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Notes on compact sets

Throughout the notes, $(\mathcal{X}, d)$ will denote a metric space. These notes are partly based on the notes [1].

Definitions

Definition 1 (Open cover)**.** *Given an index set I and a subset $\mathcal{C}$ of $\mathcal{X}$, a family $\{\mathcal{O}_i\}_{i\in I}$ of open sets $\mathcal{O}_i \subseteq \mathcal{X}$ is said to be an* open cover *of $\mathcal{C}$ if*

$$\mathcal{C} \subseteq \bigcup_{i\in I} \mathcal{O}_i.$$

The family $\{\mathcal{O}_j\}_{j\in J}$ is said to be a finite subcover *of $\{\mathcal{O}_i\}_{i\in I}$ if it is a cover of $\mathcal{C}$ and contains a finite number of open sets, i.e., $\mathcal{C} \subseteq \bigcup_{j\in J} \mathcal{O}_j$ and $J \subseteq I$ is finite.*

Definition 2 (Compactness)**.** *A subset $\mathcal{C}$ of $\mathcal{X}$ is said to be* compact *if every open cover of $\mathcal{C}$ admits a finite subcover.*

Definition 3 (Sequential compactness)**.** *A subset $\mathcal{C}$ of $\mathcal{X}$ is said to be* sequentially compact *if every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}$ admits a converging subsequence.*

Equivalence of the notions of compactness

We have defined two distinct notions of compactness. It turns out that in a metric space, these two notions are equivalent and relate to the notion of covering numbers seen in class.

Lemma 1. *The following are equivalent:*

a) The set $\mathcal{C} \subseteq \mathcal{X}$ is compact;

b) The set $\mathcal{C} \subseteq \mathcal{X}$ is sequentially compact;

c) The set $\mathcal{C} \subseteq \mathcal{X}$ is complete and, for every $\varepsilon > 0$, the ε-covering number of $\mathcal{C}$ is finite.

Remark 1. *One often encounters the denomination* 'totally bounded' *for a set which, for every $\varepsilon > 0$, has a finite ε-covering number.*

Remark 2. *In the lecture, we defined the covering number only for compact sets. The implication 'a) $\implies$ c)' in Lemma 1 ensures that the covering number a compact set is well defined and finite.*

Proof. We prove that compactness is equivalent to sequential compactness, which in turn is equivalent to being complete and totally bounded.

'$a) \implies b)$': Assume by contradiction that $\mathcal{C}$ is compact but not sequentially compact and take $\{x_n\}_{n\in\mathbb{N}}$ a sequence which admits no converging subsequence. Consider the family

$$\{\mathcal{O} \subseteq \mathcal{C} \mid \mathcal{O} \text{ is open and contains at most a finite number of elements of } \{x_n\}_{n\in\mathbb{N}}\}\,.$$

Then this family is an open cover of $\mathcal{C}$. Indeed, for every $x \in \mathcal{C}$, since no subsequence of $\{x_n\}_{n\in\mathbb{N}}$ converges to x, there exists an open neighborhood $\mathcal{O}$ of x which contains at most finitely many elements of $\{x_n\}_{n\in\mathbb{N}}$. By compactness of $\mathcal{C}$, it can then be covered by finitely many open sets $\mathcal{O}$, which each contain at most finitely many elements of $\{x_n\}_{n\in\mathbb{N}}$. We have reached a contradiction.

'$b) \implies a)$': Let's take $\mathcal{C}$ to be sequentially compact. Let $\{\mathcal{O}_i\}_{i\in I}$ be an open cover of $\mathcal{C}$. We first prove that there exists $\delta > 0$ such that, for every $x \in \mathcal{C}$, there exists $i \in I$ satisfying $\mathcal{B}(x,\delta) \subseteq \mathcal{O}_i$. We argue by contradiction, that is, we suppose that we can find a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{C}$ such that $\mathcal{B}(x_n, 1/n)$ is not contained in any $\mathcal{O}_i$. By sequential compactness, $\{x_n\}_{n\in\mathbb{N}}$ admits a subsequence which converges to some $x \in \mathcal{C}$. Because $\{\mathcal{O}_i\}_{i\in I}$ covers $\mathcal{C}$, there exists $i \in I$ such that $x \in \mathcal{O}_i$, and, since $\mathcal{O}_i$ is open, there exists $\varepsilon > 0$ such that $\mathcal{B}(x,\varepsilon) \subseteq \mathcal{O}_i$. Since $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence converging to x, there exists $n \in \mathbb{N}$ such that $\mathcal{B}(x_n, 1/n) \subseteq \mathcal{B}(x,\varepsilon) \subseteq \mathcal{O}_i$, which is in contradiction with the construction of $\{x_n\}_{n\in\mathbb{N}}$.

We have proven that there exists $\delta > 0$ such that, for every $x \in \mathcal{C}$, there exists $i \in I$ satisfying $\mathcal{B}(x,\delta) \subseteq \mathcal{O}_i$. Assume, again by way of contradiction, that there is no finite subcover of $\{\mathcal{O}_i\}_{i\in I}$. Then, one can construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{C}$ such that $x_p \notin \mathcal{B}(x_q,\delta)$ for all $p \neq q$, otherwise $\mathcal{C}$ could be covered by a finite number of balls of radius δ and *a fortiori* by a finite number open sets $\mathcal{O}_i$ containing these balls (which, we have proven above, exist). The sequence $\{x_n\}_{n\in\mathbb{N}}$ admits a convergent subsequence by sequential compactness of $\mathcal{C}$, but also satisfies $d(x_p, x_q) \geq \delta$ for all $p \neq q$ by construction, which yields a contradiction and concludes the proof.

