Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Chair for Mathematical Information Science

Verner Vlačić
Sternwartstrasse 7
CH-8092 Zürich

Mathematics of Information
The Fourier transform on $L^2(\mathbb{R})$.

These notes are based on [1, Chap. 1].

1 Failure of the Fourier integral for the sinc function

The Fourier transform on $\mathbb{R}$ is usually introduced as an operator mapping a function $f:\mathbb{R}\to\mathbb{C}$ to another function $\hat{f}:\mathbb{R}\to\mathbb{C}$ defined by the integral

$$\hat{f}(\omega)=\int_{-\infty}^{+\infty} f(x)e^{-2\pi i\omega x}\mathrm{d}x,\quad \forall\omega\in\mathbb{R}, \tag{1}$$

without paying too much attention to the sense in which the integral on the right-hand side of (1) is defined. To see why we can run into trouble if we are not being careful, we consider the following example:

Examples. 1. Let $f(x)=\frac{\sin(\pi x)}{\pi x}$ be the normalized sinc function, whose Fourier transform is commonly known to be the rectangular function $\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(\omega)$. Suppose we wish to verify this at $\omega=0$ by using the formula (1). In other words, we wish to evaluate

$$\hat{f}(0)=\int_{-\infty}^{+\infty}\frac{\sin(\pi x)}{\pi x}\mathrm{d}x. \tag{2}$$

We split the domain of integration into sets $S_+=\{x\in\mathbb{R}: f(x)\geqslant 0\}$ and $S_-=\{x\in\mathbb{R}: f(x)<0\}$ and calculate

$$\begin{aligned}\int_{S_+} f(x)\mathrm{d}x &= \sum_{n=0}^{\infty}\int_{2n}^{2n+1}\frac{\sin(\pi x)}{\pi x}\mathrm{d}x+\sum_{n=-\infty}^{0}\int_{2n-1}^{2n}\frac{\sin(\pi x)}{\pi x}\mathrm{d}x\\ &=2\sum_{n=0}^{\infty}\int_{2n}^{2n+1}\frac{\sin(\pi x)}{\pi x}\mathrm{d}x\\ &\geqslant 2\sum_{n=0}^{\infty}\int_{2n}^{2n+1}\frac{\sin(\pi x)}{(2n+1)\pi}\mathrm{d}x\\ &=2\sum_{n=0}^{\infty}\frac{2/\pi}{(2n+1)\pi}=\infty.\end{aligned}$$

Similarly, on S_- we have

$$\begin{aligned}\int_{S_-} f(x)\mathrm{d}x &= 2\sum_{n=0}^{\infty}\int_{2n+1}^{2n+2}\frac{\sin(\pi x)}{\pi x}\mathrm{d}x\\ &\leqslant 2\sum_{n=0}^{\infty}\int_{2n+1}^{2n+2}\frac{\sin(\pi x)}{(2n+2)\pi}\mathrm{d}x\\ &=2\sum_{n=0}^{\infty}\frac{-2/\pi}{(2n+2)\pi}=-\infty,\end{aligned}$$

We could now try to evaluate

$$\int_{-\infty}^{+\infty} f(x)\mathrm{d}x = \int_{S_+} \frac{\sin(\pi x)}{\pi x}\mathrm{d}x + \int_{S_-} \frac{\sin(\pi x)}{\pi x}\mathrm{d}x = \infty - \infty, \tag{3}$$

which is meaningless.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be an absolutely integrable function, that is $f \in L^1(\mathbb{R})$. Define the sets $S_+ = \{x \in \mathbb{R} : f(x) \geqslant 0\}$ and $S_- = \{x \in \mathbb{R} : f(x) < 0\}$ as in the previous example. Then

$$\int_{S_+} f(x)\mathrm{d}x = \int_{\mathbb{R}} \mathbb{1}_{S_+}(x)|f(x)|\mathrm{d}x \leqslant \int_{\mathbb{R}} |f(x)|\mathrm{d}x = \|f\|_{L^1(\mathbb{R})} < \infty$$

and

$$\int_{S_-} f(x)\mathrm{d}x = -\int_{\mathbb{R}} \mathbb{1}_{S_-}(x)|f(x)|\mathrm{d}x \geqslant -\int_{\mathbb{R}} |f(x)|\mathrm{d}x = -\|f\|_{L^1(\mathbb{R})} > -\infty,$$

so

$$\int_{\mathbb{R}} f(x)\mathrm{d}x = \int_{S_+} f(x)\mathrm{d}x + \int_{S_-} f(x)\mathrm{d}x \tag{4}$$

is a sum of finite numbers. We see that a pathology such as (3) cannot happen if $f \in L^1(\mathbb{R})$.

