Eidgenössische Technische Hochschule Zürich
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Dept. ITET
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Examination on Mathematics of Information
August 23, 2019

Please note:
- Exam duration: 180 minutes
- Maximum number of points: 100
- You are allowed to use any printed or handwritten material (i.e., books, lecture and discussion session notes, summaries), but no computers, tablets, smart phones or other electronic devices.
- Your solutions should be explained in detail and your handwriting needs to be clean and readable.
- Please do not use red or green pens. You may use pencils.
- Please note that the ETHZ "Disziplinarordnung RSETHZ 361.1" applies.

Before you start:
1. The problem statements consist of 6 pages including this page. Please verify that you have received all 6 pages.
2. Please fill in your name and your Legi-number below.
3. Please place an identification document on your desk so we can verify your identity.

During the exam:

4. For your solutions, please use only the empty sheets provided by us. Should you need more paper, please let us know.
5. Each problem consists of several subproblems. If you do not provide a solution to a subproblem, you may nonetheless assume its conclusion in the ensuing subproblems.

After the exam:

6. Please number all the sheets you want to hand in. Please specify below the number of *additional* sheets you want to hand in (excluding the sheets with the problem statements). All sheets containing problem statements must be handed in.

Family name: .... First name: ....

Legi-No.: ....

Number of additional sheets handed in: ....

Signature: ....

Throughout the exam, we use the following convention for the definition of the Fourier transform $\hat{f}$ of a function $f \in L^1(\mathbb{R})$:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \omega t} \mathrm{d}t, \quad \omega \in \mathbb{R},$$

and we use $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ to denote the corresponding L^2-Fourier transform.

Moreover, we use the discrete-time Fourier transform $(\mathrm{DTFT}) : \ell^2(\mathbb{Z}) \to L^2[0, 2\pi]$ given by

$$\mathrm{DTFT}\big\{\{a_n\}_{n\in\mathbb{Z}}\big\}(\theta) = \hat{\boldsymbol{a}}(\theta) = \sum_{n\in\mathbb{Z}} a_n e^{-in\theta}, \quad \theta \in [0, 2\pi],$$

with its inverse

$$\mathrm{DTFT}^{-1}\{\hat{\boldsymbol{a}}\} = \left(\frac{1}{2\pi} \int_0^{2\pi} \hat{\boldsymbol{a}}(\theta)\, e^{in\theta} \mathrm{d}\theta\right)_{n\in\mathbb{Z}}.$$

Problem 1

(a) Show that the function $f_1(x) = \frac{1}{1+|x|}$, $x \in \mathbb{R}$, is an element of $L^2(\mathbb{R})$ and compute $\|f_1\|_{L^2(\mathbb{R})}$. Show that f_1 is not an element of $L^1(\mathbb{R})$.

(b) Present an explicit example of a function $f_2 \in L^1(\mathbb{R})$ that is not an element of $L^2(\mathbb{R})$. [Hint: Such a function f_2 will necessarily be unbounded.]

In the remainder of Problem 1 you may use — without proof — the following fact: If $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then its Fourier transform $\hat{g} : \mathbb{R} \to \mathbb{C}$ is a continuous function.

(c) Let f be an element of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and additionally assume that the function $H_f : \mathbb{R} \to \mathbb{C}$ given by

$$H_f(\omega) = \omega\, \hat{f}(\omega), \quad \omega \in \mathbb{R},$$

is an element of $L^2(\mathbb{R})$. Define the function $G_f : \mathbb{R} \to \mathbb{C}$ by

$$G_f(\omega) = (1 + |\omega|)|\hat{f}(\omega)|, \quad \omega \in \mathbb{R}.$$

(i) Show that G_f is an element of $L^2(\mathbb{R})$.

(ii) Show that $\hat{f}$ is an element of $L^1(\mathbb{R})$ by proving the following bound:

$$\|\hat{f}\|_{L^1(\mathbb{R})} \leq \sqrt{2}\, \|G_f\|_{L^2(\mathbb{R})}.$$

[Hint: Apply the Cauchy-Schwarz inequality in the Hilbert space $L^2(\mathbb{R})$ to the functions $\omega \mapsto \frac{1}{1+|\omega|}$ and G_f.]

(iii) Show that f is a continuous function.

Problem 2

Fix $m \in \mathbb{N}$, and consider the finite-dimensional vector space $\mathbb{C}^{m \times m}$ equipped with the inner product

$$\langle A, B \rangle := \operatorname{tr}(B^H A) = \sum_{j,k=1}^{m} A_{jk} \overline{B_{jk}}, \quad A, B \in \mathbb{C}^{m \times m}. \tag{1}$$

For $\ell, n \in \{1, \ldots, m\}$, let $E^{(\ell,n)} \in \mathbb{C}^{m \times m}$ be the matrix whose entry (ℓ, n) has value 1, and all other entries have value 0, i.e.,

$$E^{(\ell,n)}_{jk} = \begin{cases} 1, & \text{if } j = \ell \text{ and } k = n \\ 0, & \text{else} \end{cases}, \quad \text{for all } (j,k) \in \{1, \ldots, m\}.$$

(a) Show that the set $\mathcal{E} := \{E^{(\ell,n)} : \ell, n \in \{1, \ldots, m\}\}$ is an orthonormal basis for $\mathbb{C}^{m \times m}$ with respect to the inner product (1). What is the dimension of the vector space $\mathbb{C}^{m \times m}$?

