ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Prof. Dr. Helmut Bölcskei
Chair for Mathematical Information Science

Examination on Mathematics of Information
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Please note:
- Exam duration: 180 minutes
- Maximum number of points: 100
- You are allowed to use any printed or handwritten material (i.e., books, lecture and discussion session notes, summaries), but no computers, tablets, smartphones or other electronic devices.
- Your solutions should be explained in detail and your handwriting needs to be clean and legible.
- Please do not use red or green pens. You may use pencils.
- Please note that the "ETH Zurich Ordinance on Disciplinary Measures" applies.

Before you start:
1. The problem statements consist of 6 pages including this page. Please verify that you have received all 6 pages.
2. Please fill in your name, student ID card number and signature below.
3. Please place your student ID card at the front of your desk so we can verify your identity.

During the exam:

4. For your solutions, please use only the empty sheets provided by us. Should you need additional sheets, please let us know.
5. Each problem consists of several subproblems. If you do not provide a solution to a subproblem, you may, whenever applicable, nonetheless assume its conclusion in the ensuing subproblems.

After the exam:

6. Please write your name on every sheet and prepare all sheets in a pile. All sheets, including those containing problem statements, must be handed in.
7. Please clean up your desk and stay seated and silent until you are allowed to leave the room in a staggered manner row by row.
8. Please avoid crowding and leave the building by the most direct route.

Family name: .... First name: ....

Legi-No.: ....

Number of additional sheets handed in: ....

Signature: ....

Problem 1 (25 points)

Recall the sparse signal recovery procedure

$$\text{(P1)} \qquad \widehat{x} = \arg\min \|\widetilde{x}\|_1 \;\text{ subject to } y = D\widetilde{x},$$

with observation vector $y \in \mathbb{R}^m$ and measurement matrix $D \in \mathbb{R}^{m\times n}$, where $m < n$. In this problem, we are concerned with recovering vectors that are *almost* sparse.

To this end, we define with $s \in \mathbb{N}$, for given $x \in \mathbb{R}^n$,

$$\sigma_s(x) := \inf\{\|x - z\|_1 \mid z \in \mathbb{R}^n,\ \|z\|_0 \le s\}.$$

Further, for $D \in \mathbb{R}^{m\times n}$ with $m < n$, define for $s < \operatorname{spark}(D)$,

$$\Delta_s(D) = \max_{\substack{S\subset[n]\\ |S|=s}} \max_{v\in\ker(D)\setminus 0} \frac{\|v_S\|_1}{\|v_{S^c}\|_1},$$

where $\ker(D) = \{v \in \mathbb{R}^n \mid Dv = 0\}$, $v_S \in \mathbb{R}^n$ denotes the vector obtained from v according to

$$(v_S)_i = \begin{cases} v_i, & i \in S \\ 0, & i \notin S \end{cases},$$

and S^c stands for the complement of the set S in $[n] = \{1, \dots, n\}$. You may assume throughout that $\Delta_s(D)$ is well-defined, i.e., that there are S with $|S| < \operatorname{spark}(D)$ and v that achieve the maximum.

(a) (2 Points) Prove that if $s < \operatorname{spark}(D)$, then for every set $S \subset [n]$ with $|S| = s$, it holds that

$$\|v_{S^c}\|_1 \neq 0, \qquad \forall v \in \ker(D)\setminus 0.$$

(b) (4 Points) Prove the inequality

$$\|(x-z)_{S^c}\|_1 \le 2\|x_{S^c}\|_1 - \|x\|_1 + \|(x-z)_S\|_1 + \|z\|_1\,, \qquad \text{for } x, z \in \mathbb{R}^n,\ S \subset [n].$$

(c) (11 Points) Fix $x \in \mathbb{R}^n$, $D \in \mathbb{R}^{m\times n}$, $s < \operatorname{spark}(D)$, and assume that $\Delta_s(D) \in (0,1)$. Prove that every solution $\widehat{x}$ of (P1) with $y = Dx$ approximates x to within error

$$\|x - \widehat{x}\|_1 \le 2\,\frac{1 + \Delta_s(D)}{1 - \Delta_s(D)}\,\sigma_s(x).$$

Hint: You may use the results from subproblems (a) and (b).