The above examples suggest that it might be a bad idea to try to work directly with integrals with infinite limits such as (1) when f is not in $L^1(\mathbb{R})$. Instead, we could interpret (1) as the limit

$$\hat{f}(\omega) = \lim_{R\to\infty} \int_{-R}^{R} f(x)e^{-2\pi i\omega x}\mathrm{d}x = \lim_{R\to\infty} \int_{-\infty}^{+\infty} \mathbb{1}_{[-R,R]}(x)f(x)e^{-2\pi i\omega x}\mathrm{d}x. \tag{5}$$

Then, as we will see in the example below, if $f \in L^2(\mathbb{R})$, then the function $\mathbb{1}_{[-R,R]}(x)f(x)e^{-2\pi i\omega x}$ is in $L^1(\mathbb{R})$, for each fixed R and ω, and so its integral is well-behaved in the sense of (4). This is true even if $f \notin L^1(\mathbb{R})$, e.g., the sinc function $f(x) = \frac{\sin(\pi x)}{\pi x}$ is in $L^2(\mathbb{R})$, but not in $L^1(\mathbb{R})$. It can be verified for the sinc function that the limit (5) indeed exists for any fixed ω, and can be evaluated to be $\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(\omega)$ by using the calculus of residues.

Our goal now is to show how the Fourier transform can be defined for functions such as $f(x) = \frac{\sin(\pi x)}{\pi x}$ for which the integral (1) is not well-defined. We will do this by applying the tools we developed in the first chapter to formalize the limit (5) for a general function $f \in L^2(\mathbb{R})$. Throught this chapter we will write $\hat{f}$ for the Fourier transform of a function $f \in L^1(\mathbb{R})$ as given by the formula (1). The Fourier transform that we will define for functions that are in $L^2(\mathbb{R})$ (but *not necessarily* in $L^1(\mathbb{R})$) will later be denoted by $\mathcal{F}$, and we call this the L^2-Fourier transform. We will also see that the transforms $\hat{\ }$ and $\mathcal{F}$ of functions in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ coincide. Before continuing, we give several more examples that illustrate the difference between the Banach space $L^1(\mathbb{R})$ and the Hilbert space $L^2(\mathbb{R})$.

Examples. 1. If $[-R, R]$ is a finite interval, then $L^2([-R, R]) \subset L^1([-R, R])$, and for every $f \in L^2([-R, R])$ we have $\|f\|_{L^1([-R,R])} \leqslant \sqrt{2R}\,\|f\|_{L^2([-R,R])}$. Indeed, we can apply the Cauchy-Schwarz inequality in the Hilbert space $L^2([-R, R])$

to the functions f and $\overline{\text{sgn}(f)}$ to yield

$$\begin{aligned}\|f\|_{L^1([-R,R])} &= \left|\int_{-R}^{R} f(x)\text{sgn}(f(x))\,\mathrm{d}x\right| = |\langle f, \overline{\text{sgn}(f)}\rangle| \\ &\leqslant \|f\|_{L^2([-R,R])}\|\overline{\text{sgn}(f)}\|_{L^2([-R,R])} \\ &\leqslant \|f\|_{L^2([-R,R])}\left(\int_{-R}^{R} 1^2\mathrm{d}x\right)^{\frac{1}{2}} \leqslant \sqrt{2R}\,\|f\|_{L^2([-R,R])} < \infty\end{aligned}$$

