Eidgenössische Technische Hochschule Zürich
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Examination on Mathematics of Information
August 29, 2018

Please note:

- Duration of exam: 180 minutes
- Maximum number of points: 100
- You are allowed to use any printed or handwritten material (i.e., books, lecture and discussion session notes, summaries), but no computers, tablets, smart phones or other electronic devices.
- Your solutions should be explained in detail and your handwriting needs to be clean and readable.
- Please do not use red or green pens. You may use pencils.
- Please note that the ETHZ "Disziplinarordnung RSETHZ 361.1" applies.

Before you start:

1. The problem statements consist of 7 pages. Please verify that you have received all 7 pages.
2. Please fill in your name and your Legi-number below.
3. Please place an identification document on your desk so we can verify your identity.

During the exam:

4. For your solutions, please use only the empty sheets provided by us. Should you need more paper, please let us know.

After the exam:

5. Please number all the sheets you want to turn in. Please specify below the number of *additional* sheets you want to turn in (excluding the sheets with the problem statements). All sheets containing problem statements must be turned in.

Surname: .... Given name: ....

Legi-No.: ....

Number of sheets turned in: ....

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1. **Problem 1**

The Hilbert space $L^2(\mathbb{R}^2)$ consists of all $F : \mathbb{R}^2 \to \mathbb{C}$ such that $\|F\|_{L^2(\mathbb{R}^2)} < \infty$, where

$$\|F\|_{L^2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} |F(x,y)|^2 \,\mathrm{d}x \,\mathrm{d}y = \int_{\mathbb{R}^2} |F(x,y)|^2 \,\mathrm{d}y \,\mathrm{d}x.$$

(a) Write down the definition of a unitary operator on a general Hilbert space.

(b) Let $f, g \in L^2(\mathbb{R})$. We define the short-time Fourier transform $V_g f : \mathbb{R}^2 \to \mathbb{C}$ of f with respect to window g by

$$(V_g f)(x,\omega) = \int_{\mathbb{R}} f(t)\overline{g(t-x)}e^{-2\pi i\omega t}\mathrm{d}t.$$

Consider the following two transformations:

(i) The asymmetric coordinate transform $\mathcal{T}_a$ is defined for a function $F : \mathbb{R}^2 \to \mathbb{C}$ by

$$\mathcal{T}_a F(x,y) = F(y, y-x).$$

Show that $\mathcal{T}_a$ is a unitary operator on $L^2(\mathbb{R}^2)$.
[Hint: First show that $\mathcal{T}_a$ maps $L^2(\mathbb{R}^2)$ functions to $L^2(\mathbb{R}^2)$ functions, and then compute the adjoint $\mathcal{T}_a^*$ explicitly.]

(ii) The partial Fourier transform $\mathcal{F}_2 : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is the unique bounded linear operator on $L^2(\mathbb{R}^2)$ with the following property:
Whenever $F \in L^2(\mathbb{R}^2)$ is such that $F(x,\cdot) \in L^1(\mathbb{R})$ for all $x \in \mathbb{R}$, meaning that

$$\int_{\mathbb{R}} |F(x,y)| \,\mathrm{d}y < \infty \quad \text{for all } x \in \mathbb{R},$$

$\mathcal{F}_2 F$ is given by the formula

$$(\mathcal{F}_2 F)(x,\omega) = \int_{\mathbb{R}} F(x,t)e^{-2\pi i\omega t}\mathrm{d}t, \quad \text{for all } (x,\omega) \in \mathbb{R}^2. \tag{1}$$

You may use—without proof—the fact that $\mathcal{F}_2$ is a unitary operator.

Show carefully that

$$V_g f = \mathcal{F}_2 \mathcal{T}_a (f \otimes \overline{g}) \quad \text{for all } f, g \in L^2(\mathbb{R}), \tag{2}$$

where for two functions $h_1, h_2 : \mathbb{R} \to \mathbb{C}$ we write $h_1 \otimes h_2$ to denote the function $(h_1 \otimes h_2)(x,y) = h_1(x)h_2(y)$.
[Please note that if you want to use (1) to compute $\mathcal{F}_2 F$ for a function $F \in L^2(\mathbb{R}^2)$, you first have to verify that F satisfies the additional assumption that $F(x,\cdot) \in L^1(\mathbb{R})$ for all $x \in \mathbb{R}$.]