(d) (8 Points) Fix $D \in \mathbb{R}^{m\times n}$, $s < \operatorname{spark}(D)$, and assume that $\Delta_s(D) \in (0,1)$. Show that one can find $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$, with $\|x\|_1 = \|z\|_1$ and $Dx = Dz$, such that

$$\|x - z\|_1 = 2\,\frac{1 + \Delta_s(D)}{1 - \Delta_s(D)}\,\sigma_s(x).$$

Problem 2

In this problem, we are going to study the so-called Zak transform of signals $f \in \mathcal{L}^2(\mathbb{R})$, defined as

$$\mathcal{Z}_f(u,\xi) = \sum_{k=-\infty}^{\infty} e^{i2\pi k\xi} f(u-k),\ \forall (u,\xi) \in [0,1]^2. \tag{1}$$

Let $g \in \mathcal{L}^2(\mathbb{R})$ and consider the set $\left\{g_{n,\ell}(x) = g(x-n)e^{i2\pi\ell x}\right\}_{(n,\ell)\in\mathbb{Z}^2}$. Suppose that g is such that $\{g_{n,\ell}\}_{(n,\ell)\in\mathbb{Z}^2}$ constitutes a Bessel sequence (see Definition 11 in the Handout). Let $\mathbb{T}$ be the analysis operator (see Definition 12 in the Handout) associated with $\{g_{n,\ell}\}_{(n,\ell)\in\mathbb{Z}^2}$.

Theorems 1, 2, and 3 in the Handout can be used in the following without proof. Further, the concepts in Definitions 5 - 14 of the Handout can be useful as well.

(a) (4 points) Let $(n,\ell) \in \mathbb{Z}^2$. Prove that

$$\mathcal{Z}_{g_{n,\ell}}(u,\xi) = e^{i2\pi\ell u} e^{-i2\pi n\xi} \mathcal{Z}_g(u,\xi),\ \forall (u,\xi) \in [0,1]^2. \tag{2}$$

(b) (4 points) Prove that $\langle \mathcal{Z}_f, \mathcal{Z}_{g_{n,\ell}} \rangle_{\mathcal{L}^2([0,1]^2)} = c_{-n,\ell}^{\mathcal{Z}_f\mathcal{Z}_g^*},\ \forall f \in \mathcal{L}^2(\mathbb{R}), \forall n,\ell \in \mathbb{Z}$, where $c_{n,\ell}^{\mathcal{Z}_f\mathcal{Z}_g^*}$ denotes the 2-dimensional Fourier series coefficients (see Theorem 1 in the Handout) of the function $\mathcal{Z}_f\mathcal{Z}_g^*$, with $\mathcal{Z}_g^*$ designating the complex conjugate of the function $\mathcal{Z}_g$.

Hint: Use the result from subproblem (a).

(c) (4 points) Let $f_1, f_2 \in \mathcal{L}^2(\mathbb{R})$. Show that

$$\langle \mathbb{T}f_1, \mathbb{T}f_2 \rangle_{\ell^2} = \sum_{(n,\ell)\in\mathbb{Z}^2} c_{n,\ell}^{\mathcal{Z}_{f_1}\mathcal{Z}_g^*} \left(c_{n,\ell}^{\mathcal{Z}_{f_2}\mathcal{Z}_g^*}\right)^*, \tag{3}$$

where $\langle \cdot, \cdot \rangle_{\ell^2}$ denotes the inner product on ℓ^2 (see Definition 10 in the Handout).

Hint: Use the result from subproblem (b).

(d) (7 points) Prove that $\{g_{n,\ell}\}_{(n,\ell)\in\mathbb{Z}^2}$ is a frame (see Definition 13 in the Handout) if there exist $A, B \in \mathbb{R}$, $0 < A < B$, such that

$$A \le |\mathcal{Z}_g(u,\xi)|^2 \le B,\ \forall (u,\xi) \in [0,1]^2. \tag{4}$$

Hint: Use Plancherel's formula (see Theorem 2 in the Handout).

(e) (6 points) Let $g \in \mathcal{L}^2(\mathbb{R})$ be such that (4) is satisfied, and denote the frame operator corresponding to $\{g_{n,\ell}\}_{(n,\ell)\in\mathbb{Z}^2}$ as $\mathbb{S} = \mathbb{T}^*\mathbb{T}$. Show that $\langle \mathcal{Z}_{\mathbb{S}f}, \mathcal{Z}_\psi \rangle_{\mathcal{L}^2([0,1]^2)} = \langle \mathcal{Z}_f |\mathcal{Z}_g|^2, \mathcal{Z}_\psi \rangle_{\mathcal{L}^2([0,1]^2)},\ \forall f,\psi \in \mathcal{L}^2(\mathbb{R})$.

Hint: Use Plancherel's formula (see Theorem 2 in the Handout).

Problem 3 (25 points)

Fix $\delta \in (0, 1/2)$. Throughout, 'log' denotes logarithm to the base 2. Fix an integer $m \geq 1$ and take $\{x_j\}_{j=1}^m$ to be an orthonormal basis for $\mathbb{R}^m$. Also fix an integer $k \geq 1$ together with a function $f\colon \mathbb{R}^m \to \mathbb{R}^k$ which, for all $1 \leq i, j \leq m$, satisfies

$$(1-\delta)\|x_i - x_j\|_2^2 \leq \|f(x_i) - f(x_j)\|_2^2 \leq (1+\delta)\|x_i - x_j\|_2^2. \tag{5}$$

In this problem, we prove a converse to the Johnson-Lindenstrauss Lemma discussed in the lecture, namely that there exists a constant $C > 0$ independent of k, m, and δ such that, if $C \log(m) > k$, there does not exist a function f satisfying (5).

