisfies the following bound²

$$\log_2 N(\epsilon; \mathcal{F}([0, 1]^d), \rho_\infty) \gtrsim \epsilon^{-d}.$$

In order to solve the problem, we first derive lower bounds on $M(\epsilon; [0, 1]^d, \|\cdot\|_\infty)$ by constructing an ϵ -packing for $[0, 1]^d$ according to

$$\begin{aligned} N &= \lceil \epsilon^{-1} \rceil, \quad \delta = \frac{1}{N - 1}, \\ x_{i_1, i_2, \dots, i_d}[k] &= i_k \delta, \quad k = 1, 2, \dots, d, \end{aligned} \tag{3}$$

where $(i_1, i_2, \dots, i_d) \in \{0, 1, \dots, N - 1\}^d$, and considering the set

$$X := \{x_{i_1, i_2, \dots, i_d} \in [0, 1]^d \mid (i_1, i_2, \dots, i_d) \in \{0, 1, \dots, N - 1\}^d, x_{i_1, i_2, \dots, i_d} \text{ satisfies (3)}\}.$$

(a) (6 points) Prove that X is an ϵ -packing for $[0, 1]^d$ with respect to $\|\cdot\|_\infty$ and that

$$M(\epsilon; [0, 1]^d, \|\cdot\|_\infty) \gtrsim \epsilon^{-d}.$$

Now, suppose that $\{x_1, x_2, \dots, x_M\} \subset [0, 1]^d$ is a maximal ϵ -packing for $[0, 1]^d$. Define $\epsilon_1 = \min_{\ell \neq j} \|x_\ell - x_j\|_\infty > \epsilon$ and arbitrarily choose $\epsilon' \in (\epsilon, \epsilon_1)$. Consider balls of radius $\epsilon'/2$ centered at the packing points according to

$$S_\ell := \{x \in [0, 1]^d \mid \|x - x_\ell\|_\infty \leq \epsilon'/2\}, \quad \ell = 1, 2, \dots, M.$$

¹For a vector $x \in \mathbb{R}^d$, we use $x[k]$ to denote its k -th component and define $\|x\|_\infty = \max_{k=1}^d |x[k]|$.

²The notation $f(\epsilon) \gtrsim g(\epsilon)$ expresses that there exist $C > 0$ and $\epsilon_0 > 0$ such that $f(\epsilon) \geq Cg(\epsilon)$, for all $\epsilon \in (0, \epsilon_0)$.

- (b) (3 points) Prove that the S_ℓ are mutually disjoint, i.e., $S_\ell \cap S_j = \emptyset$ if $\ell \neq j$.

Next, assume that $0 \in S_M$ and define the set $\mathcal{U}$ according to

$$\mathcal{U} := \{g_{\alpha_1, \alpha_2, \dots, \alpha_{M-1}} : [0, 1]^d \rightarrow \mathbb{R} \mid (\alpha_1, \alpha_2, \dots, \alpha_{M-1}) \in \{0, 1\}^{M-1}\}$$

$$g_{\alpha_1, \alpha_2, \dots, \alpha_{M-1}}(x) := \begin{cases} (2\alpha_\ell - 1)(\epsilon'/2 - \|x - x_\ell\|_\infty), & \text{for}^3 x \in S_\ell \ (\ell \neq M), \\ 0, & \text{for } x \notin \bigcup_{j=1}^{M-1} S_j. \end{cases}$$

- (c) (8 points) Prove that $g_{\alpha_1, \alpha_2, \dots, \alpha_{M-1}} \in \mathcal{F}([0, 1]^d)$ for all $(\alpha_1, \alpha_2, \dots, \alpha_{M-1}) \in \{0, 1\}^{M-1}$.

You may write g instead of $g_{\alpha_1, \alpha_2, \dots, \alpha_{M-1}}$ for notational simplicity.

Hint: Use the result from subproblem (b). When proving that g satisfies $|g(x) - g(x')| \leq \|x - x'\|_\infty$, try splitting into three cases: (1) $x, x' \in S_\ell$ for some $\ell \in \{1, 2, \dots, M-1\}$; (2) $x \in S_\ell, x' \in S_j$ for some $\ell, j \in \{1, 2, \dots, M-1\}$ with $\ell \neq j$, (draw the segment from x to x' , note that it must leave one set before entering the other, consider the boundary points intersected with the segment and apply the triangle inequality); (3) $x \in S_\ell$ for some $\ell \in \{1, 2, \dots, M-1\}$ and $x' \notin \bigcup_{j=1}^{M-1} S_j$. When $x \notin \bigcup_{j=1}^{M-1} S_j$ and $x' \notin \bigcup_{j=1}^{M-1} S_j$, the result follows immediately as $g(x) = g(x') = 0$.