(c) Using (2) deduce that $V_g f \in L^2(\mathbb{R}^2)$ for all $f, g \in L^2(\mathbb{R})$, and that

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle = \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}, \quad \text{for all } f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}). \tag{3}$$

Problem 2

A multiresolution approximation is an increasing sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed linear subspaces of $L^2(\mathbb{R})$ such that

(I) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$,

(II) for all $f \in L^2(\mathbb{R})$ and $j \in \mathbb{Z}$, $f \in V_j \iff f(2\,\cdot) \in V_{j+1}$,

(III) for all $f \in L^2(\mathbb{R})$ and $k \in \mathbb{Z}$, $f \in V_0 \iff f(\cdot - k) \in V_0$,

(IV) there exists a function $\varphi \in V_0$ such that $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is a Riesz basis of the space V_0.

Let $\varphi \in L^2(\mathbb{R})$ be given in the Fourier domain by

$$\hat{\varphi}(\xi) = \begin{cases} 1, & |\xi| \le \frac{1}{3} \\ \cos\left(\frac{\pi}{2}\nu(3|\xi|-1)\right), & \frac{1}{3} < |\xi| \le \frac{2}{3} \\ 0, & \text{otherwise} \end{cases},$$

where

$$\nu(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x \end{cases}.$$

(a) (i) Sketch $\hat{\varphi}$ on the interval $[-1, 1]$.

(ii) Show that $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal system, that is

$$\langle \varphi(\cdot - k), \varphi(\cdot - l)\rangle = \begin{cases} 0, & k \neq l \\ 1, & k = l \end{cases} \quad \text{for all } k, l \in \mathbb{Z}.$$

You may use — without proof — the fact that, for any given $g \in L^2(\mathbb{R})$, we have that $\{g(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal system if and only if $\sum_{n\in\mathbb{Z}} |\hat{g}(\xi + n)|^2 = 1$, for all $\xi \in \mathbb{R}$.

(b) Define V_0 to be the closure of $\operatorname{span}\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$, and let $V_j = \{f(2^j\cdot) : f \in V_0\}$ for $j \in \mathbb{Z}$.

(i) Let P_{V_j} denote the orthogonal projection onto V_j. You may use — without proof — that $\{\varphi_{j,k} := 2^{\frac{j}{2}}\varphi(2^j \cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_j, and that $\|P_{V_j} f\|_{L^2(\mathbb{R})}$ is given by the following expression:

$$\|P_{V_j} f\|^2_{L^2(\mathbb{R})} = \sum_{k\in\mathbb{Z}} \left| \int_{\mathbb{R}} f(x)\overline{\varphi_{j,k}(x)}\,\mathrm{d}x \right|^2, \quad \text{for all } f \in L^2(\mathbb{R}). \tag{4}$$

By applying the Plancherel identity to (4) show that

$$\|P_{V_j} f\|^2_{L^2(\mathbb{R})} + \|P_{V_j}(f(\cdot - 2^{-(j+1)}))\|^2_{L^2(\mathbb{R})} = 2\|\hat{\varphi}(2^{-j}\,\cdot\,)\hat{f}\|^2_{L^2(\mathbb{R})}, \tag{5}$$

for all $j \in \mathbb{Z}$ and $f \in L^2(\mathbb{R})$.

[Hint: The Plancherel identity states that for any two functions $h_1, h_2 \in$

$L^2(\mathbb{R})$ we have $\langle h_1, h_2\rangle = \langle \hat{h}_1, \hat{h}_2\rangle$. After applying this identity, combine the expression you obtain into one sum. Then reinterpret the resulting expression as an expansion in

$$\mathcal{E} = \{e_m(\xi) = 2^{-\frac{j+1}{2}} e^{\frac{-2\pi i \xi m}{2^{j+1}}} : m \in \mathbb{Z}\}.$$

For this you will need to use the fact that $\hat{\varphi}$ is zero outside a finite interval to replace the limits of the integrals accordingly. You may use — without proof—that $\mathcal{E}$ is an orthonormal basis for $L^2([-2^j, 2^j])$.]