2. If $R = \infty$, then the argument in the previous example breaks down since we may have $\|\overline{\text{sgn}(f)}\|_{L^2(\mathbb{R})} = \infty$. Consider the function $f(x) = \min\{1, \frac{1}{|x|}\}$. Then

$$\begin{aligned}\int_{-\infty}^{+\infty} |f(x)|^2\,\mathrm{d}x &= \int_{-1}^{1} 1^2\,\mathrm{d}x + \int_{-\infty}^{-1} \frac{1}{x^2}\,\mathrm{d}x + \int_{1}^{\infty} \frac{1}{x^2}\,\mathrm{d}x \\ &= 2 + \left.\frac{-1}{x}\right|_{-\infty}^{-1} + \left.\frac{-1}{x}\right|_{1}^{\infty} = 4 < \infty,\end{aligned}$$

so $f \in L^2(\mathbb{R})$. However,

$$\begin{aligned}\int_{-\infty}^{+\infty} |f(x)|\,\mathrm{d}x &= \int_{-1}^{1} 1\,\mathrm{d}x + \int_{-\infty}^{-1} \frac{1}{-x}\,\mathrm{d}x + \int_{1}^{\infty} \frac{1}{x}\,\mathrm{d}x \\ &= 2 + (-\log(-x))|_{-\infty}^{-1} + \log(x)|_{1}^{\infty} = \infty,\end{aligned}$$

so $f \notin L^1(\mathbb{R})$. Therefore we have shown that $L^2(\mathbb{R}) \not\subset L^1(\mathbb{R})$. In fact, we can show similarly that the function $f(x) = \mathbb{1}_{(0,1]}(x)\frac{1}{\sqrt{x}}$ is in $L^1(\mathbb{R})$ but not in $L^2(\mathbb{R})$, and so $L^1(\mathbb{R}) \not\subset L^2(\mathbb{R})$. In other words, neither of $L^1(\mathbb{R})$ is a subset of each other $L^2(\mathbb{R})$.

2 Translation, modulation, and the Fourier transform of the gaussian

For $x, \xi \in \mathbb{R}$ we define the following linear operators on $L^2(\mathbb{R})$:

$$\begin{aligned}(T_x f)(t) &= f(t-x) \qquad \forall t \in \mathbb{R}, \\ (M_\xi f)(t) &= e^{2\pi i \xi t} f(t) \qquad \forall t \in \mathbb{R},\end{aligned}$$

where T_x is called the *translation* by x, and M_ξ is called the *modulation* by ξ.

Example.
For any $x, \xi \in \mathbb{R}$ the maps T_x and M_ξ are linear maps on $L^1(\mathbb{R})$. Moreover, they are invertible with inverses $T_x^{-1} = T_{-x}$ and $M_\xi^{-1} = M_{-\xi}$. We can compute for an arbitrary $f \in L^1(\mathbb{R})$

$$\begin{aligned}\|T_t f\|_{L^1(\mathbb{R})} &= \int_{-\infty}^{\infty} |f(t-x)|\mathrm{d}t \overset{t\mapsto t+x}{=} \int_{-\infty}^{\infty} |f(t)|\mathrm{d}t = \|f\|_{L^1(\mathbb{R})} \\ \|M_\xi f\|_{L^1(\mathbb{R})} &= \int_{-\infty}^{\infty} |e^{2\pi i \xi t} f(t)|\mathrm{d}t = \int_{-\infty}^{\infty} |f(t)|\mathrm{d}t = \|f\|_{L^1(\mathbb{R})},\end{aligned}$$

so T_x and M_ξ are linear isometries on $L^1(\mathbb{R})$. A similar calculation shows that they are also linear isometries on $L^2(\mathbb{R})$.

A gaussian is a function of the form $f(t) = ae^{-bt^2}$, where $a > 0$, $b > 0$ are fixed real numbers. We now show that the Fourier transform of a gaussian with $b = \pi$ is the gaussian itself:

Lemma 1. Let $\varphi(t) = e^{-\pi t^2}$ be the normalized Gaussian. The Fourier transform of φ as given by (1) is the function φ itself.