Next, define the $m \times m$ cyclic time-shift matrix D and the $m \times m$ modulation matrix M according to:

$$D = \begin{pmatrix} 0 & & & \cdots & 0 & 1 \\ 1 & 0 & & & & 0 \\ & 1 & 0 & & & \\ & & & \ddots & & \\ & & & & 1 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} e^{-\frac{2\pi i \cdot 0}{m}} & & & & \\ & e^{-\frac{2\pi i \cdot 1}{m}} & & \text{\Large 0} & \\ & & e^{-\frac{2\pi i \cdot 2}{m}} & & \\ & \text{\Large 0} & & \ddots & \\ & & & & e^{-\frac{2\pi i (m-1)}{m}} \end{pmatrix}.$$

Now, for $\ell, n \in \{0, 1, \ldots, m-1\}$, let $G^{(\ell,n)} = \frac{1}{\sqrt{m}} M^{\ell} D^{n}$.

(b) Show that the set $\mathcal{G} := \{G^{(\ell,n)} : \ell, n \in \{0, 1, \ldots, m-1\}\}$ is an orthonormal basis for $\mathbb{C}^{m \times m}$ with respect to the inner product (1).
[Hint: Use a dimensionality argument to establish completeness.]

Given orthonormal bases $\mathcal{B}_1$ and $\mathcal{B}_2$ for $\mathbb{C}^{m \times m}$, we define their mutual coherence $\mu(\mathcal{B}_1, \mathcal{B}_2)$ as

$$\mu(\mathcal{B}_1, \mathcal{B}_2) = \max_{U \in \mathcal{B}_1, V \in \mathcal{B}_2} |\langle U, V \rangle|.$$

(c) Compute $\mu(\mathcal{E}, \mathcal{G})$.

(d) Show that

$$\mu(\mathcal{B}_1, \mathcal{B}_2) \geq \frac{1}{m},$$

for every pair of orthonormal bases $\mathcal{B}_1$ and $\mathcal{B}_2$ for $\mathbb{C}^{m \times m}$.
[Hint: Assume that the claim is false for orthonormal bases $\mathcal{B}_1 = \{U_1, \ldots, U_{m^2}\}$ and $\mathcal{B}_2 = \{V_1, \ldots, V_{m^2}\}$, and then use the energy conservation identity $\|U_1\|^2 := \langle U_1, U_1 \rangle = \sum_{n=1}^{m^2} |\langle U_1, V_n \rangle|^2$ to derive a contradiction.]

Problem 3

In this problem you may use results from the "Orthonormal Wavelets" chapter of the discussion session notes, provided you refer to them explicitly and verify that the conditions of the result you intend to use are fulfilled.

Let $\beta : \mathbb{R} \to [0,1]$ be an increasing continuous function with the following properties

1. $\beta(x) = 0$, for $x \leq 0$,
2. $\beta(x) = 1$, for $x \geq 1$, and
3. $\beta(x) + \beta(1-x) = 1$, for $x \in [0,1]$.

The function $\varphi \in L^2(\mathbb{R})$ is specified via its L^2-Fourier transform according to

$$(\mathcal{F}\varphi)(\omega) = \begin{cases} 1, & |\omega| \leq \frac{1}{3} \\ \cos\left(\frac{\pi}{2}\beta(3|\omega|-1)\right), & \frac{1}{3} < |\omega| \leq \frac{2}{3} \\ 0, & \text{else} \end{cases}, \quad \omega \in \mathbb{R}.$$

For $j,k \in \mathbb{Z}$ define $\varphi_{j,k}$ as $\varphi_{j,k}(x) = 2^{\frac{j}{2}}\varphi(2^j x - k)$, $x \in \mathbb{R}$. Furthermore, for every $j \in \mathbb{Z}$, define $\mathcal{V}_j = \overline{\operatorname{span}\{\varphi_{j,k} : k \in \mathbb{Z}\}}$. You may use — without proof — the following facts throughout the problem:

(F1) $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and

(F2) $\mathcal{V}_{j+1} = \{f(2\,\cdot) : f \in \mathcal{V}_j\}$, for all $j \in \mathbb{Z}$.

(a) Show that, for every $j \in \mathbb{Z}$, $\{\varphi_{j,k} : k \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$.

(b) Show that $\varphi_{-1,0} \in \mathcal{V}_0$ by establishing the existence of a sequence $\{h[k]\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that

$$\varphi_{-1,0} = \sum_{k\in\mathbb{Z}} h[k]\varphi(\,\cdot - k),$$

and deduce that $\mathcal{V}_j \subset \mathcal{V}_{j+1}$, for all $j \in \mathbb{Z}$.

