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Handout
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Definition 1 (Spark). *The spark of a matrix A, denoted by $\operatorname{spark}(A)$, is defined as the cardinality of the smallest set of linearly dependent columns.*

Definition 2 (Metric). *A metric $d\colon \mathcal{X}\times\mathcal{X}\to\mathbb{R}$ on a non-empty set $\mathcal{X}$ is a function satisfying the following properties:*

- $d(x,x')\geq 0$, *for all* x, x';
- $d(x,x')=0$, *if and only if* $x=x'$;
- $d(x,x')=d(x',x)$, *for all* x, x';
- $d(x,x')\leq d(x,\tilde{x})+d(\tilde{x},x')$, *for all* x, x', $\tilde{x}$.

Definition 3 (Covering number). *Let $(\mathcal{X},d)$ be a compact metric space and $\varepsilon\in\mathbb{R}_+$. An ε-covering of $\mathcal{X}$ with respect to the metric d is a set $\{x_1,\dots,x_N\}\subset\mathcal{X}$ such that for all $x\in\mathcal{X}$, there exists an $i\in\{1,\dots,N\}$ so that $d(x,x_i)\leq\varepsilon$. The ε-covering number $N(\varepsilon;\mathcal{X},d)$ is the cardinality of the smallest ε-covering.*

Definition 4 (Packing number). *Let $(\mathcal{X},d)$ be a compact metric space and $\varepsilon\in\mathbb{R}_+$. An ε-packing of $\mathcal{X}$ with respect to the metric d is a set $\{x_1,\dots,x_M\}\subset\mathcal{X}$ such that $d(x_i,x_j)>\varepsilon$, for all distinct i,j. The ε-packing number $M(\varepsilon;\mathcal{X},d)$ is the cardinality of the largest ε-packing.*

Lemma 1 (Volume ratio estimate of metric entropy). *Consider a pair of norms $\|\cdot\|$ and $\|\cdot\|'$ on $\mathbb{R}^d$, and let $\mathcal{B}$ and $\mathcal{B}'$ be their corresponding unit balls, i.e., $\mathcal{B}:=\{x\in\mathbb{R}^d \mid \|x\|\leq 1\}$ and $\mathcal{B}':=\{x\in\mathbb{R}^d \mid \|x\|'\leq 1\}$. Then, the ε-covering number $N(\varepsilon;\mathcal{B},\|\cdot\|')$ and the ε-packing number $M(\varepsilon;\mathcal{B},\|\cdot\|')$ of $\mathcal{B}$ in the $\|\cdot\|'$-norm satisfy*

$$\left(\frac{1}{\varepsilon}\right)^d\frac{vol(\mathcal{B})}{vol(\mathcal{B}')}\leq N(\varepsilon;\mathcal{B},\|\cdot\|')\leq M(\varepsilon;\mathcal{B},\|\cdot\|')\leq\frac{vol\left(\frac{2}{\varepsilon}\mathcal{B}+\mathcal{B}'\right)}{vol(\mathcal{B}')}.$$

Definition 5. *$\mathcal{L}^2(\mathbb{R})$ denotes the space of square-integrable functions on $\mathbb{R}$, i.e., the set of all functions f satisfying*

$$\int_{\mathbb{R}}|f(x)|^2dx<\infty.$$

We define the norm $\|f\|_{\mathcal{L}^2(\mathbb{R})}=\sqrt{\int_{\mathbb{R}}|f(x)|^2dx}$, for $f\in\mathcal{L}^2(\mathbb{R})$.

Definition 6. *$\mathcal{L}^2([0,1]^2)$ denotes the space of square-integrable functions on $[0,1]^2$, i.e., the set of all functions f satisfying*

$$\iint_{[0,1]^2} |f(x,y)|^2 \, dxdy < \infty.$$

We define the norm $\|f\|_{\mathcal{L}^2([0,1]^2)} = \sqrt{\iint_{[0,1]^2} |f(x,y)|^2 \, dxdy}$, for $f \in \mathcal{L}^2([0,1]^2)$.

Definition 7. *Let $f, g \in \mathcal{L}^2(\mathbb{R})$. We define the inner product on $\mathcal{L}^2(\mathbb{R})$ as*

$$\langle f, g \rangle_{\mathcal{L}^2(\mathbb{R})} = \int_{\mathbb{R}} f(x) g^*(x) dx \tag{1}$$

Definition 8. *Let $f, g \in \mathcal{L}^2([0,1]^2)$. We define the inner product on $\mathcal{L}^2([0,1]^2)$ as*

$$\langle f, g \rangle_{\mathcal{L}^2([0,1]^2)} = \iint_{[0,1]^2} f(x,y) g^*(x,y) \, dxdy \tag{2}$$