- (d) (4 points) Prove that $\mathcal{U}$ is an ϵ -packing for $\mathcal{F}([0, 1]^d)$ with respect to ρ_∞ .

- (e) (9 points) Prove the following metric entropy lower bound

$$\log_2 N(\epsilon; \mathcal{F}([0, 1]^d), \rho_\infty) \gtrsim \epsilon^{-d}.$$

Hint: Use the results from subproblems (a) and (d).

³Note that by subproblem (b), for every $x \in \bigcup_{j=1}^{M-1} S_j$, there is a unique $\ell \in \{1, 2, \dots, M-1\}$ such that $x \in S_\ell$.

Problem 4 (30 points)

For $m, n \in \mathbb{N}$ with $m < n$, let $A \in \mathbb{R}^{m \times n}$ be of full row rank, and fix $b \in \mathbb{R}^m \setminus \{0\}$. We consider the minimization problem

$$\min_{x \in \mathbb{R}^n} (\|x\|_1 - \|x\|_2) \quad \text{subject to } Ax = b. \quad (P_{1-2})$$

A vector $x^* \in \mathbb{R}^n$ is said to be a *local minimizer* of (P_{1-2}) if there exists an $r > 0$ such that x^* is a solution to⁴

$$\min_{x \in B_r(x^*)} (\|x\|_1 - \|x\|_2) \quad \text{subject to } Ax = b. \quad (P_{1-2}^{x^*, r})$$

Denoting by N_{loc} the number of local minimizers of (P_{1-2}) , the goal of this problem is to show that $N_{\text{loc}} \leq \sum_{k=1}^m \binom{n}{k}$.

(a) (4 points) Show that $\mathbb{R}^n \rightarrow \mathbb{R}, x \mapsto \|x\|_1 - \|x\|_2$ is non-convex.

Hint: A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if for all $x, x' \in \mathbb{R}^n$, $t \in [0, 1]$, it holds that $f(tx + (1-t)x') \leq tf(x) + (1-t)f(x')$. Show that $\mathbb{R}^n \rightarrow \mathbb{R}, x \mapsto \|x\|_1 - \|x\|_2$ violates this definition.

Let now $x^* \in \mathbb{R}^n$ be a local minimizer of (P_{1-2}) , and set⁵ $S^* := \text{supp}(x^*)$. For $v \in \mathbb{R}^n$, define $T_v(x^*) = x^* + v$.

(b) (5 points) Show that if $v \in \mathbb{R}^n$ is such that $\text{supp}(v) \subseteq S^*$ and $\|v\|_2 < \min_{i \in S^*} |x_i^*|$, then

$$\|x^*\|_1 = \frac{1}{2} \|T_v(x^*)\|_1 + \frac{1}{2} \|T_{-v}(x^*)\|_1.$$

(c) (5 points) Show that if $v \in \ker(A) \setminus \{0\}$, then

$$\|x^*\|_2 < \frac{1}{2} \|T_v(x^*)\|_2 + \frac{1}{2} \|T_{-v}(x^*)\|_2.$$

Hint: You may use without proof that for $x, y \in \mathbb{R}^n$ with $y \neq 0$, $\langle x, y \rangle = \|x\|_2 \|y\|_2$ if and only if $x = cy$ for some $c \geq 0$. Here, $\langle x, y \rangle = y^\top x$ denotes the standard inner product on $\mathbb{R}^n$.

(d) (8 points) Use the results of subproblems (b) and (c) to show that the columns of A indexed by S^* are linearly independent.

(e) (3 points) Show that x^* is m -sparse, i.e., $|\text{supp}(x^*)| \leq m$.

Hint: Use the result of subproblem (d).

⁴Here, and throughout this problem, $B_r(x^*) := \{x \in \mathbb{R}^n : \|x - x^*\|_2 < r\}$ denotes the open ball of radius r centered at x^* .

⁵For $x \in \mathbb{R}^n$, $\text{supp}(x) := \{i \in \{1, \dots, n\} : x_i \neq 0\}$.

- (f) (5 points) Show that $N_{\text{loc}} \leq \sum_{k=1}^m \binom{n}{k}$.
Hint: Use the results of subproblems (d) and (e).