Proof. Note that $\varphi \in L^1(\mathbb{R})$ and $|\varphi(t)e^{-2\pi i\omega t}| \leqslant \varphi(t)$, so $\varphi(t)e^{-2\pi i\omega t}$ is in $L^1(\mathbb{R})$ for every fixed ω, and thus φ has a Fourier transform defined by the integral (1). What is more, the function $-2\pi ti\varphi(t)e^{-2\pi i\omega t}$ obtained by differentiating $\varphi(t)e^{-2\pi i\omega t}$ with respect to ω satisfies $|-2\pi ti\varphi(t)e^{-2\pi i\omega t}| \leqslant 2\pi t\,\varphi(t) \in L^1(\mathbb{R})$, and so we can integrate under the integral sign to obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\omega}\hat{\varphi}(\omega) &= \frac{\mathrm{d}}{\mathrm{d}\omega}\int_{-\infty}^{+\infty} e^{-\pi t^2}e^{-2\pi i\omega t}\mathrm{d}t \\
&= \int_{-\infty}^{+\infty} \frac{\partial}{\partial\omega}\left(e^{-\pi t^2}e^{-2\pi i\omega t}\right)\mathrm{d}t = \int_{-\infty}^{+\infty} -2\pi i t e^{-\pi t^2}e^{-2\pi i\omega t}\mathrm{d}t \\
&= \int_{-\infty}^{+\infty} i\left(\frac{\mathrm{d}}{\mathrm{d}t}e^{-\pi t^2}\right)e^{-2\pi i\omega t}\mathrm{d}t \\
&= \left(ie^{-\pi t^2}e^{-2\pi i\omega t}\right)\Big|_{t=-\infty}^{t=+\infty} - \int_{-\infty}^{+\infty} ie^{-\pi t^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}e^{-2\pi i\omega t}\right)\mathrm{d}t \\
&= 0 - \int_{-\infty}^{+\infty} ie^{-\pi t^2}\left(-2\pi i\omega\, e^{-2\pi i\omega t}\right)\mathrm{d}t \\
&= -2\pi\omega\,\hat{\varphi}(\omega),
\end{aligned}$$

for all $\omega \in \mathbb{R}$. From this it follows

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\hat{\varphi}(\omega)e^{\pi\omega^2}\right) = \left(\frac{\mathrm{d}\hat{\varphi}}{\mathrm{d}\omega}(\omega) + 2\pi\omega\,\hat{\varphi}(\omega)\right)e^{\pi\omega^2} = 0\cdot e^{\pi\omega^2} = 0, \quad \text{for all } \omega \in \mathbb{R},$$

so the function $\omega \mapsto \hat{\varphi}(\omega)e^{\pi\omega^2}$ is constant on $\mathbb{R}$. Therefore

$$\hat{\varphi}(\omega)e^{\pi\omega^2} = \hat{\varphi}(0)e^{\pi 0^2} = \int_{-\infty}^{+\infty} e^{-\pi t^2}\mathrm{d}t = 1, \quad \text{for all } \omega \in \mathbb{R},$$

and so $\hat{\varphi}(\omega) = e^{-\pi\omega^2} = \varphi(\omega)$, as desired. □

The following examples show that many operations involving T_x, M_ξ and the Fourier transform applied to φ can be carried out explicitly.

Examples. 1. Let $f \in L^1(\mathbb{R})$ and $(x, \xi) \in \mathbb{R}^2$. We compute $\widehat{M_\xi T_x f}$ in terms of $\hat{f}$:

$$\begin{aligned}
\widehat{M_\xi T_x f}(\omega) &= \int_{-\infty}^{+\infty} e^{2\pi i\xi t}f(t-x)e^{-2\pi i\omega t}\mathrm{d}t \\
&\overset{t\mapsto t+x}{=} e^{-2\pi i(\omega-\xi)x}\int_{-\infty}^{+\infty} f(t)e^{-2\pi i(\omega-\xi)t}\mathrm{d}t \\
&= e^{2\pi i\xi x}e^{-2\pi i\omega x}\hat{f}(\omega-\xi) \\
&= e^{2\pi i\xi x}(M_{-x}T_\xi\hat{f})(\omega),
\end{aligned}$$

so $\widehat{M_\xi T_x f} = e^{2\pi i \xi x} M_{-x} T_\xi \hat{f}$.