(c) Show that $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ is a multiresolution approximation (MRA) with scaling function φ.

Problem 4

We consider functions of the form

$$x = \sum_{n\in\mathbb{Z}} c_n\, \phi(\,\cdot - nT), \tag{2}$$

where $T > 0$, $\{c_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, and $\phi : \mathbb{R} \to \mathbb{R}$ are such that (2) converges unconditionally in $L^2(\mathbb{R})$, but otherwise arbitrary. In many cases such functions can be recovered from samples taken at integer multiples of T, even though they do not need to be bandlimited.

(a) Fix $T > 0$ and consider the function

$$x(t) = \begin{cases} 1, & 0 \le t < T \\ 2, & T \le t < 2T \\ 1, & 2T \le t < 3T \\ 4, & 3T \le t < 4T \\ 0, & \text{else} \end{cases}, \quad t \in \mathbb{R}. \tag{3}$$

(i) Sketch x on the interval $[-T, 5T]$, and show that x can be written in the form (2) by specifying suitable $\{c_n\}_{n\in\mathbb{Z}}$ and ϕ.

(ii) By explicitly computing the Fourier transform of x in (3), show that this x is not bandlimited. You may use—without proof—the fact that the set of zeros of a trigonometric polynomial is discrete.

(iii) Note that x can be reconstructed from the samples $\{x(nT)\}_{n\in\mathbb{Z}}$ taken at integer multiples of T (provided that ϕ is known). Seeing that x is not bandlimited, explain why this does not contradict the sampling theorem (Theorem 1.4.1 in the lecture notes).

(b) Fix $T > 0$ and let $\phi \in L^2(\mathbb{R})$ be such that $\{\phi(nT)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. Furthermore, suppose that ϕ satisfies the following condition:

$$\text{there exists an } \alpha > 0 \text{ s.t. } \left|\sum_{n\in\mathbb{Z}} \phi(nT)\, e^{-in\theta}\right| \ge \alpha, \quad \text{for all } \theta \in [0, 2\pi). \tag{$*$}$$

Now, consider functions x of the form (2) with $\{c_n\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$.

(i) Let $\boldsymbol{x} = \{x(nT)\}_{n\in\mathbb{Z}}$. Find elements $\boldsymbol{\phi}^n$ of $\ell^2(\mathbb{Z})$, for $n \in \mathbb{Z}$, such that

$$\boldsymbol{x} = \sum_{n\in\mathbb{Z}} c_n\, \boldsymbol{\phi}^n. \tag{4}$$

(ii) Provide an expression for the coefficients $\{c_n\}_{n\in\mathbb{Z}}$ given $\{\phi(nT)\}_{n\in\mathbb{Z}}$ and the samples $\{x(nT)\}_{n\in\mathbb{Z}}$. You may use—without proof—the fact that the series (4) converges unconditionally, and that $\{x(nT)\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$.

(c) Sometimes one needs to sample x at a higher rate in order to recover $\{c_n\}_{n\in\mathbb{Z}}$. Fix $T > 0$ and let

$$\phi(t) = \frac{1}{2}\sigma_{\frac{T}{2}}(t) + \frac{1}{4}\left[\sigma_{\frac{T}{2}}\left(t - \tfrac{T}{2}\right) - \sigma_{\frac{T}{2}}\left(t + \tfrac{T}{2}\right)\right] - \frac{1}{2}\left[\sigma_{\frac{T}{2}}(t - T) + \sigma_{\frac{T}{2}}(t + T)\right], \ t \in \mathbb{R},$$

where $\sigma_{\frac{T}{2}}(u) = \frac{\sin(2\pi u/T)}{2\pi u/T}$, $u \in \mathbb{R}$, is the scaled cardinal sine function. Consider functions x of the form (2) with $\{c_n\}_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$.

(i) Evaluate ϕ at $\{\frac{kT}{2}\}_{k\in\mathbb{Z}}$. Does ϕ satisfy Condition $(*)$? Explain your answer.

(ii) Provide an expression for the coefficients $\{c_n\}_{n\in\mathbb{Z}}$ given $\{\phi(nT)\}_{n\in\mathbb{Z}}$ and the samples $\{x(nT)\}_{n\in\mathbb{Z}}$ and $\{x(nT+\frac{T}{2})\}_{n\in\mathbb{Z}}$.
[Hint: Set $\Phi_1(\theta) := \sum_{n\in\mathbb{Z}} \phi(nT)\, e^{-in\theta}$ and $\Phi_2(\theta) := \sum_{n\in\mathbb{Z}} \phi(nT + T/2)\, e^{-in\theta}$. Show that there exist $\alpha, \beta_1, \beta_2 > 0$ such that $\beta_1|\Phi_1(\theta)|^2 + \beta_2|\Phi_2(\theta)|^2 \geq \alpha$, for all $\theta \in [0, 2\pi)$.]