'$b) \implies c)$': Let's take $\mathcal{C}$ to be sequentially compact and $\{x_n\}_{n\in\mathbb{N}}$ a Cauchy sequence in $\mathcal{C}$. By sequential compactness, $\{x_n\}_{n\in\mathbb{N}}$ has a converging subsequence. Since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy and has a converging subsequence, it is convergent in $\mathcal{C}$ and therefore $\mathcal{C}$ is complete.

Given $\varepsilon > 0$, the family

$$\{\mathcal{B}(x,\varepsilon) \mid x \in \mathcal{C}\}$$

is an open cover of $\mathcal{C}$. We have already proven that sequential compactness implies compactness. One can thus find a finite subcover to $\{\mathcal{B}(x,\varepsilon) \mid x \in \mathcal{C}\}$. The cardinality of a minimal such subcover is, by definition, the ε-covering number of $\mathcal{C}$, which is therefore finite.

'$c) \implies b)$': Let's assume that $\mathcal{C}$ is complete and has finite $(1/m)$-covering number for every integer $m \geq 1$. We consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{C}$ and want to show that it admits a converging subsequence. Let $\{y_1^{(1)}, \dots, y_{N_1}^{(1)}\}$ be a 1-covering of $\mathcal{C}$, then, by finiteness of the covering, one can extract a subsequence $\{x_n^{(1)}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that all its elements are in $\mathcal{B}(y_\ell^{(1)}, 1)$ for some $1 \leq \ell \leq N_1$. Likewise, one can find a subsequence $\{x_n^{(m)}\}_{n\in\mathbb{N}}$ of $\{x_n^{(m-1)}\}_{n\in\mathbb{N}}$ such that all its elements are contained in $\mathcal{B}(y_\ell^{(m)}, 1/m)$, where $\{y_1^{(m)}, \dots, y_{N_m}^{(m)}\}$ is a $(1/m)$-covering of $\mathcal{C}$ and $1 \leq \ell \leq N_m$. The sequence $\{x_k^{(k)}\}_{k\in\mathbb{N}}$ is therefore a Cauchy sequence. By completeness of $\mathcal{C}$, it is a converging subsequence of $\{x_n\}_{n\in\mathbb{N}}$, which completes the proof. □

Examples

Corollary 1 (Finite metric space)**.** *A metric space $(\mathcal{X}, d)$ containing a finite number of elements is compact.*

Proof. Since $\mathcal{X}$ is finite, for any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathcal{X}$ there must exists a point $x \in \mathcal{X}$ such that $\{x_n\}_{n\in\mathbb{N}}$ passes infinitely many times through x. Writing $\{n_k\}_{k\in\mathbb{N}}$ the indices for which $x_{n_k} = x$, we have constructed a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ which converges (to x). Therefore $\mathcal{X}$ is sequentially compact. From the implication '$b) \implies a)$' in Lemma 1, the set $\mathcal{X}$ is compact. □

Example 1. *Given $n \in \mathbb{N}$, let the Hamming cube be defined as*

$$\mathbb{H}^n := \{0,1\}^n,$$

and let

$$\begin{aligned} d\colon \mathbb{H}^n \times \mathbb{H}^n &\to \mathbb{R}_+ \\ (x,y) &\mapsto \#\{i \in \{1,\ldots,n\} \mid x_i \neq y_i\} \end{aligned}$$

be a metric on $\mathbb{H}^n$. Then $\mathbb{H}^n$ is finite and therefore compact. From the implication '$a) \implies c)$' in Lemma 1, the set $\mathbb{H}^n$ has a finite covering number. We refer the reader to [Problem 4, Exam Winter 2021/2022] for an estimation of this covering number.

Corollary 2 (Closed bounded sets in $\mathbb{R}^n$)**.** *A closed bounded subset $\mathcal{C}$ of $\mathbb{R}^n$ equipped with the Euclidean norm $\|\cdot\|_2$ is compact.*

Remark 3. *Since $\mathbb{R}^n$ is a finite dimensional vector space, all the norms on $\mathbb{R}^n$ are equivalent. Therefore, the result above generalizes readily to $\mathbb{R}^n$ equipped with any norm $\|\cdot\|$.*

Proof. Consider a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}$. It converges to some $x \in \mathbb{R}^n$ by completeness of $\mathbb{R}^n$. Since $\mathcal{C}$ is closed, $x \in \mathcal{C}$ which proves that $\mathcal{C}$ is complete.

Fix $\varepsilon > 0$. Since $\mathcal{C}$ is bounded, it is included in the cube $[-R,R]^n$. From the lecture, we know that the ε-covering number of this cube is finite. To show that it implies that $\mathcal{C}$ is totally bounded, we make the following observation. Since the ε-covering number of the cube is finite, there exist $x_1,\ldots,x_N \in \mathbb{R}^n$ such that $[-R,R]^n \subseteq \cup_{i=1}^N \mathcal{B}(x_i,\varepsilon/2)$. Then, for every $i \in \{1,\ldots,N\}$ such that $\mathcal{C} \cup \mathcal{B}(x_i,\varepsilon/2) \neq \emptyset$, choose $c_i \in \mathcal{C} \cup \mathcal{B}(x_i,\varepsilon/2)$. It follows from $\mathcal{B}(x_i,\varepsilon/2) \subset \mathcal{B}(c_i,\varepsilon)$ that $\mathcal{C} \subseteq \cup_i \mathcal{B}(c_i,\varepsilon)$. The set $\mathcal{C}$ is thus totally bounded.

$\mathcal{C}$ is complete and totally bounded, therefore, from the implication '$c) \implies a)$' in Lemma 1, $\mathcal{C}$ is compact. □

References

[1] Giada Franz. *Equivalent Notions of Compactness.*