(a) (5 points) Prove that $\{f(x_j)\}_{j=1}^m \subseteq \mathcal{B}(y, 2)$, where $y := (1/m)\sum_{j=1}^m f(x_j)$ and $\mathcal{B}(y, 2)$ is the open ball with respect to the $\|\cdot\|_2$-norm centered at y and of radius 2.

(b) (7 points) Prove that $\{f(x_j)\}_{j=1}^m$ is a 1-packing (as defined in the Handout, Definition 3) of $(\mathcal{B}(y, 2), \|\cdot\|_2)$.

(c) (8 points) Prove that the 1-packing number $M(1; \mathcal{B}(y, 2), \|\cdot\|_2)$ of $(\mathcal{B}(y, 2), \|\cdot\|_2)$ satisfies

$$C \log\left(M(1; \mathcal{B}(y, 2), \|\cdot\|_2)\right) \leq k,$$

where $C > 0$ is a constant that does not depend on any of k, m, δ.

Hint: Use the volume ratio estimate provided in the Handout (Lemma 1).

(d) (5 points) Conclude that there exists a constant $C > 0$ independent of k, m, and δ such that, if $C \log(m) > k$, there does not exist a function $f\colon \mathbb{R}^m \to \mathbb{R}^k$ satisfying (5).

Problem 4 (25 points)

In this problem, we want to generalize the volume ratio estimate provided in the Handout (Lemma 1) to the Hamming cube. Fix the integer $n \geq 1$, define the Hamming cube as $\mathbb{H}^n := \{0,1\}^n$, and consider the map

$$\begin{aligned} d \colon \mathbb{H}^n \times \mathbb{H}^n &\to \mathbb{N}_0 \\ (x,y) &\mapsto \#\{i \in \{1,\dots,n\} \mid x_i \neq y_i\}. \end{aligned}$$

We use the notation $[n]$ to designate the set of integers $\{1,\dots,n\}$ and $\mathbb{N}_0$ stands for the non-negative integers.

(a) (4 points) Prove that d is a metric on $\mathbb{H}^n$.

(b) (6 points) Given $x \in \mathbb{H}^n$ and an integer $m \in [n]$, we define the ball $\mathcal{B}(x,m)$, centered at x and of radius m with respect to the metric d, to be the subset of $\mathbb{H}^n$ given by

$$\mathcal{B}(x,m) := \{y \in \mathbb{H}^n \mid d(x,y) \leq m\}.$$

Compute the cardinality of the ball $\mathcal{B}(x,m)$.

(c) (6 points) Fix $m \in [n]$. An m-covering of $\mathbb{H}^n$ with respect to the metric d is a set $\{x_1,\dots,x_N\} \subset \mathbb{H}^n$ such that for all $x \in \mathbb{H}^n$, there exists an $i \in \{1,\dots,N\}$ so that $d(x,x_i) \leq m$. The m-covering number $N(m;\mathbb{H}^n,d)$ is the cardinality of the smallest m-covering. Prove that

$$N(m;\mathbb{H}^n,d) \geq \frac{2^n}{\sum_{k=0}^{m} \binom{n}{k}}.$$

(d) (4 points) Fix $m \in [n]$. An m-packing of $\mathbb{H}^n$ with respect to the metric d is a set $\{x_1,\dots,x_M\} \subset \mathbb{H}^n$ such that $d(x_i,x_j) > m$, for all distinct i,j. The m-packing number $M(m;\mathbb{H}^n,d)$ is the cardinality of the largest m-packing. Prove that, for a maximal m-packing $\{x_j\}_{j=1}^{M}$, the balls $\{\mathcal{B}(x_j,\lfloor m/2 \rfloor)\}_{j=1}^{M}$ are, indeed, disjoint subsets of $\mathbb{H}^n$.

(e) (4 points) Deduce from the statement in subproblem (d) that

$$M(m;\mathbb{H}^n,d) \leq \frac{2^n}{\sum_{k=0}^{\lfloor m/2 \rfloor} \binom{n}{k}}.$$

(f) (1 point) Prove that

$$\frac{2^n}{\sum_{k=0}^{m} \binom{n}{k}} \leq N(m;\mathbb{H}^n,d) \leq M(m;\mathbb{H}^n,d) \leq \frac{2^n}{\sum_{k=0}^{\lfloor m/2 \rfloor} \binom{n}{k}}.$$

Hint: You can use, without proof, that, for the Hamming cube, $N(m;\mathbb{H}^n,d) \leq M(m;\mathbb{H}^n,d)$.