2. Let $\varphi(x) = e^{-\pi x^2}$ be the normalized gaussian. For any given $(x, \xi), (u, \eta) \in \mathbb{R}^2$ we compute the inner product $\langle M_\xi T_x \varphi, M_\eta T_u \varphi \rangle$ in the space $L^2(\mathbb{R})$:

$$\begin{aligned}
\langle M_\xi T_x \varphi, M_\eta T_u \varphi \rangle &= \int_{-\infty}^{+\infty} e^{2\pi i \xi t} e^{-\pi(t-x)^2} \overline{e^{2\pi i \eta t} e^{-\pi(t-u)^2}} \mathrm{d}t \\
&= \int_{-\infty}^{+\infty} e^{-\pi\left[(t-x)^2 + (t-u)^2\right]} e^{-2\pi i(\eta-\xi)t} \mathrm{d}t \\
&= \int_{-\infty}^{+\infty} e^{-\pi\left[\frac{1}{2}(2t-x-u)^2 + \frac{1}{2}(u-x)^2\right]} e^{-2\pi i(\eta-\xi)t} \mathrm{d}t \\
&\overset{y=\frac{2t-x-u}{\sqrt{2}}}{=} e^{-\frac{\pi}{2}(u-x)^2} \int_{-\infty}^{+\infty} e^{-\pi y^2} e^{-2\pi i(\eta-\xi)\frac{y}{\sqrt{2}}} e^{-\pi i(\eta-\xi)(x+u)} \frac{1}{\sqrt{2}} \mathrm{d}y \\
&= \frac{1}{\sqrt{2}} e^{-\frac{\pi}{2}(u-x)^2} e^{-\pi i(\eta-\xi)(x+u)} \hat{\varphi}\left(\frac{\eta-\xi}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{2}} e^{-\pi i(\eta-\xi)(x+u)} \varphi\left(\frac{u-x}{\sqrt{2}}\right) \varphi\left(\frac{\eta-\xi}{\sqrt{2}}\right)
\end{aligned}$$

3 Plancherel's Theorem

We use the calculations in the previous examples to show the following Lemma which will be our first step towards defining the Fourier transform on $L^2(\mathbb{R})$.

Lemma 2. Let $\mathcal{X} = \operatorname{span}\{M_\xi T_x \varphi : (x, \xi) \in \mathbb{R}^2\}$. Then we have the following

(i) $\mathcal{X}$ is a subspace of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

(ii) The Fourier transform $f \mapsto \hat{f}$ is a bijection from $\mathcal{X}$ to itself, and

(iii) $\|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$, for all $f \in \mathcal{X}$.

Proof. (i) Note that, for any $(x, \xi) \in \mathbb{R}^2$, the function $M_\xi T_x \varphi$ is in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, since $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and M_ξ are T_x linear isometries on both of $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$. Therefore any linear combination of such functions is in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and so $\mathcal{X} \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.
(ii) Now let $f = \sum_{k=1}^n c_k M_{\xi_k} T_{x_k} \varphi$ be an element of $\mathcal{X}$, where $c_k \in \mathbb{C}$ are complex coefficients. Then

$$\hat{f} = \sum_{k=1}^n c_k \widehat{M_{\xi_k} T_{x_k}} \varphi = \sum_{k=1}^n c_k e^{2\pi i \xi_k x_k} M_{-x_k} T_{\xi_k} \varphi \in \mathcal{X}.$$

This shows that the image of $\mathcal{X}$ under the Fourier transform lies in $\mathcal{X}$. To show surjectivity, we need to show that every $f \in \mathcal{X}$ is the Fourier transform of some element of $\mathcal{X}$. Let f be as above and define $g = \sum_{k=1}^n c_k e^{2\pi i \xi_k x_k} M_{x_k} T_{-\xi_k} \varphi$. Then

$$\hat{g} = \sum_{k=1}^n c_k e^{2\pi i \xi_k x_k} \cdot e^{-2\pi i \xi_k x_k} M_{\xi_k} T_{x_k} \varphi = f,$$

as desired.