(ii) For the remainder of the question you may use—without proof—that

$$\lim_{j\to\infty} \|\hat{f} - \hat{\varphi}(2^{-j}\cdot)\hat{f}\|_{L^2(\mathbb{R})} = 0, \quad \text{and}$$
$$\lim_{j\to-\infty} \|\hat{\varphi}(2^{-j}\cdot)\hat{f}\|_{L^2(\mathbb{R})} = 0,$$

for any $f \in L^2(\mathbb{R})$. Use these facts together with (5) to show that

$$\lim_{j\to\infty} \|P_{V_j} f\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}.$$

Deduce that $\lim_{j\to\infty} \|f - P_{V_j} f\|_{L^2(\mathbb{R})} = 0$.
[Hint: Recall the following facts, which you may use without proof, about orthonormal projections: ${P_{V_j}}^2 = P_{V_j}$, $P^*_{V_j} = P_{V_j}$, and $\|P_{V_j} g\|_{L^2(\mathbb{R})} \le \|g\|_{L^2(\mathbb{R})}$, for all $g \in L^2(\mathbb{R})$.]

(iii) Use (5) to show that $\lim_{j\to-\infty} \|P_{V_j} f\|_{L^2(\mathbb{R})} = 0$.

(c) Use the results you have obtained so far to prove that $\{V_j\}_{j\in\mathbb{Z}}$ defined in (b) is a multiresolution approximation of $L^2(\mathbb{R})$.
[Hint: Use (b)(ii) and (b)(iii) to prove item (I) in the definition of a multiresolution approximation. To show that $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$, it suffices to establish that for any $\epsilon > 0$, there exist a $j \in \mathbb{Z}$ and a function $\tilde{f} \in V_j$, such that $\|f - \tilde{f}\|_{L^2(\mathbb{R})} < \epsilon$. To show this you can use $\lim_{j\to\infty} \|f - P_{V_j} f\|_{L^2(\mathbb{R})} = 0$.]

Problem 3

For this problem we use the two-indices notation $f_{x,y}$ to denote time-frequency shifts, that is, if $f : \mathbb{R} \to \mathbb{C}$ is a function and $x, y \in \mathbb{R}$, then we shall write $f_{x,y}(t) = e^{2\pi i y t} f(t - x)$.

You may use—without proof—that for all $f, h \in L^2(\mathbb{R})$ the function $(x, y) \mapsto \langle f, h_{x,y}\rangle$ is an $L^2(\mathbb{R}^2)$ function, that is

$$\int_{\mathbb{R}^2} |\langle f, h_{x,y}\rangle|^2 \,\mathrm{d}x\,\mathrm{d}y < \infty.$$

Furthermore, you may also use—without proof—the following identity:

$$\int_{\mathbb{R}^2} \langle f, g_{x,y}\rangle \overline{\langle u, v_{x,y}\rangle} \,\mathrm{d}x\,\mathrm{d}y = \langle f, u\rangle \overline{\langle g, v\rangle} \quad \text{for all } f, g, u, v \in L^2(\mathbb{R}). \tag{IR}$$

(a) Consider a Weyl-Heisenberg system $\mathcal{G} = \{g_{mT,nF}\}_{m,n\in\mathbb{Z}}$ with time-frequency parameters $T > 0$ and $F > 0$. Assume that $\mathcal{G}$ is a frame for $L^2(\mathbb{R})$. Let $\mathbb{S}$ be the corresponding frame operator and $\tilde{g} = \mathbb{S}^{-1}g$ the canonical dual function. We know that $\tilde{\mathcal{G}} = \{\tilde{g}_{mT,nF}\}_{m,n\in\mathbb{Z}}$ is the canonical dual frame to $\mathcal{G}$, and that the following reconstruction formula holds:

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, \tilde{g}_{mT,nF}\rangle g_{mT,nF} \quad \text{for all } f \in L^2(\mathbb{R}).$$

Using this reconstruction formula, prove that

$$\langle f, h\rangle = \sum_{m,n\in\mathbb{Z}} \langle f, \tilde{g}_{mT+x,nF+y}\rangle \langle g_{mT+x,nF+y}, h\rangle \tag{6}$$

for all $f, h \in L^2(\mathbb{R})$ and all $x, y \in \mathbb{R}$.
[Hint: Expand $f_{-x,-y}$ using the reconstruction formula, and then take the inner product of both sides with $h_{-x,-y}$.]