Definition 9. *Let $\mathcal{K}$ be a countable set and $\{\alpha_k\}_{k\in\mathcal{K}}$ a sequence of elements taken from $\mathbb{R}$. $\{\alpha_k\}_{k\in\mathcal{K}}$ is an ℓ^2-summable sequence, and we write $\{\alpha_k\}_{k\in\mathcal{K}} \in \ell^2$, if*

$$\sum_{k\in\mathcal{K}} |\alpha_k|^2 < \infty. \tag{3}$$

We define the norm on ℓ^2 as

$$\|\{\alpha_k\}_{k\in\mathcal{K}}\|_{\ell^2} = \sqrt{\sum_{k\in\mathcal{K}} |\alpha_k|^2}. \tag{4}$$

Definition 10. *Let $\mathcal{K}$ be a countable set, $\{\alpha_k\}_{k\in\mathcal{K}} \in \ell^2$ and $\{\beta_k\}_{k\in\mathcal{K}} \in \ell^2$. We define the inner product on ℓ^2 as*

$$\langle \{\alpha_k\}_{k\in\mathcal{K}}, \{\beta_k\}_{k\in\mathcal{K}} \rangle_{\ell^2} = \sum_{k\in\mathcal{K}} \alpha_k \beta_k^*. \tag{5}$$

Definition 11. *Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot \rangle$, $\mathcal{K}$ a countable set, and $\{g_k\}_{k\in\mathcal{K}}$ a sequence of elements taken from $\mathcal{H}$. $\{g_k\}_{k\in\mathcal{K}}$ is a Bessel sequence if*

$$\sum_{k\in\mathcal{K}} |\langle x, g_k \rangle|^2 < \infty, \forall x \in \mathcal{H} \tag{6}$$

Definition 12. *Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot \rangle$, $\mathcal{K}$ a countable set, and $\{g_k\}_{k\in\mathcal{K}}$ a Bessel sequence of elements taken from $\mathcal{H}$. We define the analysis operator $\mathbb{T}$ correlated to $\{g_k\}_{k\in\mathcal{K}}$ as $\mathbb{T}x = \{\langle x, g_k \rangle\}_{k\in\mathcal{K}}$.*

Definition 13. *Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot \rangle$ and norm $\| \cdot \|$, $\mathcal{K}$ a countable set, and $\{g_k\}_{k\in\mathcal{K}}$ a Bessel sequence of elements taken from $\mathcal{H}$. We say that $\{g_k\}_{k\in\mathcal{K}}$ is a frame for $\mathcal{H}$ if there exist $A, B \in \mathbb{R}$ with $0 < A < B$ such that $A\|x\|^2 \leq \langle \mathbb{T}x, \mathbb{T}x \rangle_{\ell^2} = \sum_{k\in\mathcal{K}} |\langle x, g_k \rangle|^2 \leq B\|x\|^2, \forall x \in \mathcal{H}$.*

Theorem 1. *(2-dimensional Fourier series) Every function $h \in \mathcal{L}^2([0,1]^2)$ can be represented as a 2-dimensional Fourier series according to*

$$h(u,\xi) = \sum_{(n,\ell)\in\mathbb{Z}^2} c^h_{n,\ell}\, e^{i2\pi\ell u}\, e^{i2\pi\xi n},\ \forall(u,\xi)\in[0,1]^2, \tag{7}$$

where $\{c^h_{n,\ell}\}_{(n,\ell)\in\mathbb{Z}^2}$ denotes the 2-dimensional Fourier series coefficients of h, which are given by

$$c^h_{n,\ell} = \iint_{[0,1]^2} e^{-i2\pi\ell u} e^{-i2\pi\xi n} h(u,\xi)\, dud\xi,\ \ \forall(n,\ell)\in\mathbb{Z}^2. \tag{8}$$

Theorem 2. *(Plancherel's formula) Let $f_1, f_2 \in \mathcal{L}^2([0,1]^2)$. We have*

$$\langle\{c^{f_1}_{n,\ell}\}_{(n,\ell)\in\mathbb{Z}^2}, \{c^{f_2}_{n,\ell}\}_{(n,\ell)\in\mathbb{Z}^2}\rangle_{\ell^2} = \langle f_1, f_2\rangle_{\mathcal{L}^2([0,1]^2)}, \tag{9}$$

where $\{c^{f_1}_{n,\ell}\}_{(n,\ell)\in\mathbb{Z}^2}$ and $\{c^{f_2}_{n,\ell}\}_{(n,\ell)\in\mathbb{Z}^2}$ denote the 2-dimensional Fourier series coefficients of f_1 and f_2, respectively.

Definition 14. *The Zak transform of the signal $f \in \mathcal{L}^2(\mathbb{R})$ is defined as*

$$\mathcal{Z}_f(u,\xi) = \sum_{k=-\infty}^{\infty} e^{i2\pi k\xi} f(u-k),\ \forall(u,\xi)\in[0,1]^2. \tag{10}$$

Theorem 3. *$\mathcal{Z}$ is a unitary operator between $\mathcal{L}^2(\mathbb{R})$ and $\mathcal{L}^2([0,1]^2)$, i.e.,*

$$\langle x, y\rangle_{\mathcal{L}^2(\mathbb{R})} = \langle \mathcal{Z}_x, \mathcal{Z}_y\rangle_{\mathcal{L}^2([0,1]^2)}. \tag{11}$$