(iii) We first compute for any given $(x,\xi),(u,\eta)\in\mathbb{R}^2$:

$$\begin{aligned}
\langle \widehat{M_\xi T_x\varphi}, \widehat{M_\eta T_u\varphi}\rangle &= \langle e^{2\pi i\xi x} M_{-x}T_\xi\hat{\varphi}, e^{2\pi i\eta u} M_{-u}T_\eta\hat{\varphi}\rangle \\
&= e^{2\pi i(\xi x-\eta u)}\langle M_{-x}T_\xi\varphi, M_{-u}T_\eta\varphi\rangle \\
&= e^{2\pi i(\xi x-\eta u)}\frac{1}{\sqrt{2}}e^{-\pi i(-u+x)(\xi+\eta)}\varphi\left(\frac{\eta-\xi}{\sqrt{2}}\right)\varphi\left(\frac{-u+x}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{2}}e^{-\pi i(\eta-\xi)(x+u)}\varphi\left(\frac{u-x}{\sqrt{2}}\right)\varphi\left(\frac{\eta-\xi}{\sqrt{2}}\right) \\
&= \langle M_\xi T_x\varphi, M_\eta T_u\varphi\rangle .
\end{aligned}$$

Now let $f\in\mathcal{X}$ be as above. Then

$$\begin{aligned}
\|\hat{f}\|^2_{L^2(\mathbb{R})} &= \left\langle \sum_{k=1}^n c_k\,\widehat{M_{\xi_k}T_{x_k}}\varphi, \sum_{l=1}^n c_l\,\widehat{M_{\xi_l}T_{x_l}}\varphi \right\rangle \\
&= \sum_{k,l=1}^n c_k\overline{c_l}\left\langle \widehat{M_{\xi_k}T_{x_k}}\varphi, \widehat{M_{\xi_l}T_{x_l}}\varphi\right\rangle \\
&= \sum_{k,l=1}^n c_k\overline{c_l}\,\langle M_{\xi_k}T_{x_k}\varphi, M_{\xi_l}T_{x_l}\varphi\rangle \\
&= \left\langle \sum_{k=1}^n c_k\, M_{\xi_k}T_{x_k}\varphi, \sum_{l=1}^n c_l\, M_{\xi_l}T_{x_l}\varphi \right\rangle = \|f\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

□

Lemma 3. Let $\mathcal{X}$ be as in the previous lemma. Then $\mathcal{X}$ is dense in $L^2(\mathbb{R})$, that is $\overline{\mathcal{X}}=L^2(\mathbb{R})$.

Proof. Fix an $f\in L^2(\mathbb{R})$ and an $\epsilon>0$. Since $\int_{-\infty}^{+\infty}|f(x)|^2\mathrm{d}x<\infty$, we have

$$\|f-\mathbb{1}_{[-R,R]}f\|^2_{L^2(\mathbb{R})}=\int_{-\infty}^{+\infty}\mathbb{1}_{\mathbb{R}\setminus[-R,R]}(t)|f(t)|^2\mathrm{d}t\to 0,$$

as $R\to\infty$. Fix an $R>0$ large enough so that $\|f-\mathbb{1}_{[-R,R]}f\|_{L^2(\mathbb{R})}<\epsilon/2$. Also fix a large enough $A>0$ so that $A>R$ and

$$\frac{2e^{-2\pi A^2}}{1-e^{-2\pi A^2}}\left(e^{\pi R^2}\|f\|_{L^2(\mathbb{R})}+\epsilon\right)^2<\frac{\epsilon^2}{8}. \tag{6}$$

The significance of this choice will become apparent later. Now consider the function $g(t)=\mathbb{1}_{[-R,R]}(t)f(t)e^{\pi t^2}$. Note that

$$\int_{-A}^{A}|g(t)|^2\mathrm{d}t=\int_{-A}^{A}\mathbb{1}_{[-R,R]}(t)|f(t)|^2e^{2\pi t^2}\mathrm{d}t\leqslant e^{2\pi R^2}\int_{-A}^{A}|f(t)|^2\mathrm{d}t\leqslant e^{2\pi R^2}\|f\|^2_{L^2(\mathbb{R})}<\infty, \tag{7}$$

so $g\in L^2([-A,A])$. Thus the function g can be approximated by its partial Fourier series on $[-A,A]$, that is, there exists an $N\in\mathbb{N}$ and coefficients $(c_k)_{k=-N}^N$ such that

$$s(t)=\sum_{k=-N}^{N}c_ke^{\pi ikt/A}$$

satisfies $\|g-s\|^2_{L^2([-A,A])} < \epsilon^2/8$. We now bound $\|s\|_{L^2([-A,A])}$ by using (7) as follows:

$$\|s\|_{L^2([-A,A])} \leqslant \|g\|_{L^2([-A,A])} + \|s-g\|_{L^2([-A,A])} \leqslant e^{\pi R^2}\|f\|_{L^2(\mathbb{R})} + \sqrt{\frac{\epsilon^2}{8}} < e^{\pi R^2}\|f\|_{L^2(\mathbb{R})} + \epsilon.$$