(b) By integrating both sides of (6) over $(x, y) \in [0, T) \times [0, F)$ for a fixed pair of functions f and h, show that $\langle g, \tilde{g}\rangle = TF$. Justify the validity of any manipulations you do by verifying the following absolute convergence property:

$$\sum_{m,n\in\mathbb{Z}} \int_0^F \int_0^T |\langle f, \tilde{g}_{mT+x,nF+y}\rangle \langle g_{mT+x,nF+y}, h\rangle| \,\mathrm{d}x\,\mathrm{d}y < \infty.$$

[Hint: For both calculations, rewrite the resulting expression as an integral over $\mathbb{R}^2$ which does not involve infinite summation.]

(c) Assume the following result from the lectures without proof:

Lemma. *Denote by $\mathcal{K}$ a countable index set. Let $\{h_k\}_{k\in\mathcal{K}}$ be a frame for a Hilbert space $\mathcal{H}$ and $\{\tilde{h}_k\}_{k\in\mathcal{K}}$ its canonical dual frame. For a fixed $z \in \mathcal{H}$, let $c_k = \langle z, \tilde{h}_k\rangle$ so that*

$z = \sum_{k \in \mathcal{K}} c_k h_k$. *If it is possible to find scalars* $\{a_k\}_{k \in \mathcal{K}}$ *such that* $z = \sum_{k \in \mathcal{K}} a_k h_k$, *then we must have*

$$\sum_{k \in \mathcal{K}} |a_k|^2 = \sum_{k \in \mathcal{K}} |c_k|^2 + \sum_{k \in \mathcal{K}} |c_k - a_k|^2.$$

Find two distinct sets $\{a_{m,n}\}_{m,n \in \mathbb{Z}}$ such that $g = \sum_{m,n \in \mathcal{K}} a_{m,n} g_{mT,nF}$, and then use the Lemma to deduce that $TF \leq 1$.

Problem 4

Define the following local-averaging operator

$$(\mathcal{A}x)_n = \int_{n-1/2}^{n+1/2} x(t)\mathrm{d}t, \quad n \in \mathbb{Z},$$

that takes in a function $x \in L^2(\mathbb{R})$ and yields a sequence $\{(\mathcal{A}x)_n\}_{n\in\mathbb{Z}}$ of local averages.

(a) Verify that $\mathcal{A}$ is a bounded linear operator from $L^2(\mathbb{R})$ to $\ell^2(\mathbb{Z})$ and compute the adjoint $\mathcal{A}^* : \ell^2(\mathbb{Z}) \to L^2(\mathbb{R})$ of $\mathcal{A}$.

(b) Show that $\|\mathcal{A}^*y\|_{L^2(\mathbb{R})} = \|y\|_{\ell^2(\mathbb{Z})}$ for all $y \in \ell^2(\mathbb{Z})$.

(c) Define $\mathrm{Im}(\mathcal{A}^*) = \{\mathcal{A}^*y : y \in \ell^2(\mathbb{Z})\}$. You may use — without proof — that $\mathrm{Im}(\mathcal{A}^*)$ is a closed subspace of $L^2(\mathbb{R})$, and thus a Hilbert space in its own right. For each $n \in \mathbb{Z}$ let $e_n = \mathbb{1}_{[n-1/2,\, n+1/2]}$ be the indicator function of the interval $[n-1/2, n+1/2]$. Show that $\mathcal{G} := \{e_n : n \in \mathbb{Z}\}$ is a subset of $\mathrm{Im}(\mathcal{A}^*)$, and that $\mathcal{G}$ is a frame for $\mathrm{Im}(\mathcal{A}^*)$. Show that $\mathcal{A}$ can be interpreted as the analysis operator associated with the frame $\mathcal{G}$.