Now define the function h by

$$h(t) := s(t)e^{-\pi t^2} = \sum_{k=-N}^{N} c_k e^{\pi i k t/A}\varphi(t) = \sum_{k=-N}^{N} c_k M_{\frac{k}{2A}}\varphi(t),$$

so that we have $h \in \mathcal{X}$. By using the $2A$-periodicity of s and fact that $\mathbb{1}_{[-R,R]}f$ is zero outside the interval $[-A, A]$ we can estimate

$$\begin{aligned}
\|\mathbb{1}_{[-R,R]}f - h\|^2_{L^2(\mathbb{R})} &= \sum_{n\in\mathbb{Z}} \int_{(2n-1)A}^{(2n+1)A} |\mathbb{1}_{[-R,R]}(t)f(t) - h(t)|^2 \mathrm{d}t \\
&= \int_{-A}^{A} |\mathbb{1}_{[-R,R]}(t)f(t) - s(t)e^{-\pi t^2}|^2 \mathrm{d}t + \sum_{n\in\mathbb{Z}\setminus\{0\}} \int_{(2n-1)A}^{(2n+1)A} |s(t)e^{-\pi t^2}|^2 \mathrm{d}t \\
&\leqslant \int_{-A}^{A} |g(t) - s(t)|^2 e^{-2\pi t^2} \mathrm{d}t + \sum_{n\in\mathbb{Z}\setminus\{0\}} \int_{(2n-1)A}^{(2n+1)A} |s(t)|^2 e^{-2\pi(2|n|-1)^2A^2} \mathrm{d}t \\
&\leqslant \|g-s\|^2_{L^2([-A,A])} + \sum_{n\in\mathbb{Z}\setminus\{0\}} e^{-2\pi|n|A^2} \int_{-A}^{A} |s(t+2nA)|^2 \mathrm{d}t \\
&= \|g-s\|^2_{L^2([-A,A])} + 2\frac{e^{-2\pi A^2}}{1-e^{-2\pi A^2}} \|s\|^2_{L^2([-A,A])} \\
&< \frac{\epsilon^2}{8} + 2\frac{e^{-2\pi A^2}}{1-e^{-2\pi A^2}} \left(e^{\pi R^2}\|f\|_{L^2(\mathbb{R})} + \epsilon\right)^2 \\
&< \frac{\epsilon^2}{8} + \frac{\epsilon^2}{8} = \frac{\epsilon^2}{4},
\end{aligned}$$

where the last line is due to (6). Therefore

$$\|f-h\|_{L^2(\mathbb{R})} \leqslant \|f - \mathbb{1}_{[-R,R]}f\|_{L^2(\mathbb{R})} + \|\mathbb{1}_{[-R,R]}f - h\|_{L^2(\mathbb{R})} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We have shown that an arbitrary element f of $L^2(\mathbb{R})$ can be approximated to arbitrary precision ϵ by an element $h \in \mathcal{X}$, and so $\mathcal{X}$ is dense in $L^2(\mathbb{R})$. □

With the previous two Lemmas we are finally in the position to define the Fourier transform on $L^2(\mathbb{R})$:

Theorem 1. Let $\mathcal{X}$ be as in Lemma 2. Then the Fourier transform $f \mapsto \hat{f}$ on $\mathcal{X}$ extends to a unitary operator $\mathcal{F}$ on $L^2(\mathbb{R})$.

Proof. Since $\mathcal{X} \subset L^1(\mathbb{R})$, we can define $\mathcal{F}f := \hat{f}$, for $f \in \mathcal{X}$. Now let f be an arbitrary element of $L^2(\mathbb{R})$. By Lemma 3 there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{X}$ such that $\|f_n - f\|_{L^2(\mathbb{R})} \to 0$ as $n \to \infty$. Since the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges, we know it is a Cauchy sequence. Now, by Lemma 2 (iii) we have $\|\hat{f}_n - \hat{f}_m\|_{L^2(\mathbb{R})} = \|f_n - f_m\|_{L^2(\mathbb{R})}$ for $m, n \in N$, so $\{\hat{f}_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence. Since the space $L^2(\mathbb{R})$ is complete, it follows that $\{\hat{f}_n\}_{n\in\mathbb{N}}$ converges to some element of $L^2(\mathbb{R})$.

We set $\mathcal{F}f = \lim_{n\to\infty} \hat{f}_n$. As $\hat{\cdot}$ and lim are linear, it follows immediately that $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined in this way is linear. Moreover,

$$\|\mathcal{F}f\|_{L^2(\mathbb{R})} = \|\lim_{n\to\infty} \hat{f}_n\|_{L^2(\mathbb{R})} = \lim_{n\to\infty} \|\hat{f}_n\|_{L^2(\mathbb{R})} = \lim_{n\to\infty} \|f_n\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})},$$

so $\mathcal{F}$ is an isometry. In order to conclude that $\mathcal{F}$ is unitary, it remains to show that $\mathcal{F}$ is surjective. Fix an arbitrary element f of $L^2(\mathbb{R})$ and let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ such that $\lim_{n\to\infty} \|f_n - f\|_{L^2(\mathbb{R})} = 0$ as before. By Lemma 2 (ii) we can find another sequence $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ such that $f_n = \hat{g}_n$, for all $n \in \mathbb{N}$. Then, by Lemma 2 (iii) we have

$$\|g_n - g_m\|_{L^2(\mathbb{R})} = \|\hat{g}_n - \hat{g}_m\|_{L^2(\mathbb{R})} = \|f_n - f_m\|_{L^2(\mathbb{R})},$$

so $\{g_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and so it converges to some $g \in L^2(\mathbb{R})$. Then, by definition of $\mathcal{F}$,

$$\mathcal{F}g = \lim_{n\to\infty} \hat{g}_n = \lim_{n\to\infty} f_n = f,$$

where the limits are in $L^2(\mathbb{R})$. We have shown that an arbitrary $f \in L^2(\mathbb{R})$ is in the image of $\mathcal{F}$, and so $\mathcal{F}$ is surjective. Thus $\mathcal{F}$ is a surjective linear isometry on a Hilbert space, and therefore it is unitary. □

With some extra work, it can be shown that the space $\mathcal{X}$ in the statement of the previous theorem can be replaced with the whole of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, i.e., we have the following theorem.

Theorem 2 (Plancherel)**.** The Fourier transform $f \mapsto \hat{f}$ on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ extends to a unitary operator $\mathcal{F}$ on $L^2(\mathbb{R})$.

Thus the Fourier transform of functions f which are in $L^2(\mathbb{R})$ but not in $L^1(\mathbb{R})$ (such as $\frac{\sin(\pi x)}{\pi x}$) can be obtained as the limit of the L^1-Fourier transforms (as given by the formula (1)) of any functions in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ that converge to f. In practice, this means that the Fourier transform can often[1] be calculated just like we did heuristically in (5):

$$\begin{aligned}\mathcal{F}f(\omega) &= \lim_{R\to\infty} \mathcal{F}(\mathbb{1}_{[-R,R]}f)(\omega) = \lim_{R\to\infty} \widehat{(\mathbb{1}_{[-R,R]}f)}(\omega)\\ &= \lim_{R\to\infty} \int_{-\infty}^{+\infty} (\mathbb{1}_{[-R,R]}f)(x)e^{-2\pi i\omega x}\mathrm{d}x\\ &= \lim_{R\to\infty} \int_{-R}^{R} f(x)e^{-2\pi i\omega x}\mathrm{d}x,\end{aligned}$$

as $\mathbb{1}_{[-R,R]}f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for all $R > 0$ and $\|f - \mathbb{1}_{[-R,R]}f\|_{L^2(\mathbb{R})} \to 0$ as $R \to \infty$.

References

[1] K. Gröchenig, *Foundations of Time-Frequency Analysis.* Boston, MA, U.S.A.: Birkhäuser, 2001.

[1] The limit is always valid, but in some cases it may be difficult to compute its exact value.